# Week02 - Part 01

## Review: Binary Linear Classification Learning Model

- **Training Set:** $\mathcal{D} = \{(\underline{x}_1, y_1), \cdots, (\underline{x}_N, y_N)\}$

  $\underline{x}_n \in \mathcal{X}, \quad \underline{x}_n = (x_{n1}, x_{n2}, \cdots, x_{nd}), \quad y_n \in \{-1, +1\} = \mathcal{Y}$

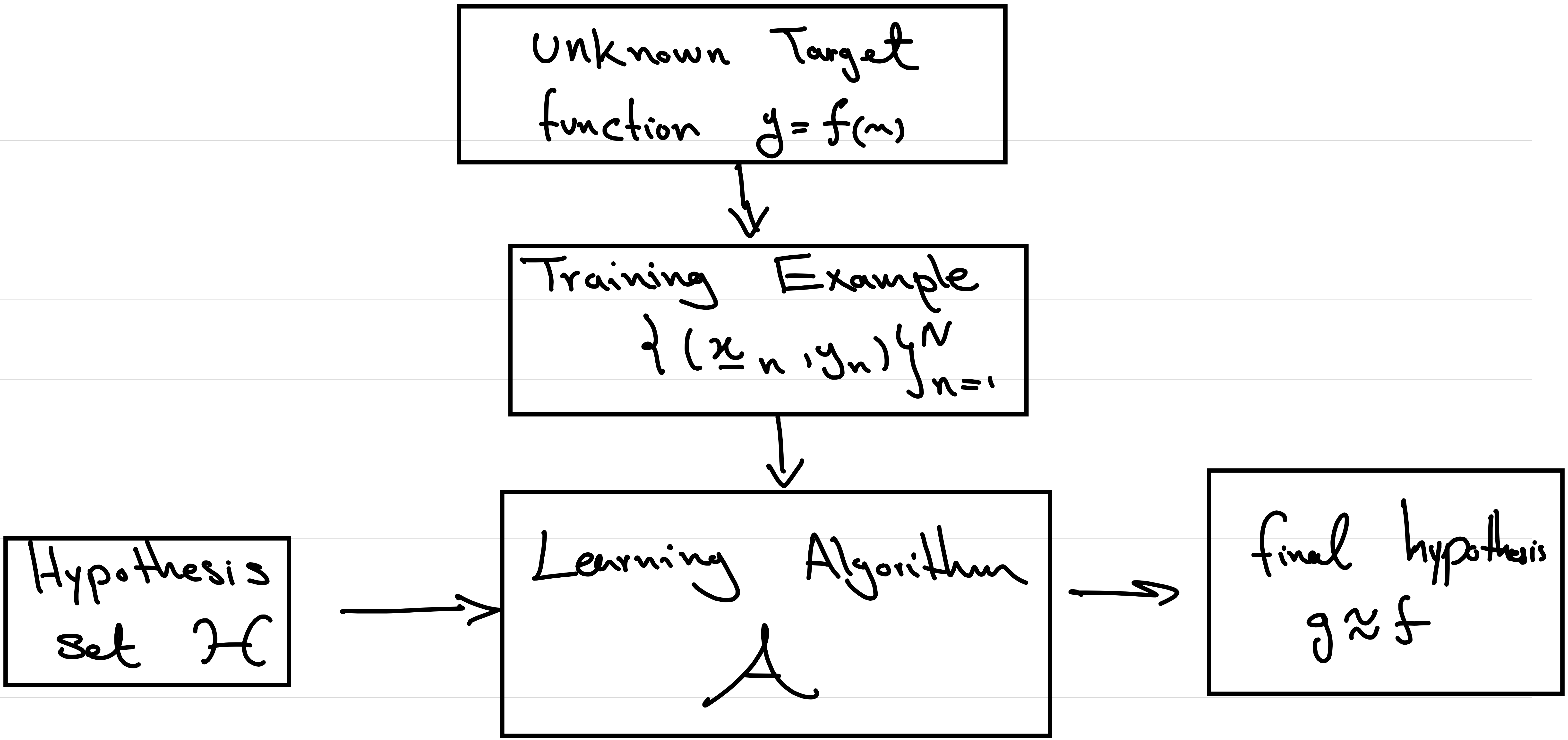
- **Task:** Given any $\underline{x} \in \mathcal{X}$, output $y \in \{-1, +1\} = \mathcal{Y}$

- **Hypothesis (Decision Rule):** $h(\underline{x}) = \text{Sign}\left(\sum_{i=1}^{d} w_i x_i + b\right)$

- **Training:**

$$E_{in}(\underline{w}, b) = \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}\left(f(\underline{x}_n) \neq h(\underline{x}_n)\right) = \sum_{n=1}^{N} \mathbb{1}\left(y_n \neq \text{Sign}\left(\sum_{i=1}^{d} w_i x_{ni} + b\right)\right)$$

# Simple Learning Model Diagram



Last lecture:

- We saw that finding a linear classifier that minimizes $E_{in}$ is NP-hard
- However, if the dataset is linearly separable, we have an algorithm that can find the perfect linear classifier efficiently.
- That algorithm is Perceptron Learning Algorithm (PLA)

Today:

- Perceptron Learning Algorithm

# Perceptron Learning Algorithm

- Efficiently finds a perfect discriminator for linearly separable data set.
- To have cleaner math, we change our notation a bit

Old formulation of the decision rule:

# New formulation of Binary Linear Classification

- **Training Set:** $\mathcal{D} = \{(\underline{x}_1, y_1), \cdots, (\underline{x}_N, y_N)\}$

$\underline{x}_n \in \{1\} \times \mathcal{X}$, $\quad \underline{x}_n = (x_{n0} = 1, x_{n1}, x_{n2}, \cdots, x_{nd})$, $\quad y_n \in \{-1, +1\} = \mathcal{Y}$

- **Hypothesis set:** $h_{\underline{w}} \in \mathcal{H}$, where $h_{\underline{w}}(\underline{x}) = \text{sign}(\underline{w}^T \underline{x})$

weight vector: $\underline{w} = (w_0, w_1, \cdots, w_d) \in \mathbb{R}^{d+1}$

- **Training:** Minimize $E_{in}(\underline{w}) = \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}(y_n \neq h_{\underline{w}}(\underline{x}))$

# Perceptron Learning Algorithm (PLA)

**Input:** training set $D$ that is linearly separable

**Output:** $\underline{w} \in \mathbb{R}^{d+1}$ that achieves $E_{in}(\underline{w}) = 0$

**Initialization:** choose arbitrary $\underline{w}$, e.g., $\underline{w} = \underline{0}$

**Step 1:** check if $E_{in}(\underline{w}) = 0$. If yes, stop and return $\underline{w}$.

**Step 2:** Let $(\underline{x}_n, y_n)$ be a miss-classified point, i.e., $y_n \neq \hat{y}_n$ (including the points on the boundary)

If $y_n = +1$, $\underline{w} \leftarrow \underline{w} + \underline{x}_n$

If $y_n = -1$, $\underline{w} \leftarrow \underline{w} - \underline{x}_n$

**Go to Step 1.**

<demo: vinizinho's PLA visualization>

# Why Does PLA Work? (Intuitive explanation)

Let's have a closer look at datapoint $(\underline{x}_n, y_n)$

| $y_n$ | $\underline{w}^T x_n$ | correct classification? | |
|---|---|---|---|
| | | | |

Let's have a closer look at the updating rule.

- Suppose $(\underline{x}_n, y_n)$ is missclassified.

$$\underline{w}_{new} =$$

- Now let's see what is the impact of updating $\underline{w}$ on how we classify $\underline{x}_n$

$$y_n \, \underline{w}_{new}^T \, \underline{x}_n =$$

- Observe that $y_n \underline{w}_{new}^T \underline{x}_n \boxed{?} y_n \underline{w}^T \underline{x}_n$

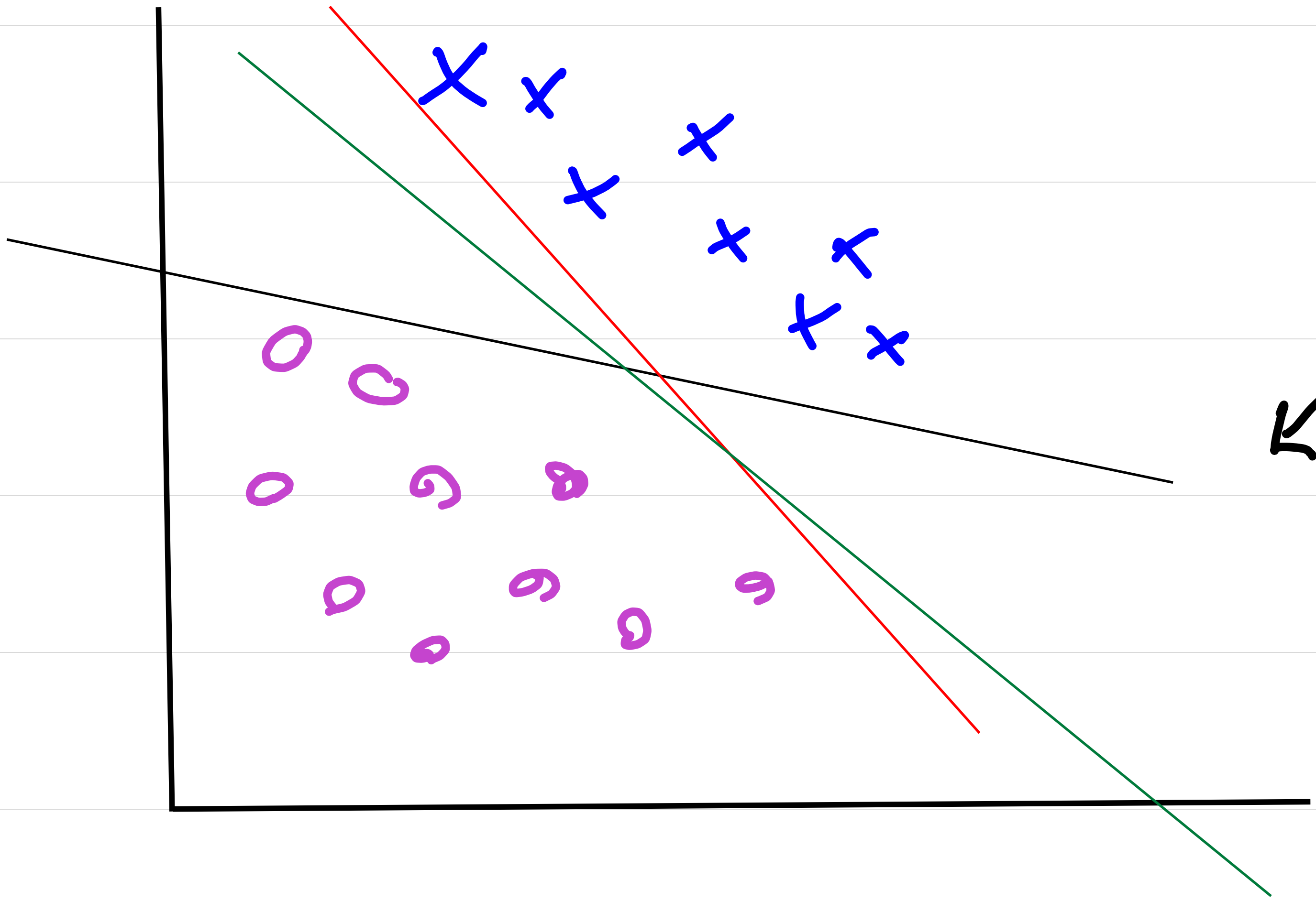
Note: What we saw was an <u>induitive</u> explanation. Although PLA update rule give us a better classifier for the missclassified point $\underline{x}_n$, it may cause new missclassification for other points?

■ We need more than intuitive explanation to show that PLA indeed works.
(This was proved by Rosenblatt, 1957)

Rosenblatt Theorem: Given a linearly separable dataset, PLA terminates in a finite # of steps yielding $E_{in}(\underline{w}) = 0$

(If you are interested, the proof is in Problem 1.3 of LFD)

Remark: The output of PLA is not unique.



which line is better?

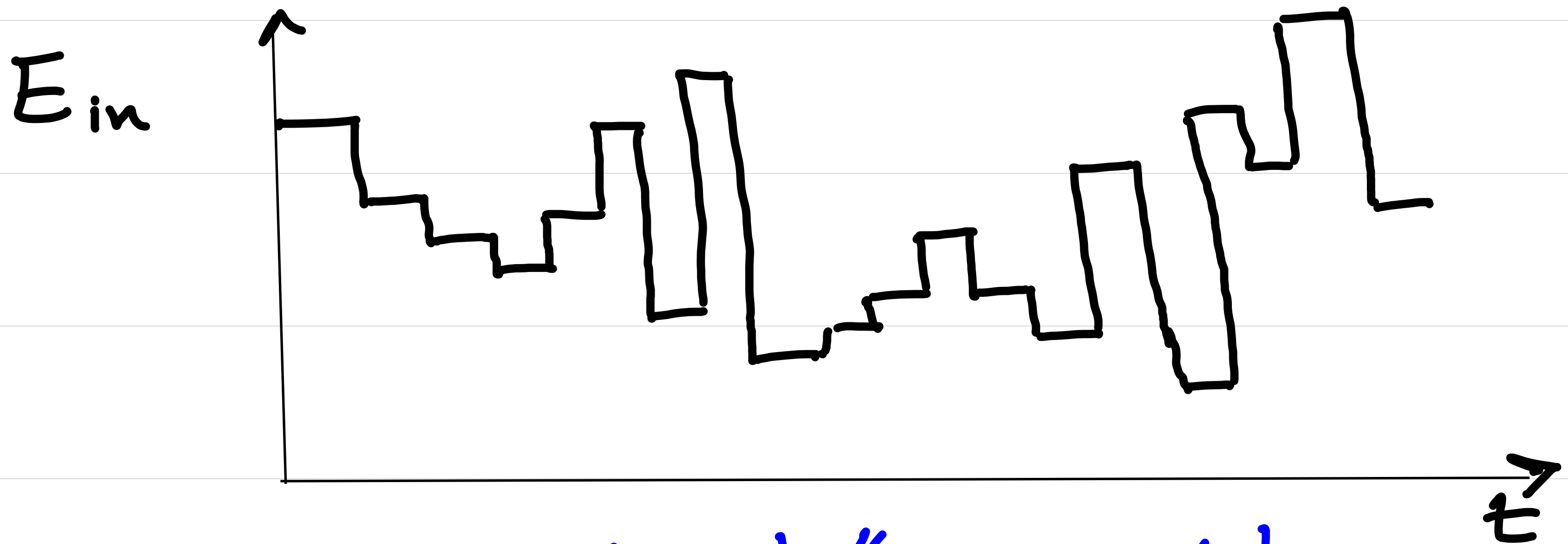
- So far we have only considered linearly separable dataset and saw that PLA works for such dataset.

- What if the dataset is NOT linearly separable?
    - What would happen if we use PLA for such datasets?

- How can we modify PLA to work with non-separable dataset?

# Pocket Algorithm

- Pocket Algorithm extends PLA for dataset that are not linearly separable.




iteration

Keep the "best" weight vector $\underline{w}$ upto iteration $t$ in the pocket.

## Pocket Algorithm:

Pick time horizon $T$

Set pocketed weight vector $\underline{w}^*$ to $\underline{w}(0)$ in PLA.

**for** $t = 1, 2, \dots, T$ :

  Run PLA for one update to obtain $\underline{w}(t)$

  Evaluate $E_{in}(\underline{w}(t))$

  **if** $E_{in}(\underline{w}(t)) < E_{in}(\underline{w}^*)$ **then**

    Set $\underline{w}^* = \underline{w}(t)$

  **End if**

**End for**

**Return** $\underline{w}^*$

- So far, we saw binary classification.
- Can we use perceptron idea to do classification with more than two classes (i.e. multiary classification)?
- Multiary classification