CSCD84: Artificial Intelligence

Worksheet: Probability Review

Q1

Suppose the variables A and B are Boolean variables (*i.e.*, A can have the value a or $\neg a$) and A is independent of B. Determine the missing values in the joint distribution for $P(A, B)$ below.

$P(\neg a, \neg b)$	0.1
$P(\neg a, b)$	0.3
$P(a, \neg b)$	
$P(a, b)$	

Answer. A is independent of B. Hence, $P(A = x, B = y) = P(A = x)P(B = y)$. Therefore, $\frac{P(\neg a, \neg b)}{(\neg a, b)} = \frac{P(B=\neg b)}{P(B=b)} = \frac{0.1}{0.3} = \frac{1}{3}$. Hence, $\frac{P(a,\neg b)}{P(a,b)} = \frac{P(B=\neg b)}{P(B=b)} = \frac{1}{3} \Rightarrow P(a, b) = 3P(a, \neg b)$. These four probabilities should sum to 1. Therefore, $4P(a, \neg b) = 0.6 \Rightarrow P(a, \neg b) = \frac{P(a,b)}{3} = 0.15$. The completed table would be

$P(\neg a, \neg b)$	0.1
$P(\neg a, b)$	0.3
$P(a, \neg b)$	0.15
$P(a, b)$	0.45

Q2

Suppose A, B and C are Boolean variables and that B is independent of C given A. Determine the missing values in the joint distribution for $P(A, B, C)$ below.

$P(\neg a, \neg b, \neg c)$	0.01
$P(\neg a, \neg b, c)$	0.02
$P(\neg a, b, \neg c)$	0.03
$P(\neg a, b, c)$	
$P(a, \neg b, \neg c)$	0.01
$P(a, \neg b, c)$	0.1
$P(a, b, \neg c)$	
$P(a, b, c)$	

Answer. Since B is independent of C given A, $P(A = x, B = y, C = z) = P(B = y|A = x, C = z)P(A = x, C = z) = P(B = y|A = x)P(A = x, C = z)$. Now, let's rewrite the table

$P(B = \neg b|A = \neg a)P(A = \neg a, C = \neg c)$	0.01
$P(B = \neg b|A = \neg a)P(A = \neg a, C = c)$	0.02
$P(B = b|A = \neg a)P(A = \neg a, C = \neg c)$	0.03
$P(B = b|A = \neg a)P(A = \neg a, C = c)$	
$P(B = \neg b|A = a)P(A = a, C = \neg c)$	0.01
$P(B = \neg b|A = a)P(A = a, C = c)$	0.1
$P(B = b|A = a)P(A = a, C = \neg c)$	
$P(B = b|A = a)P(A = a, C = c)$	

Rows 1 and 2 imply that $P(B = \neg b|A = \neg a) = 1/4$, $P(B = b|A = \neg a) = 3/4$, and $P(A = \neg a, C = \neg c) = 1/25$. These results together with row 2 imply that $P(A = \neg a, C = c) = 2/25$. Therefore, the value for row 4 is $2/25 \times 3/4 = 0.06$. Moreover, $P(A = \neg a) = 1 - P(A = a) = 3/25$. Rows 5 and 6 and the fact that $P(A = a) = 22/25$ imply that $P(A = a, C = c) = 10P(A = a, C = \neg c) = 20/25$. Therefore, $P(B = \neg b|A = a) = 1/8$ and $P(B = b|A = a) = 7/8$. Thus, the value for rows7 and 8 are $7/8 \times 2/25$ and $7/8 \times 20/25$, respectively. The completed table would be

$P(\neg a, \neg b, \neg c)$	0.01
$P(\neg a, \neg b, c)$	0.02
$P(\neg a, b, \neg c)$	0.03
$P(\neg a, b, c)$	0.06
$P(a, \neg b, \neg c)$	0.01
$P(a, \neg b, c)$	0.1
$P(a, b, \neg c)$	0.07
$P(a, b, c)$	0.7

Q3

Given that $P(A \mid B) < P(A)$, show that $P(B \mid A) < P(B)$.

Answer. $P(A \mid B) < P(A) \Rightarrow \frac{P(A,B)}{P(B)} < P(A) \Rightarrow \frac{P(A,B)}{P(A)} < P(B) \Rightarrow P(B \mid A) < P(B)$