International Institute of Information Technology, Hyderabad
MA4.101-Real Analysis (Monsoon-2025)

Practice Problems 4 and Solutions

Some definitions and facts.

Adherent points. Let $X \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is said to be an adherent point of X if for all $\varepsilon > 0$ there exists $y \in X$ such that $|y - x| \leq \varepsilon$. Equivalently, $(x - \varepsilon, x + \varepsilon) \cap X \neq \emptyset$ for all $\varepsilon > 0$.

Limit points. Let $X \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is said to be a limit point of X if for all $\varepsilon > 0$ there exists $y \in X \setminus \{x\}$ such that $|y - x| \leq \varepsilon$. Equivalently, $(x - \varepsilon, x + \varepsilon) \cap (X \setminus \{x\}) \neq \emptyset$ for all $\varepsilon > 0$.

Closure and closed sets. Let $X \subseteq \mathbb{R}$. The closure $\overline{X}$ of X may be defined as either: (1) the set of all adherent points of X, or (2) the union of X with its limit points. A set X is said to be closed if $\overline{X} = X$.

Continuity of functions. Let $X \subseteq \mathbb{R}$, let $x_0 \in X$, and let $f : X \to \mathbb{R}$ be a function. The function f is said to be continuous at x_0 if either of the following equivalent conditions holds: (1) For all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in X$ with $|x - x_0| < \delta$ we have $|f(x) - f(x_0)| < \varepsilon$. (2) For all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in X$ with $|x - x_0| \leq \delta$ we have $|f(x) - f(x_0)| \leq \varepsilon$. Thus the use of $<$ or $\leq$ in these definitions is irrelevant.

Derivative of functions. Let $X \subseteq \mathbb{R}$, let x_0 be a limit point of X, and let $f : X \to \mathbb{R}$ be a function defined as $f(x) = x^\alpha$, where $\alpha \in \mathbb{R}$. Then the derivative of f at x_0 is given by $f'(x_0) = \alpha\, x_0^{\alpha - 1}$.

Question 1. Answer the following questions.

(A). Prove that the union of any finite collection of bounded subsets of $\mathbb{R}$ is still a bounded set.

(B). Prove or disprove that union of an infinite collection of bounded subsets of $\mathbb{R}$ is a still a bounded set.

Solution (A). Let $B_1, B_2, \ldots, B_n \subset \mathbb{R}$ be bounded sets. Then for each i, there exist a real number M_i such that $|x| \leq M_i$ for all $x \in B_i$. Define

$$M := \max\{M_1, M_2, \ldots, M_n\}.$$

Now, for any $x \in B_1 \cup \cdots \cup B_n$, x belongs to some B_k, so

$$|x| \leq M_k \leq M.$$

Hence the finite union $B_1 \cup \cdots \cup B_n$ is bounded.

Solution (B). Consider an infinite collection of bounded sets $\{B_i\}_{i=1}^{\infty} \subset \mathbb{R}$. Even though each B_i is bounded individually, the union may be unbounded. For example, take $B_n = (-n-1, n+1)$ for all $n \in \mathbb{N}$. Then each B_n is bounded by $n+1$. But the union is given by

$$\bigcup_{i=1}^{\infty} B_i = \mathbb{R},$$

which is unbounded.

Question 2 (Continuous version of the Squeeze Theorem). Let $X \subseteq \mathbb{R}$, let $E \subseteq X$, and let $x_0 \in \mathbb{R}$ be a *limit point* of E. Let $f, g, h : X \to \mathbb{R}$ be functions such that

$$f(x) \leq g(x) \leq h(x) \quad \text{for all } x \in E.$$

Suppose that

$$\lim_{x \to x_0; x \in E} f(x) = \lim_{x \to x_0; x \in E} h(x) = L$$

for some real number L. Show that

$$\lim_{x \to x_0; x \in E} g(x) = L.$$

Solution. By definition of limit at a limit point, we have for every $\varepsilon > 0$: there exist $\delta_1, \delta_2 > 0$ such that

$$0 < |x - x_0| < \delta_1, \quad x \in E \implies |f(x) - L| < \varepsilon$$

and

$$0 < |x - x_0| < \delta_2, \quad x \in E \implies |h(x) - L| < \varepsilon.$$

Set

$$\delta := \min\{\delta_1, \delta_2\}.$$

Then for any $x \in E$ with $0 < |x - x_0| < \delta$, we have

$$L - \varepsilon < f(x) \le g(x) \le h(x) < L + \varepsilon.$$

This immediately implies

$$|g(x) - L| < \varepsilon \quad \text{for all } x \in E \text{ with } 0 < |x - x_0| < \delta.$$

Since $\varepsilon > 0$ was arbitrary, by definition of limit at a limit point, we conclude

$$\lim_{x \to x_0; x \in E} g(x) = L.$$

Question 3. Give examples of

(A). A continuous, bounded function $f : (1, 2) \to \mathbb{R}$ which attains its minimum but does not attain its maximum.

(B). A continuous function $f : [0, \infty) \to \mathbb{R}$ which is bounded and attains its maximum somewhere, but does not attain its minimum.

(C). A function $f : [-1, 1] \to \mathbb{R}$ which is bounded but does not attain its minimum or maximum.

(D). A function $f : [-1, 1] \to \mathbb{R}$ which has no upper or lower bound.

Solution (A). Take

$$f(x) := (x - 1.5)^2 \qquad (x \in (1, 2)).$$

Checks:

- f is continuous on $(1, 2)$ (a polynomial).
- f is bounded on $(1, 2)$: for $x \in (1, 2)$ we have $0 \le f(x) \le (0.5)^2 = 0.25$.
- $f(1.5) = 0$ and for all $x \neq 1.5$ we have $f(x) > 0$; hence f attains its minimum at $x = 1.5$.

- The supremum of f on $(1,2)$ is $\sup_{x\in(1,2)} f(x) = 0.25$, but this value is only approached as $x \to 1^+$ or $x \to 2^-$; since $1, 2 \notin (1,2)$, no $x \in (1,2)$ satisfies $f(x) = 0.25$. Thus f does not attain a maximum on $(1,2)$.

Solution (B). Take

$$f(x) = \frac{1}{1+x}, \qquad x \in [0,\infty).$$

Checks:

- f is continuous on $[0,\infty)$.
- $0 < f(x] \le 1$ for all $x \ge 0$, so f is bounded.
- $f(0) = 1$, so f attains its maximum at $x = 0$.
- $\inf_{[0,\infty)} f = 0$, but $f(x) > 0$ for every finite x, and $\lim_{x\to\infty} f(x) = 0$. Thus 0 is not attained, so f does not attain a minimum on $[0,\infty)$.

Solution (C). Define $f : [-1,1] \to \mathbb{R}$ by

$$f(x) := \begin{cases} x, & x \in (-1,1), \\ 0, & x = -1 \text{ or } x = 1. \end{cases}$$

Checks:

- **Boundedness:** For all $x \in [-1,1]$, we have $-1 < f(x) < 1$ for $x \in (-1,1)$ and $f(\pm 1) = 0$, so f is bounded.
- **Maximum:** $\sup f = 1$, which would be attained at $x \to 1^-$, but $f(1) = 0 \neq 1$, so the maximum is not attained.
- **Minimum:** $\inf f = -1$, which would be attained at $x \to -1^+$, but $f(-1) = 0 \neq -1$, so the minimum is not attained.

Solution (D). Define $f : [-1,1] \to \mathbb{R}$ by

$$f(x) := \begin{cases} \frac{1}{x} \text{ when } & x \neq 0 \\ 1 \text{ when } & x = 0 \end{cases} .$$

Checks:

- **No upper bound:** As $x \to 0^+$, $f(x) \to +\infty$.
- **No lower bound:** As $x \to 0^-$, $f(x) \to -\infty$.
- f is defined on $[-1, 1]$ and it is not continuous on $x = 0$ and hence not continuous on the full closed interval, thus the maximum principle does not apply.

Question 4. Prove the following statements.

(A). Let $a < b$ and $c < d$ $f : [a, b] \to [c, d]$ be continuous and suppose $f(a) = c$ and $f(b) = d$. Then
$$f([a, b]) = [c, d].$$

(B). Let $a < b$ and $f, g : [a, b] \to \mathbb{R}$ be two continuous functions. If
$$\big(f(a) - g(a)\big)\big(f(b) - g(b)\big) \leq 0,$$
then there exists $c \in [a, b]$ with $f(c) = g(c)$.

Solution (A). Since f maps into $[c, d]$ we have $f([a, b]) \subseteq [c, d]$. Conversely take any $y \in [c, d]$. Because $c = f(a) \leq y \leq f(b) = d$, the number y lies between $f(a)$ and $f(b)$. By the Intermediate Value Theorem there exists $x \in [a, b]$ such that $f(x) = y$. Hence $y \in f([a, b])$. Therefore $[c, d] \subseteq f([a, b])$, and combining inclusions yields $f([a, b]) = [c, d]$.

Solution (B). Define $h(x) := f(x) - g(x)$. Then h is continuous on $[a, b]$. The hypothesis says $h(a)h(b) \leq 0$. If $h(a) = 0$ take $c = a$; if $h(b) = 0$ take $c = b$. Otherwise $h(a)$ and $h(b)$ have opposite signs, so by the Intermediate Value Theorem there exists $c \in (a, b)$ with $h(c) = 0$, i.e. $f(c) = g(c)$.

Question 5. The set of rational numbers $\mathbb{Q}$ is countable, meaning that there exists a bijection $q : \mathbb{N} \to \mathbb{Q}$, so that every rational number appears exactly once in the infinite sequence (q(0), q(1), q(2), q(3), ...) and every rational number occurs at some finite index in this list. Let $g(q(n)) := 2^{-n}$, then $\sum_{r \in \mathbb{Q}} g(r) = \sum_{n \in \mathbb{N}} 2^{-n}$ is absolutely convergent. Now define a function $f : \mathbb{R} \to \mathbb{R}$ by
$$f(x) \; := \sum_{\substack{r \in \mathbb{Q} \\ r < x}} g(r).$$

Now answer the following questions.

(A). Show that for any two real numbers $x < y$, we have $f(x) < f(y)$.

(B). Let r be a rational number. Prove that f fails to be continuous at the point r.

(C). Let x be an irrational number. Prove that f is continuous at x.

Solution. The series $\sum_{n=0}^{\infty} 2^{-n}$ converges (indeed equals 2), hence for every x the subseries that defines $f(x)$ converges absolutely and $f(x)$ is a well-defined real number.

(1) If $x < y$ then $f(x) < f(y)$.

Fix $x < y$. Because the rationals are dense in $\mathbb{R}$, there exists a rational number r with $x < r < y$. Write $r = q(n)$ for some $n \in \mathbb{N}$. By the definition of f,

$$f(x) = \sum_{\substack{s\in\mathbb{Q}\\ s<x}} g(s), \qquad f(y) = \sum_{\substack{s\in\mathbb{Q}\\ s<y}} g(s).$$

The set $\{s \in \mathbb{Q} : s < x\}$ is a proper subset of $\{s \in \mathbb{Q} : s < y\}$: indeed it is missing at least the rational r. Therefore the sum for $f(y)$ equals the sum for $f(x)$ plus a sum of nonnegative terms which includes the term $g(r) = 2^{-n} > 0$. Hence

$$f(y) \;=\; f(x) + \sum_{\substack{s\in\mathbb{Q}\\ x\le s<y}} g(s) \;\ge\; f(x) + g(r) \;=\; f(x) + 2^{-n} \;>\; f(x).$$

Thus $f(y) > f(x)$, as required.

(2) f is discontinuous at every rational point. Let $r \in \mathbb{Q}$. Write $r = q(n)$ for some n. We will show there is a jump of size at least 2^{-n} immediately to the right of r.

By definition,

$$f(r) = \sum_{\substack{s\in\mathbb{Q}\\ s<r}} g(s).$$

If $x > r$ then the set $\{s \in \mathbb{Q} : s < x\}$ contains $\{s \in \mathbb{Q} : s < r\}$ together with r itself (and possibly more rationals). Thus for every $x > r$,

$$f(x) = \sum_{\substack{s\in\mathbb{Q}\\ s<x}} g(s) = \Big(\sum_{\substack{s\in\mathbb{Q}\\ s<r}} g(s)\Big) + g(r) + \sum_{\substack{s\in\mathbb{Q}\\ r<s<x}} g(s) \ge f(r) + g(r) = f(r) + 2^{-n}.$$

Therefore any right-hand limit at r satisfies

$$\lim_{x\to r^+} f(x) \;\ge\; f(r) + 2^{-n}.$$

In particular the right-hand limit is strictly greater than $f(r)$, so f has a jump at r and is not continuous at r. One can also note that the left-hand limit equals $f(r)$: for $x < r$ the sums defining $f(x)$ use only rationals $< x$, which are all contained in the set of rationals $< r$, and as $x \to r^-$ these sums increase to the sum over all rationals $< r$; hence the left limit is $f(r)$. Together with the strictly larger right limit this shows a discontinuity.

(3) f is continuous at every irrational point. Fix an irrational x_0. We will prove $\lim_{x\to x_0} f(x) = f(x_0)$. Let us first define an indicator function $\mathbf{1}_{\{q(k)<x\}}$ as

$$\mathbf{1}_{\{q(k)<x\}} = \begin{cases} 1 & \text{if } q(k) < x, \\ 0 & \text{otherwise.} \end{cases}$$

For each $N \in \mathbb{N}$ define the finite partial sum function

$$f_N(x) := \sum_{k=0}^{N} g\big(q(k)\big)\, \mathbf{1}_{\{q(k)<x\}} = \sum_{k=0}^{N} 2^{-k}\, \mathbf{1}_{\{q(k)<x\}}.$$

Then noting that $0 \le \mathbf{1}_{\{q(k)<x\}} \le 1$, we have

$$\begin{aligned} |f(x) - f_N(x)| &= \left| \sum_{k=0}^{\infty} 2^{-k}\, \mathbf{1}_{\{q(k)<x\}} - \sum_{k=0}^{N} 2^{-k}\, \mathbf{1}_{\{q(k)<x\}} \right| \\ &= \sum_{k=N+1}^{\infty} 2^{-k}\, \mathbf{1}_{\{q(k)<x\}} \\ &\le \sum_{k=N+1}^{\infty} 2^{-k} = 2^{-N}, \end{aligned}$$

for all $x \in \mathbb{R}$. Now since x_0 is irrational, none of the values $q(0), \dots, q(N)$ equals x_0. Since none of the indicator functions switches from 0 to 1 at x_0, therefore, each $\mathbf{1}_{\{q(k)<x\}}$ is constant in some interval around x_0 and hence f_N is continuous at x_0.

For any given $\varepsilon > 0$ we may proceed as follows. Choose N large enough that the tail is small:

$$2^{-N} < \frac{\varepsilon}{3}.$$

Having fixed such N, use continuity of the continuous function f_N at x_0 to find $\delta > 0$ such that

$$|f_N(x) - f_N(x_0)| < \frac{\varepsilon}{3} \quad \text{whenever } |x - x_0| < \delta.$$

Now for such x we estimate

$$\begin{aligned}|f(x)-f(x_0)| &\le |f(x)-f_N(x)|+|f_N(x)-f_N(x_0)|+|f_N(x_0)-f(x_0)|\\ &\le 2^{-N}+\frac{\varepsilon}{3}+2^{-N}<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.\end{aligned}$$

Thus whenever $|x-x_0|<\delta$ we have $|f(x)-f(x_0)|<\varepsilon$. Since $\varepsilon>0$ was arbitrary, this proves $\lim_{x\to x_0} f(x)=f(x_0)$, i.e. f is continuous at the irrational x_0.

Question 6. Let $f,g:(0,\infty)\to\mathbb{R}$ be defined by $f(x)=x^q$ and $g(x)=x^m$, where q is a rational number and m is any integer.

(A). Show that g is differentiable on $(0,\infty)$ and for all $x_0\in(0,\infty)$

$$g'(x_0)=mx_0^{m-1}.$$

(B). Show that f is differentiable on $(0,\infty)$ and that

$$f'(x)=qx^{q-1}.$$

(C). Show that

$$\lim_{x\to 1;\, x\in(0,\infty)\setminus\{1\}}\frac{x^q-1}{x-1}=q.$$

Solution. Using induction, we can prove that for any $x_0\in(0,\infty)$

$$g'(x_0)=mx^{m-1}.$$

The base case $m=0$ is trivial as left hand side is derivative of 1, which is zero and right hand side is zero as well. If the formula holds for m, then

$$(x^{m+1})'=(x\cdot x^m)'=x'\,x^m+x\,(x^m)'=1\cdot x^m+x\cdot mx^{m-1}=(m+1)x^m.$$

For negative integers $m=-k$ one can apply the reciprocal rule to $x^{-k}=1/x^k$ (using the already proved integer power rule) to obtain

$$\frac{d}{dx}x^{-k}=-kx^{-k-1}.$$

This completes part (A).

(B) Rational exponents. Write the rational q in lowest terms as $q=\frac{p}{n}$ with integers $p \in \mathbb{Z}$ and $n \in \mathbb{Z}_{>0}$. We work on $(0, \infty)$. Define two functions $r, h : (0, \infty) \to (0, \infty)$ by

$$r(x) = x^{1/n}, \;\; h(x) = x^n.$$

Then, $(r \circ h)(x) = r(h(x)) = r(x^n) = x$. Now for $x \in (0, \infty)$, applying chain rule we have

$$(r \circ h)'(x) = r'(h(x))h'(x).$$

But $(r \circ h)'(x) = 1$ as $(r \circ h)(x) = x$. Then

$$r'(h(x)) = \frac{1}{h'(x)} = \frac{1}{nx^{n-1}}.$$

Replacing $y = x^n = h(x)$, we have $x = y^{1/n}$. Thus

$$r'(y) = \frac{1}{n\, y^{(n-1)/n}}.$$

Now $f(x) = x^{p/n} = \left(x^{1/n}\right)^p = r(x)^p$. Applying the chain rule again, we get

$$f'(x) = p\, r(x)^{\,p-1} \cdot r'(x) = p\, r(x)^{\,p-1} \cdot \frac{1}{n\, x^{(n-1)/n}}.$$

But $r(x)^{\,p-1} = x^{(p-1)/n}$, so

$$f'(x) = \frac{p}{n}\, x^{(p-1)/n-(n-1)/n} = \frac{p}{n}\, x^{(p-n)/n} = \frac{p}{n}\, x^{\,p/n-1} = q\, x^{q-1}.$$

Thus, f is differentiable on $(0, \infty)$ and $f'(x) = qx^{q-1}$.

(C) Limit at 1**.** By Definition of derivative, we have

$$f'(1) = \lim_{x \to 1; x \in (0,\infty)\setminus\{1\}} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1; x \in (0,\infty)\setminus\{1\}} \frac{x^q - 1}{x - 1}.$$

From part (B) we have $f'(1) = q \cdot 1^{\,q-1} = q$. Therefore

$$\lim_{x \to 1; x \in (0,\infty)\setminus\{1\}} \frac{x^q - 1}{x - 1} = q,$$

as required. (Alternatively one could apply L'Hôpital's rule to the 0/0 form.)

Question 7. Give an example of a function $f : (-1, 1) \to \mathbb{R}$ which is differentiable, and whose derivative equals 0 at 0, but such that 0 is neither a local minimum nor a local maximum.

Solution. A simple example is

$$f(x) = x^3, \qquad x \in (-1, 1).$$

Then f is differentiable on $(-1, 1)$ and

$$f'(x) = 3x^2, \qquad \text{so } f'(0) = 0.$$

However 0 is neither a local minimum nor a local maximum: for $x > 0$ we have $f(x) = x^3 > 0$ and for $x < 0$ we have $f(x) = x^3 < 0$. Thus values of f are both larger and smaller than $f(0) = 0$ in every neighborhood of 0; hence 0 is not an extremum.

Question 8. Let $a < b$. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Define the function

$$H(t) := (b - t)\big(f(a) - f(t)\big) - (t - a)\big(f(b) - f(t)\big), \qquad t \in [a, b].$$

(A). Show that $H(a) = H(b) = 0$.

(B). Use Rolle's theorem to prove that there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(a) + f(b) - 2f(c)}{2c - a - b},$$

whenever $2c - a - b \neq 0$.

(C). Whenever $2c - a - b = 0$ then

$$f'\left(\frac{a + b}{2}\right) = \frac{f(a) + f(b)}{2}.$$

Solution.

(A) Endpoint values. Evaluate H at the endpoints $t = a$ and $t = b$.
At $t = a$,

$$H(a) = (b - a)\big(f(a) - f(a)\big) - (a - a)\big(f(b) - f(a)\big) = 0 - 0 = 0.$$

At $t = b$,

$$H(b) = (b - b)\big(f(a) - f(b)\big) - (b - a)\big(f(b) - f(b)\big) = 0 - 0 = 0.$$

Thus $H(a) = H(b) = 0$.

Regularity of H. Since f is continuous on $[a, b]$ and differentiable on (a, b), and the functions $t \mapsto b - t$ and $t \mapsto t - a$ are polynomials, it follows that H is continuous on $[a, b]$ and differentiable on (a, b). Therefore the hypotheses of Rolle's theorem apply to H.

(ii) Apply Rolle's theorem and compute $H'(t)$. By Rolle's theorem there exists $c \in (a, b)$ such that $H'(c) = 0$. We compute $H'(t)$ for $t \in (a, b)$. It is convenient to expand $H(t)$ first:

$$\begin{aligned} H(t) &= (b - t)f(a) - (b - t)f(t) - (t - a)f(b) + (t - a)f(t) \\ &= (b - t)f(a) - (t - a)f(b) + \big(- (b - t) + (t - a)\big)f(t) \\ &= (b - t)f(a) - (t - a)f(b) + (2t - a - b)f(t). \end{aligned}$$

Differentiate term-by-term (using the product rule for the last term):

$$\begin{aligned} \frac{d}{dt}\big[(b - t)f(a)\big] &= -f(a), \\ \frac{d}{dt}\big[- (t - a)f(b)\big] &= -f(b), \\ \frac{d}{dt}\big[(2t - a - b)f(t)\big] &= (2t - a - b)f'(t) + 2f(t). \end{aligned}$$

Hence, for $t \in (a, b)$,

$$H'(t) = -f(a) - f(b) + 2f(t) + (2t - a - b)f'(t).$$

Setting $t = c$ and using $H'(c) = 0$ gives

$$0 = -f(a) - f(b) + 2f(c) + (2c - a - b)f'(c).$$

Rearranging, we obtain the claimed identity

$$(2c - a - b)\, f'(c) = f(a) + f(b) - 2f(c).$$

(iii) Division case and special midpoint case. If $2c - a - b \neq 0$, we may divide both sides by $2c - a - b$ to get the explicit formula

$$f'(c) = \frac{f(a) + f(b) - 2f(c)}{2c - a - b}.$$

If instead $2c - a - b = 0$, then $c = \dfrac{a+b}{2}$ is the midpoint of $[a,b]$. In that case the identity reduces to

$$0 = f(a) + f(b) - 2f\left(\tfrac{a+b}{2}\right),$$

i.e.

$$f\left(\tfrac{a+b}{2}\right) = \frac{f(a)+f(b)}{2},$$

so f takes the arithmetic mean of its endpoint values at the midpoint. Thus the conclusion in either case gives a precise relation between the value $f(c)$ (or f at the midpoint) and the derivative $f'(c)$.

Question 9. Define the function $f : (0,4] \to \mathbb{R}$ by

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x \neq 2, \\ 0, & x = 2. \end{cases}$$

(A) Show that f is continuous at every point $x_0 \neq 2$.

(B) Compute $\lim\limits_{x\to 2} f(x)$.

(C) Conclude that f is discontinuous at $x = 2$.

Solution.

(A) Continuity at $x_0 \neq 2$. If $x_0 \neq 2$ then in a neighbourhood of x_0 we have $x \neq 2$, so the formula

$$f(x) = \frac{x^2-4}{x-2}$$

applies. Since $x^2 - 4 = (x-2)(x+2)$, for all $x \neq 2$ we have

$$f(x) = x + 2.$$

The function $x \mapsto x + 2$ is a polynomial and hence continuous everywhere. Therefore

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} (x+2) = x_0 + 2 = f(x_0).$$

Thus f is continuous at all $x_0 \neq 2$.

(B) The limit as $x \to 2$. For all $x \neq 2$,

$$f(x) = x + 2,$$

so

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}(x+2) = 4.$$

(C) Discontinuity at $x = 2$. We have

$$f(2) = 0, \qquad \lim_{x\to 2} f(x) = 4.$$

Since the limit exists but is not equal to the value of the function, f is *not* continuous at 2. The discontinuity is a **removable discontinuity**, because redefining $f(2)$ to be 4 would make the function continuous at $x = 2$.

Another approach for (A) and (B). (A) Continuity at $x_0 \neq 2$. Fix $x_0 \in (0, 4]$ with $x_0 \neq 2$. For every $x \neq 2$ we may simplify

$$\frac{x^2 - 4}{x - 2} = x + 2,$$

so for x in any neighbourhood of x_0 not containing 2 we have $f(x) = x + 2$ and $f(x_0) = x_0 + 2$. Let $\varepsilon > 0$ be given. Choose

$$\delta := \min\left\{\frac{|x_0 - 2|}{2},\ \varepsilon\right\}.$$

If $|x - x_0| < \delta$ then $|x - x_0| < |x_0 - 2|/2$, hence

$$|x - 2| \geq |x_0 - 2| - |x - x_0| > \frac{|x_0 - 2|}{2} > 0,$$

so $x \neq 2$ and the formula $f(x) = x + 2$ holds. Therefore

$$|f(x) - f(x_0)| = |(x+2) - (x_0 + 2)| = |x - x_0| < \delta \leq \varepsilon.$$

Since this δ works for every $\varepsilon > 0$, f is continuous at x_0.

Now we show $\lim_{x\to 2} f(x) = 4$. Let $\varepsilon > 0$. For every x with $0 < |x - 2| < \varepsilon$ (so $x \neq 2$) we have $f(x) = x + 2$, hence

$$|f(x) - 4| = |(x+2) - 4| = |x - 2| < \varepsilon.$$

Thus choosing $\delta := \varepsilon$ gives the desired implication: if $0 < |x-2| < \delta$ then $|f(x) - 4| < \varepsilon$. Therefore $\lim_{x\to 2} f(x) = 4$.