§ Lecture 22.0

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Limits are local: Let $X \subseteq \mathbb{R}$. Let x_0 be an adherent point of X. Let $f: X \to \mathbb{R}$ be a function and $L \in \mathbb{R}$.

Let $\delta > 0$. Then

$$\lim_{x \to x_0;\, x \in X} f(x) = L \iff \lim_{x \to x_0;\, x \in X \cap (x_0-\delta, x_0+\delta)} f(x) = L.$$

Proof:

$\Rightarrow$ Since $\displaystyle\lim_{x \to x_0;\, x \in X} f(x) = L$

if $(a_n)_{n \geq 0}$, where $a_n \in X$ be a sequence with $a_n \to x_0$

then $f(a_n) \to L$. $\longrightarrow (1)$

Now note that x_0 is an adherent point of $X \cap (x_0-\delta, x_0+\delta)$, thus there exists a sequence $(a_n)_{n \geq 0}$ with $a_n \in X \cap (x_0-\delta, x_0+\delta)$ that converges to x_0.

For this sequence $f(a_n) \to L$ (from 1)

$$\Rightarrow \lim_{x \to x_0;\, x \in X \cap (x_0-\delta, x_0+\delta)} f(x) = L.$$

$\Leftarrow$ Let $\displaystyle\lim_{x \to x_0;\, x \in X \cap (x_0-\delta, x_0+\delta)} f(x) = L$

Let $(a_n)_{n \geq 0}$ with $a_n \in X$ such that $a_n \to x_0$.

That is $\exists N$ s.t. $\forall\ n \geq N$

$$|a_n - x_0| \leq \varepsilon \quad \forall \varepsilon > 0.$$

Take $\delta > \varepsilon$, then $|a_n - x_0| < \delta$

$$\Rightarrow a_n \in X \cap (x_0-\delta, x_0+\delta) \quad \forall n \geq N.$$

Then $f(a_n) \to L$ with $(f(a_n))_{n \geq N}$.

$\Rightarrow \quad f(a_n) \to L \quad$ for $(f(a_n))_{n \geq 0}$.

Effectively:

$$\lim_{x \to x_0;\ x \in X} f(x) = \lim_{x \to x_0;\ x \in X \cap (x_0 - \delta, x_0 + \delta)} f(x)$$

Continuous functions:

Let $X \subseteq \mathbb{R}$. $f: X \to \mathbb{R}$ be a function. Let $x_0 \in X$. We say that f is continuous at x_0 iff

$$\lim_{x \to x_0;\ x \in X} f(x) = f(x_0).$$ [limit exists and equals $f(x_0)$].

We say that $f(x)$ is discontinuous at x_0 iff it is not continuous at x_0.

Example: Let $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Claim 1: $f(x)$ has no limit at any $x_0 \in \mathbb{R}$.

Claim 2: $f(x)$ is discontinuous at every $x_0 \in \mathbb{R}$.

Answer: Since x_0 is a real number, then there exists a sequence $(q_n)_{n \geq 0}$ of rationals that converges to x_0. That is

$$\lim_{n \to \infty} q_n = x_0$$

Then $f(q_n) = 1$ as $q_n \in \mathbb{Q}$. $\forall n$

or $\lim_{n \to \infty} f(q_n) = 1$.

Now choose $\quad p_n = q_n + \frac{\sqrt{2}}{n} \notin \mathbb{Q}$

Then $\lim p_n = \lim q_n = x_0$

But $\lim_{n\to\infty} p_n = \lim_{n\to\infty} x_n = x_0$.

But $f(p_n) = 0 \quad \forall n$

$$\lim_{n\to\infty} f(p_n) = 0$$

Thus we found two sequences of $\mathbb{R}$ that converge to x_0

but corresponding sequences do not converge to same

numbers.

$\Rightarrow \lim_{x\to x_0;\, x\in\mathbb{R}} f(x)$ doesn't exist.

(2) Since limit doesn't exist at any point of $\mathbb{R}$, the

function is discontinuous.

Note that

$\lim_{x\to x_0;\, x\in\mathbb{R}} f(x)$ doesn't exist.

But $\lim_{x\to x_0;\, x\in\mathbb{Q}} f(x) = 1 = f(x_0)$

$\lim_{x\to x_0;\, x\notin\mathbb{Q}} f(x) = 0 = f(x_0)$

Thus $f|_{\mathbb{Q}}$ and $f|_{\mathbb{R}\setminus\mathbb{Q}}$ are continuous functions respectively

on $\mathbb{Q}$ & $\mathbb{R}\setminus\mathbb{Q}$.

Ex2: $sgn: \mathbb{R}\to\mathbb{R}$

$$sgn(x) = \begin{cases} 1 & x>0 \\ 0 & x=0 \\ -1 & x<0 \end{cases}$$

let $x_0 > 0$ then

$$\lim_{x\to x_0;\, x\in\mathbb{R}} sgn(x) = \lim_{x\to x_0;\, x\in(x_0-\delta,\, x_0+\delta)} sgn(x)$$

$$= 1 = sgn(x_0)$$

$$\lim_{x\to -x_0;\, x\in\mathbb{R}} sgn(x) = \lim_{x\to x_0;\, x\in(-x_0-\delta,\, -x_0+\delta)} sgn(x)$$

$= -1$

$= \text{sgn}(-x_0)$.

$\lim\limits_{x \to 0;\, x \in \mathbb{R}} \text{sgn}(x) = ?$

Take sequence $\left(\frac{1}{n+1}\right)_{n \geq 0} \to 0$

$f\left(\frac{1}{n+1}\right) \to 1$

Take another sequence $\left(-\frac{1}{n+1}\right)_{n \geq 0} \to 0$

$f\left(-\frac{1}{n+1}\right) \to -1$

Thus limit doesn't exist.

§ Lecture 22.1

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Proposition (Equivalent definitions of continuity):

Let $X \subseteq \mathbb{R}$. $f: X \to \mathbb{R}$ be a function. $x_0 \in X$. Then following statements are equivalent.

① f is continuous at x_0.

② For every sequence $(a_n)_{n\geq 0}$ where $a_n \in X$ with

$\lim_{n\to+\infty} a_n = x_0$, $\lim_{n\to+\infty} f(a_n) = f(x_0)$.

③ For every $\varepsilon > 0$, $\exists$ a $\delta > 0$ s.t. $|f(x) - f(x_0)| < \varepsilon$

$\forall x \in X$ with $|x - x_0| < \delta$.

④ For every $\varepsilon > 0$, $\exists$ a $\delta > 0$ s.t. $|f(x) - f(x_0)| \leq \varepsilon$ $\forall x \in X$

with $|x - x_0| \leq \delta$.

Proof: ① $\Leftrightarrow$ ② (From sequential version of limits)

② $\Rightarrow$ ③

Assume ②. Suppose ③ is false.

Then $\exists \varepsilon_0 > 0$ s.t. $\forall \delta > 0$ $\exists x \in X$ with $|x - x_0| < \delta$

but $|f(x) - f(x_0)| \geq \varepsilon_0$.

In particular for $\delta = 1/n+1$, choose $x_n \in X$ s.t.

$|x_n - x_0| < \frac{1}{n+1}$ and

$|f(x_n) - f(x_0)| \geq \varepsilon_0$

Then $x_n \to x_0$
but $f(x_n) \not\to f(x_0)$

contradicts ②.

$\Rightarrow$ ③ is true.

③ $\Rightarrow$ ②

Assume ③. Let $a_n \in X$ and $(a_n)_{n\geq 0}$ be a sequence with $a_n \to x_0$. Fix $\varepsilon > 0$. From ③

$\exists \delta > 0$ s.t. $\forall x \in X$ $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.

Since $a_n \to x_0$ we have

$$|a_n - x_0| < \delta \quad \forall n \geq N.$$

Thus for $n \geq N$ $|f(a_n) - f(x_0)| < \varepsilon$.

$$\Rightarrow f(a_n) \to f(x_0).$$

③ $\Rightarrow$ ④ $\forall \varepsilon > 0$ $\exists \delta' > 0$

$|x - x_0| < \delta' \Rightarrow |f(x) - f(x_0)| < \varepsilon$

choose $\delta = \delta'/2$

Take any $x \in X$ with $|x - x_0| \leq \delta$.

Then $|x - x_0| \leq \delta < \delta' \Rightarrow |f(x) - f(x_0)| < \varepsilon$

$\Rightarrow |f(x) - f(x_0)| \leq \varepsilon$.

④ $\Rightarrow$ ③ $\quad$ ($\{x : |x - x_0| \leq \delta\} \subset \{|x - x_0| < \delta'\}$.

Fix $\varepsilon > 0$. Apply ④ for $\varepsilon' = \varepsilon/2$

$\exists \delta > 0$ s.t.

$|x - x_0| \leq \delta \Rightarrow |f(x) - f(x_0)| \leq \varepsilon'$

Take any x with $|x - x_0| \leq \delta \Rightarrow |f(x) - f(x_0)| \leq \varepsilon' < \varepsilon$.

Claim (Arithmatic preserve continuity)

$f, g : X \to \mathbb{R}$ continuous at x_0. Then

① $f \pm g$

② f/g if $g(x) \neq 0$ $\forall x \in X$ etc. are continuous at x_0.

Left and right limits:

$X \subset \mathbb{R}$ $f : X \to \mathbb{R}$ be a function $x \in \mathbb{R}$ at x_0

$X \subseteq \mathbb{R}$. $f: X \to \mathbb{R}$ be a function $x_0 \in \mathbb{R}$. If x_0

is an adherent point of $X \cap (x_0, \infty)$ then

right limit $f(x_0+)$ of f at x_0 is defined as

$$f(x_0+) = \lim_{x \to x_0;\, x \in X \cap (x_0, \infty)} f(x). = \lim_{x \to x_0^+} f(x)$$

If x_0 is an adherent point of $X \cap (-\infty, x_0)$, then

left limit $f(x_0-)$ of f at x_0 is defined as

$$f(x_0-) = \lim_{x \to x_0;\, x \in X \cap (-\infty, x_0)} f(x). = \lim_{x \to x_0^-} f(x)$$

Proposition: Let $X \subseteq \mathbb{R}$. $x_0 \in \mathbb{R}$ and let x_0 be an

adherent point of both $X \cap (x_0, \infty)$ and $X \cap (-\infty, x_0)$.

If $f(x_0+)$ and $f(x_0^-)$ both exists and are equal to $f(x_0)$

then $f(x)$ is continuous at x_0.

Proof:

We need to prove that $\forall \varepsilon > 0 \ \exists \delta$ s.t.

$\forall \ x \in X$ with $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.

Given

$$\lim_{x \to x_0^+} f(x) = f(x_0) = \lim_{x \to x_0^-} f(x).$$

$$\Downarrow$$

$\forall \varepsilon > 0 \ \exists \delta_+ > 0$ s.t. $\forall \ x \in X \cap (x_0, \infty)$

if $|x - x_0| < \delta_+ \Rightarrow |f(x) - f(x_0)| < \varepsilon$. $\to (1)$

$\forall \varepsilon > 0 \ \exists \delta_- > 0$ s.t. $\forall \ x \in X \cap (-\infty, x_0)$

if $|x - x_0| < \delta_- \Rightarrow |f(x) - f(x_0)| < \varepsilon$. $\to (2)$

Define $\delta = \min\{\delta_+, \delta_-\} > 0$

Consider an arbitrary $x \in X$ with $|x - x_0| < \delta$.

Then

<u>Case 1</u> $x > x_0$

$x \in X \cap (x_0, \infty)$

Since $|x - x_0| < \delta \leq \delta_+$

$\Rightarrow |f(x) - f(x_0)| < \varepsilon$ (from ①)

<u>Case 2</u> $x < x_0$

Then $x \in X \cap (-\infty, x_0)$

Since $|x - x_0| < \delta \Rightarrow |x - x_0| < \delta_-$

$\Rightarrow |f(x) - f(x_0)| < \varepsilon$ (from 2)

<u>Case 3</u>: $x = x_0$

Then $|x - x_0| = 0 < \delta$

and

$|f(x) - f(x_0)| = 0 < \varepsilon$.

Trivially satisfied.

Thus

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x \in X$ with $|x - x_0| < \delta$

$|f(x) - f(x_0)| < \varepsilon$.

$\Rightarrow \lim_{x \to x_0} f(x) = f(x_0)$ and function is continuous at x_0.

The converse is true as well.

§ Lecture 22.2

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Bounded functions: $X \subseteq \mathbb{R}$ $f: X \to \mathbb{R}$

f is bounded from above if $\exists M \in \mathbb{R}$ s.t.

$$f(x) \leq M \quad \forall x \in X.$$

f is bounded from below if $\exists M \in \mathbb{R}$ s.t.

$$f(x) \geq -M \quad \forall x \in X.$$

f is bounded if $\exists M \in \mathbb{R}$ s.t.

$$|f(x)| \leq M \quad \forall x \in X.$$

Lemma: Let $a<b$ be real numbers. $f: [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$. Then f is a bounded function.

Proof: Suppose f is not bounded. Thus for every real number M $\exists$ $x \in [a,b]$ s.t.

$$|f(x)| \geq M.$$

In particular ~~take~~ $M = n \in \mathbb{N}$. Then

$$S_n = \{ x \in [a,b] \cdot |f(x)| \geq n \} \quad \forall n.$$

is nonempty. Let $x_n \in S_n$.

Thus we can choose a sequence $(x_n)_{n \geq 0}$ s.t. $|f(x_n)| \geq n \cdot \forall n \in \mathbb{N}$.

The sequence (x_n) lies in $[a,b]$ so it is closed and bounded.

From Heine-Borel theorem there exists a subsequence $(x_{n_j})_{j=0}^{\infty}$ converges to some limit $L \in [a,b]$, where $n_0 < n_1 - - -$

Also $n_j \geq j$ $\forall j \in \mathbb{N}$ [Induction $n_0 \geq 0$]

$n_j \geq j$

$n_{j+1} > n_j \geq j$

$\Rightarrow n_{j+1} > j$

$\Rightarrow n_{j+1} \geq j+1$

Since f is continuous on $[a,b]$, it is continuous at L. Thus

$$x_{n_j} \to L \Rightarrow \lim_{j\to\infty} f(x_{n_j}) = f(L)$$

Thus sequence $(f(x_{n_j}))_j$ is convergent.

$\Rightarrow (f(x_{n_j}))_j$ is bounded.

on Other hand we know that

$$|f(x_{n_j})| \geq n_j \geq j$$

That it is not bounded. A contradiction.

Definition: Let $f: X \to \mathbb{R}$ be a function. Let $x_0 \in X$.

We say that f attains its maximum (minimum) at x_0 if $f(x_0) \geq f(x)$ $\forall x \in X$

iy ($f(x_0) \leq f(x)$ $\forall x \in X$).

Proposition: Let $a<b$ be real numbers and let $f: [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$.

Then f attains its maximum at some point $x_{max} \in [a,b]$ and minimum at some point $x_{min} \in [a,b]$.

Nontriviality: $f:(0,1) \to \mathbb{R}$ defined by

$$f(x) = x.$$

$\sup f(x) = 1$ but no point $(0,1)$ satisfy $f(x) = 1$.