ECE421 - Week 1 - Part 3 - linear classification

- It's time to rigorously define a simple Learning Model
- Consider the example of credit approval
  - Bank needs to determine whether to approve credit to a customer or not (Yes/No)
  - Input:

    Output (label):

    Let $\mathcal{X}$ denote the input space (i.e. the set of all possible $\underline{x}$)

    Let $\mathcal{Y}$ denote the output space (in this example $\mathcal{Y} = \{+1, -1\}$)
  - Unknown Target function:

Note: a bar under a parameter indicates that it is a vector.

- Historical data set $\mathcal{D} = \{(\underline{x}_1, y_1), (\underline{x}_2, y_2), \dots, (\underline{x}_N, y_N)\}$

- Goal: is to design a learning algorithm that uses $\mathcal{D}$ to pick a mapping $g: \mathcal{X} \rightarrow \mathcal{Y}$ that approximates f
- The algorithm chooses g from

• What is $\mathcal{H}$ for linear classification problem?

- We can describe $\mathcal{H}$ through a functional form that is shared among all $h \in \mathcal{H}$
  - In linear classification $h \in \mathcal{H}$ can be described as

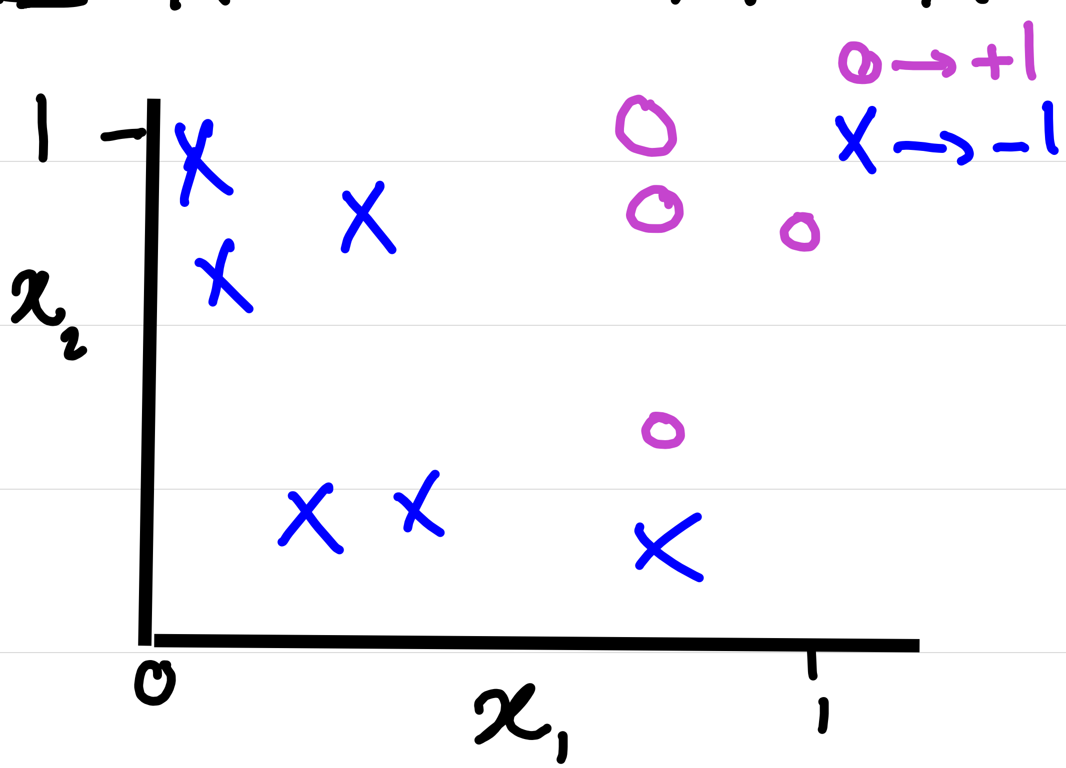
$$h(x) =$$

- **Training**: In linear classification, the goal is to find "good" $g \in \mathcal{H}$ (i.e., a "good" $\underline{w}$ and b), given the data set.

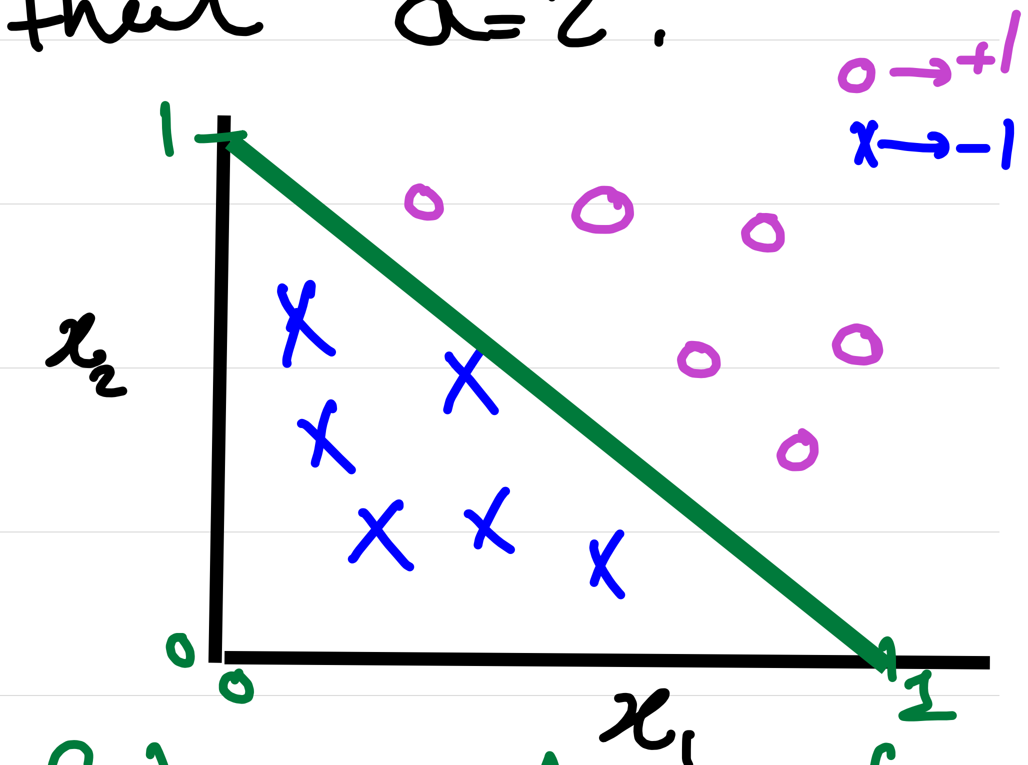
- But "good" to do what?
    - with a good model,
    - In fact, we want the $\underline{w}$ and b that give us the "minimum" possible error

■ Prediction: Given new customer $\underline{x}$, use $\sum_{i=1}^{d} w_i x_i \underset{\hat{y}=-1}{\overset{\hat{y}=+1}{\gtrless}} -b$ to determine $\hat{y}$.

■ Assume for the sake of illustration that $d=2$.



Draw the decision boundary for $\underline{w}=(0,1)$ and $b=-1/2$. Use arrow to show the positive prediction side.



Find $\underline{w}$ and b for the decision boundary above. Note the direction of arrow indicating +1 prediction side.

Basic Setup of Learning Problem of Supervised Learning

Input : Data points : $\underline{x} = (x_1, \dots, x_d) \in \mathcal{X}$

e.g., customer $\underline{x} \in \mathbb{R}^4$

Output : Label $y \in \mathcal{Y}$

Classification : if the label has discrete values

Regression : if the label is Continuous

Unknow Mapping : Target function $f : \mathcal{X} \longrightarrow \mathcal{Y}$

$$y = f(\underline{x})$$

Learning Task. Given training data

$$\mathcal{D} = \{(\underline{x}_1, y_1), (\underline{x}_2, y_2), \ldots, (\underline{x}_N, y_N)\}$$

produce a function $g: \mathcal{X} \rightarrow \mathcal{Y}$ to make predictions on new inputs (i.e., $\hat{y} = g(\underline{x})$)

How do we do this? We have to assume a model

Learning Model: Hypothesis Set: $\mathcal{H} = \{h_1, h_2, \ldots, h_m\}$

each being a candidate function ← $h_i : \mathbb{R}^d \rightarrow \mathbb{R}$, $y = h_i(\underline{x})$

e.g.: $\underline{x} \xrightarrow{\text{sign}(\underline{w}^T \underline{x} + b)} \pm 1$

Learning Algorithm: Select $g \in \mathcal{H}$ using the training set

Summary.

Unknown target
$y = f(\underline{x})$

$\downarrow$

training Examples
$\{(\underline{x}_1, y_1), \ldots, (\underline{x}_N, y_N)\}$

$\downarrow$

Hypothesis Set $\mathcal{H}$ $\longrightarrow$ Learning Algorithm $\mathcal{A}$ $\longrightarrow$ Find hypothesis $g \approx f$

Prediction: $y = g(\underline{x})$ "testing"

Basic setup of Learning Problem of Binary Linear Classification

- **Training Set:** $\mathcal{D} = \{(\underline{x}_1, y_1), \dots, (\underline{x}_N, y_N)\}$

  $\underline{x}_n \in \mathcal{X}, \quad \underline{x}_n = (x_{n1}, x_{n2}, \dots, x_{nd}), \quad y_n \in \{-1, +1\} = \mathcal{Y}$

- **Task:** Given any $\underline{x} \in \mathcal{X}$, output $y \in \{-1, +1\} = \mathcal{Y}$

- **Hypothesis (Decision Rule):**

weight vector: $\underline{w} = (w_1, \dots, w_d) \in \mathbb{R}^d$

bias: $b \in \mathbb{R}$

Given any data point $\underline{x} = (x_1, \dots, x_d)$,

if $\sum_{i=1}^{d} w_i x_i + b > 0$, then $\hat{y} = +1$

if $\sum_{i=1}^{d} w_i x_i + b < 0$, then $\hat{y} = -1$

if $\sum_{i=1}^{d} w_i x_i + b = 0$, output either $+1$ or -1 (Unimportant)

$$h(\underline{x}) = \text{Sign}\left(\sum_{i=1}^{d} w_i x_i + b\right)$$

■ Training: Compare decision rule with training data, to choose the "best" parameter values for decision rule — "best" hypothesis

Given $\mathcal{D}$ find $(\underline{w}, b)$ to minimize the training error: Average error on training set.

$$\underbrace{E_{in}}_{\text{in-sample error}}(\underline{w}, b) = \frac{1}{N}\sum_{n=1}^{N} \mathbb{1}\left(f(\underline{x}_n) \neq h(\underline{x}_n)\right) = \sum_{n=1}^{N} \mathbb{1}\left(y_n \neq \underbrace{\text{Sign}\left(\sum_{i=1}^{d} w_i x_{ni} + b\right)}_{\hat{y}_n}\right)$$

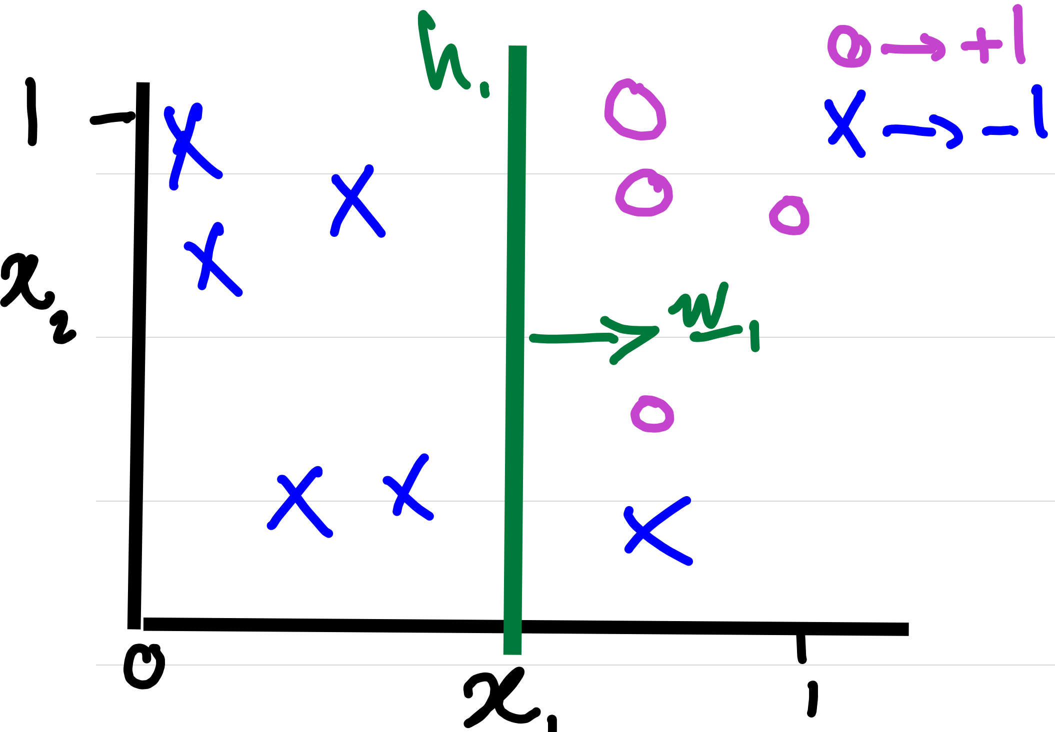
$E_{in}(h)$

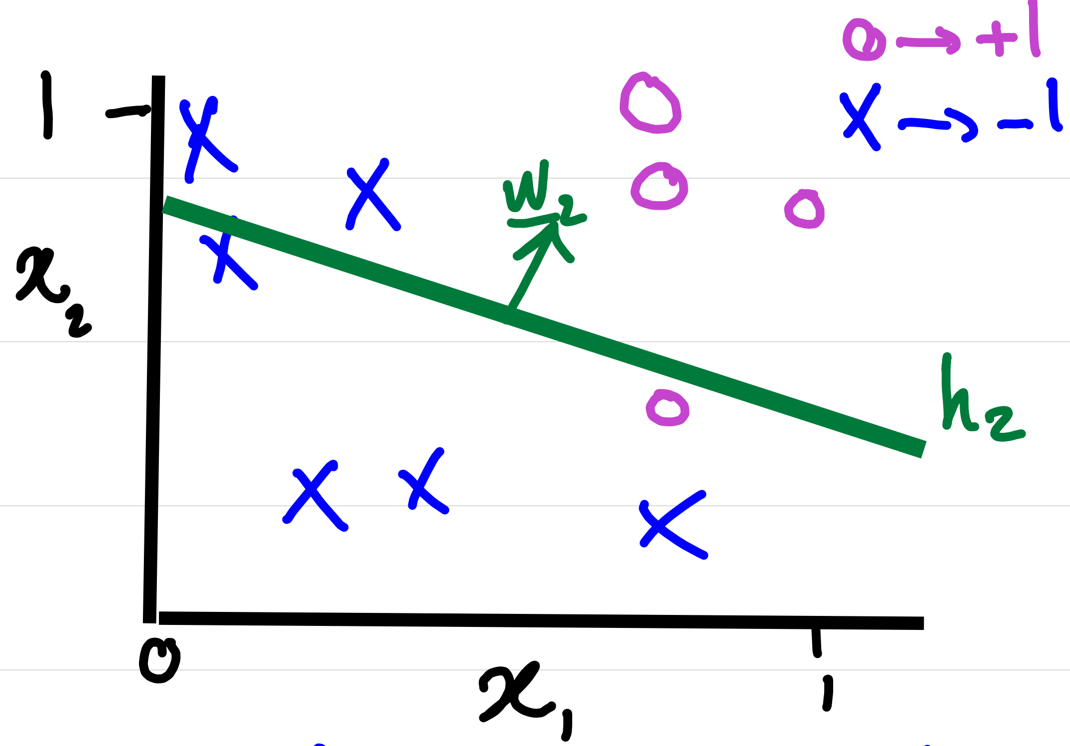
$\mathbb{1}(\cdot)$: Indicator function

y_n: true label for $\underline{x}_n$

$\hat{y}_n$: output of decision rule on example $\underline{x}_n$

x_{ni}: the i-th cordinate of the n-th input, i.e. $\underline{x}_n$





. . . .

$E_{in}(h_1) =$

$E_{in}(h_2) =$

How hard is it to find the best decision boundary, i.e. solving

$$\min_{\underline{w},b} \frac{1}{N}\sum_{n=1}^{N} \mathbb{1}\left(y_n \neq \text{sign}\left(\sum_{i=1}^{d} w_i x_{ni} + b\right)\right)$$

Bad news:

Good news:

Perceptron Learning Algorithm

- Efficiently finds a perfect discriminator for linearly separable data set.
- To have cleaner math, we change our notation a bit

New formulation of Binary Linear Classification

- **Training Set:** $\mathcal{D} = \{(\underline{x}_1, y_1), \cdots, (\underline{x}_N, y_N)\}$

$\underline{x}_n \in \{1\} \times \mathcal{X}$, $\underline{x}_n = (x_{n0} = 1, x_{n1}, x_{n2}, \cdots, x_{nd})$, $y_n \in \{-1, +1\} = \mathcal{Y}$

- **Hypothesis set:** $h_{\underline{w}} \in \mathcal{H}$, where $h_w(\underline{x}) = \text{sign}(\underline{w}^T \underline{x})$

weight vector: $\underline{w} = (w_0, w_1, \cdots, w_d) \in \mathbb{R}^{d+1}$

- **Training:** Minimize $E_{in}(\underline{w}) = \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}(y_n \neq h_{\underline{w}}(\underline{x}))$

Perceptron Learning Algorithm (PLA)

Input: training set D that is linearly separable

Output: $\underline{w} \in \mathbb{R}^{d+1}$ that achieves $E_{in}(\underline{w}) = 0$

Initialization: choose arbitrary $\underline{w}$, e.g., $\underline{w} = \underline{0}$

Step 1: check if $E_{in}(\underline{w}) = 0$. If yes, stop and return $\underline{w}$.

Step 2: Let $(\underline{x}_n, y_n)$ be a miss-classified point,

i.e., $y_n \neq \hat{y}_n$ (including the points on the boundary)

If $y_n = +1$, $\underline{w} \leftarrow \underline{w} + \underline{x}_n$

If $y_n = -1$, $\underline{w} \leftarrow \underline{w} - \underline{x}_n$

Go to Step 1.

<demo: vinizinho's PLA visualization>