§ Lecture 1.1

Thursday, 31 July 2025 08:33

What is real analysis:

Analysis of the real numbers, sequences and series of real numbers, and real valued functions.

① Rigorous study

② Theoretical foundation for calculus

collection of computational algorithms to manipulate functions.

Questions like:

① What is the largest real number? Are there more real numbers than rational numbers?

② Which sequences have limits? Can you add infinitely many real numbers to get a finite real number?

③ What does it mean for a function to be continuous, differentiable, integrable, bounded?

But why do we care?

You already know the rules like L'Hôpital rule, chain rule or integration by part.

But not knowing why these rules work, or whether there are exceptions, one can get in trouble!

Examples:

Let $S = 1 + \frac{1}{2} + \frac{1}{4} + \cdots$

$2S = 2 + 1 + \frac{1}{2} + \cdots$

$= 2 + S$

$\Rightarrow \boxed{S = 2}$

Let $S = 1 + 2 + 4 + 8 + \cdots$

$$2S = 2 + 4 + 8 + \cdots \quad +1 \quad -1$$

$$= S - 1$$

$$\Rightarrow \boxed{S = -1}$$

Can it be correct?

But (a) $S = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$

$$= 1 - (1 - 1 + 1 - 1 \cdots)$$

$$S = 1 - S \Rightarrow S = \tfrac{1}{2}.$$

(b) $S = (1-1) + (1-1) + \cdots$

$$= 0 + 0 + \cdots = 0$$

(c) $S = 1 + (-1+1) + (-1+1) + \cdots$

$$= 1 + 0 + 0 \cdots = 1$$

which is correct?

* $S = \lim_{n\to\infty} x^n$

$n \to m+1$

$$S = \lim_{m+1\to\infty} x^{m+1}$$

$$= x \lim_{m+1\to\infty} x^m$$

$$= xS$$

$S = xS \Rightarrow$ either $S = 0$ or $x = 1$

or $\lim_{n \to \infty} x^n = 0 \quad \forall x \neq 1.$

$*$ $$\lim_{x \to 0} \lim_{y \to 0} \frac{x^2}{x^2+y^2} \stackrel{??}{=} \lim_{y \to 0} \lim_{x \to 0} \frac{x^2}{x^2+y^2}$$

$$\lim_{x \to 0} 1 = \lim_{y \to 0} \frac{0}{y^2}$$

$$1 = 0.$$

§ Lecture 1.2

Monday, 4 August 2025 11:21

Number systems:

$\mathbb{N} \rightarrow$ Natural $(0, 1, \cdots)$

$\mathbb{Z} \rightarrow$ Integers $(\cdots -1, 0, 1, \cdots)$

$\mathbb{Q} \rightarrow$ Rationals $\left(\frac{m}{n}\right)$ $m, n \in \mathbb{Z}$

$\mathbb{R} \rightarrow$ Reals

$\mathbb{C} \rightarrow$ complex

How to actually define natural numbers?

$\Updownarrow$

(using computer & building computer)

Peano axioms:

Axiom 1: 0 is a natural number.

Axiom 2: If n is a natural number then $n{+}{+}$ is also a natural number.

$n{+}{+}$ is the increment operation.

$$\mathbb{N} = 0,\ 0{+}{+},\ (0{+}{+})\underbrace{{+}{+}}_{},\ \dots$$

not Addition !!

$$0 \qquad 1 \qquad \underset{1{+}{+}}{\overset{2}{\shortparallel}}$$

Definition

Is it enough ?

Ex:
$0{+}{+} = 1$
$1{+}{+} = 2$
$2{+}{+} = 0 \overset{??}{=} 3$

Axiom 3: 0 is not successor of any natural number, i.e. $n{+}{+} \neq 0$ for any n.

Other system:

$0{+}{+} = 1,\quad 1{+}{+} = 2,\quad 2{+}{+} = 1,\ \dots$

$0{+}{+} = 1,\quad 1{+}{+} = 2,\quad 2{+}{+} = 2,\ \dots$

Axiom 4: Different natural numbers must have different successors, for $n \neq m$; $n{+}{+} \neq m{+}{+}$

equivalently if $n{+}{+} = m{+}{+} \Rightarrow n = m$.

* Suppose $N := \{0, 0.5, 1, \ldots\}$

[We want some axiom that tells only number in N are those which "can be obtained" from 0 and increment op.]

Axiom 5 : (Principle of mathematical induction)

Let $P(n)$ be any property pertaining to a natural number n. Suppose that $P(0)$ is true, and suppose that $P(n)$ true $\Rightarrow$ $P(n{+}{+})$ is true as well.

Then $P(n)$ is true for all natural numbers!

Common way to use :

Suppose the property is

$P(n) = n$ "is not half integer"

(1) $P(0) = 0$ " " is true.

(2) Let $P(n) = n$ true $\Rightarrow$ $P(n++)$ = true

$\Rightarrow$ From axioms $P(n) = n$ is not half integer for all natural numbers.

$\Rightarrow$ This is just an example !

In fact, it is an infinite number of axioms.

Proposition.

A certain property $P(n)$ is true for all natural numbers.

Proof: First base case $n = 0$.

(1) Prove $P(0)$ is true.

(2) Assume $P(n)$ is proven to be true

$\Rightarrow$ From here prove that $P(n++)$ is true.

$\Rightarrow P(n)$ is true for all natural numbers.

Assumption 1: There exists a number system $\mathbb{N}$, whose elements we will call natural numbers, for which Axioms 1–5 are true.

Uniqueness: $\{0, 1, 2, \cdots\}$ Hindu–Arabic

$\{0, I, II, \cdots\}$ Roman

Isomorphic.

These are the only assumptions about numbers.

Q: Assuming Peano axioms 1–5, prove that all natural numbers are finite.

$\Rightarrow$ Infinity is not a natural number.

Q. Consider a number system that has infinity as number, prove that they cannot obey principle of induction!!

⟹ This is axiomatic approach to natural numbers!

We can't answer
- what these numbers are made of
- what do they measure

⟹ We just defined an increment operation.

This is what abstract mathematics looks only properties of the objects.

Historically, the numbers could be treated axiomatically is very recent. (Peano)
1858–1932

Beads on abacus ? $\frac{1}{2}$, $3+4i$, e^{-2}.

mass of a physical object }

With the aid of axioms, we can define sequences recursively.

Proposition: Suppose for each natural number n, we have function $f_n : \mathbb{N} \to \mathbb{N}$. Then we can assign a unique natural number a_n to each natural number n, s.t. $a_0 = c$ and $a_{n++} = f_n(a_n)$ $\forall n \in \mathbb{N}$.

↳ natural.

Proof: $a_0 = c$.

None of the other definitions $a_{n++} := f_n(a_n)$ will redefine a_0, because of Axiom 3. (0 is not successor any natural number)

suppose inductively we get a single value a_n.

Then it gives

$$a_{n++} = f_n(a_n).$$

None of the other definitions $a_{m++} := f_m(a_m)$

will redefine the value a_{n++} because of Axiom 4.

Unambigous definition:

$$f_n(m) = c \quad \forall n, m \in \mathbb{N}.$$

Then $a_0 = c = a_1 = \dots = \dots$ all are equal. It is still a well defined sequence.

§ Lecture 1.3

Monday, 4 August 2025 18:55

Addition:

Def: Let m be a natural number. To add 0

define $0+m := m$

Suppose inductively we defined $n+m$. Then

$$(n{+}{+})+m := (n+m){+}{+}.$$

$0+m = m$

$1+m = (0{+}{+})+m$

$= (0+m){+}{+}$

$= m{+}{+}$

$2+m = (1{+}{+})+m = (1+m){+}{+} = (m{+}{+}){+}{+}.$

$2+3 = (3{+}{+}){+}{+} = 4{+}{+} = 5.$

→ $n+m$ is defined for all natural n.

$\Rightarrow$ $n+m$ is defined for all natural n.

$$\boxed{\begin{aligned} a_n &= n+m \\ f_n(a_n) &= a_n{+}{+} = a_{n++} \end{aligned}}$$

Notice we defined $0+m := m$

what about $m+0 = ?$

<u>Lemma 1:</u> $n+0 = n \quad \forall n \in \mathbb{N}.$

<u>Proof:</u> Induction: from definition

$$0+m = m \Rightarrow 0+0 = 0 \quad \text{(Base case)}$$

~~Now~~ assume $n+0 = n$

we want to show that $\boxed{(n{+}{+})+0 = n{+}{+}}$

Now $$(n{+}{+})+0 = (n+0){+}{+}$$
$$= n{+}{+}$$

$\Rightarrow$ $n+0 = n.$

Lemma 2: For all $n, m \in \mathbb{N}$ $\quad n+(m++) = (n+m)++$

Proof: Induct on n keeping m fixed.

(a) Base case is $\quad 0+(m++) = (0+m)++ = m++$.

By definition

(b) Let $\quad n+(m++) = (n+m)++$

Goal: $\quad (n++)+(m++) = ((n++)+m)++$

$$LHS = (n++)+(m++) = (n+(m++))++$$

$$= ((n+m)++)++$$

$$\overset{\text{Definition}}{=} (n++ + m)++$$

$$\left(n++ = (n+0)++ = n+0++ = n+1. \right)$$

Proposition: $n+m = m+n \quad \forall m, n \in \mathbb{N}$.

Proof:- Induction on n keeping m fixed.

(a) Base case: $0+m = m+0$

$m = m$.

(b) Let $n+m = m+n$

then $(n++)+m = m+(n++)$

L.H.S $(n+m)++$

R.H.S. $(m+n++) = (m+n)++ = (n+m)++$.

Q. Addition is associative:

$$(a+b)+c = a+(b+c).$$

Prop: Cancellation law: Let $a,b,c \in \mathbb{N}$. s.t.

$$a+b = a+c \Rightarrow b=c$$

Proof: "Subtraction or negative numbers" Not allowed.

Induction on a for fixed b:

(a) Base case: $0+b = 0+c$, i.e. if $a=0$ then $b=c$.

(b) Let $a+b = a+c \Rightarrow b=c$. Then we need

$$(a++)+b = (a++)+c \Rightarrow b=c.$$

LHS: $(a++)+b = (a+b)++$

$(a++ +c) = (a+c)++$

$(a+b) = (a+c)$ (Axiom 4).

$\Rightarrow b=c.$

Notion of order:

Let $n, m \in \mathbb{N}$. We say that n is greater than or equal to m [$n \geq m$ or $m \leq n$] iff

$n = m + a$ for some $a \in \mathbb{N}$.

We say that n is strictly greater than m

[$n > m$ or $m < n$] iff

$n \geq m$ & $n \neq m$. "Properties" [ ]

Trichotomy of order:

1. Either one of them is correct for any two $m, n \in \mathbb{N}$.

[$m > n$, $m = n$, $m < n$.]

§ Lecture 1.4

Monday, 4 August 2025 20:01

Just like addition is iterated increment of ++, multiplication is iterated addition.

Def: Let m be a natural number. Define multiplication by 0 as

$$0 \times m := 0$$

Suppose we define $n \times m$. Then

$$(n{+}{+}) \times m = (n \times m) + m.$$

$$0 \times m = 0$$

$$1 \times m = (0{+}{+}) \times m = (0 \times m) + m$$
$$= 0 + m = m$$

$$2 \times m = (1{+}{+}) \times m = (1 \times m) + m$$
$$= m + m$$

Claims:

1. $m \times n = n \times m \quad \forall\, n, m \in \mathbb{N}.$

2. $n \times m = 0$ iff either $n = 0$ or $m = 0$.

3. $m(n+l) = mn + ml$

   $(n+l)\,m = nm + lm = mn + ml$

4. $m \times (n \times l) = (m \times n) \times l$

5. If $a, b \in \mathbb{N}$ and $c \neq 0 \in \mathbb{N}$ then s.t.

   $a < b \Rightarrow ac < bc.$

   If $ac = bc \quad \forall c \neq 0 \in \mathbb{N}$ then $a = b$.

Exponentiation:

Let m be a natural number. To raise m to the power 0, we define $m^0 := 1$.

$(0^0 = 1)$

Suppose m^n has been defined for $n \in \mathbb{N}$. Then

$$m^{n++} := m^n \times m.$$

$$2^0 = 1$$

$$2^1 = 2^{0++} = 2^0 \times 2 = 1 \times 2 = 2$$

$$2^2 = 2^{1++} = 2^1 \times 2 = 2 \times 2 = (1++) \times 2$$

$$= 1 \times 2 + 2$$

$$= 2 + 2$$

$$= 4.$$