

§ Lecture 21.0

Thursday, 30 October 2025 10:47

Convergence of functions at a point:

Let $X \subseteq \mathbb{R}$. $f: X \rightarrow \mathbb{R}$. Let x_0 be an adherent

point of X . We say that f converges to $L \in \mathbb{R}$

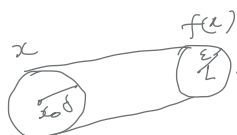
at x_0 and write

$$\lim_{x \rightarrow x_0} f(x) = L \quad \text{iff}$$

$\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$$|f(x) - L| \leq \varepsilon \quad \forall x \in X \text{ such that}$$

$$|x - x_0| < \delta.$$



Proposition: Following statements are equivalent.

(a) f converges to L at x_0 in X .

(b) For every sequence $(a_n)_{n=0}^{\infty}$ which consists

entirely of elements of X and converges to x_0 ,

the sequence $(f(a_n))_{n=0}^{\infty}$ converges to L .

Proof: (a) \Rightarrow (b)

Fix any sequence (a_n) with $a_n \in X$ and

$$\lim_{n \rightarrow \infty} a_n = x_0.$$

We must prove that $\lim_{n \rightarrow \infty} f(a_n) = L$.

By (a) $\forall \varepsilon > 0 \exists \delta$ s.t.

$$\forall x \in X, |x - x_0| < \delta \Rightarrow |f(x) - L| \leq \varepsilon.$$

Since $\lim_{n \rightarrow \infty} a_n = x_0$

$\exists N$ s.t. $\forall n \geq N$

$$|a_n - x_0| < \delta \quad (\text{For any } \delta > 0)$$

Because $a_n \in X$ & $|a_n - x_0| < \delta$ ($\forall n \geq N$)

$$\Rightarrow |f(a_n) - L| \leq \varepsilon \quad \forall n \geq N$$

$$\Rightarrow \lim_{n \rightarrow \infty} f(a_n) = L.$$

(b) \Rightarrow (a)

We prove by contradiction. Suppose (a) is false.

Then $\exists \varepsilon_0 > 0$ s.t.

$\forall \delta > 0 \quad \exists x \in X$ with $|x - x_0| < \delta$ s.t.

$$|f(x) - L| > \varepsilon_0 \quad \rightarrow \textcircled{1}$$

Using eq (1) we now build a sequence (x_n) in X

converging to x_0 but failing to make $(f(x_n))$

converge to L .

For each $n \geq 0$ apply $\textcircled{1}$ with $\delta = \frac{1}{n+1}$

to obtain $x_n \in X$ with $|x_n - x_0| < \frac{1}{n+1}$ and

$$|f(x_n) - L| > \varepsilon_0.$$

Then since

$$x_0 - \frac{1}{n+1} < x_n < x_0 + \frac{1}{n+1}$$

$$x_0 \leq \lim_{n \rightarrow \infty} x_n \leq x_0$$

$$\lim_{n \rightarrow \infty} x_n = x_0.$$

But $|f(x_n) - L| > \varepsilon_0 \quad \forall n \geq 0.$

$\Rightarrow (f(x_n))$ doesn't converge to L .

This contradicts (b). \Rightarrow (b) is not true.

thus $\neg(a) \Rightarrow \neg(b)$ or $(b) \Rightarrow (a)$.

Remark (1):

The limits of function $f: X \rightarrow \mathbb{R}$ are defined at
adherent points of X .

suppose you don't do that! suppose x_0 is
not an adherent point X .

x_0 is an adherent point of $X \Leftrightarrow$

$$\forall \varepsilon > 0 \quad \exists x \in X \text{ s.t. } |x - x_0| \leq \varepsilon.$$

x_0 is not an adherent point of $X \Leftrightarrow$

$$\exists \delta > 0 \quad \forall x \in X \quad |x - x_0| > \delta.$$

$$\text{or } X \cap [x_0 - \varepsilon, x_0 + \varepsilon] = \emptyset$$

$$\text{let } \delta = \varepsilon_0/2 < \varepsilon_0$$

Then there are no $x \in X$ satisfying

$$|x - x_0| < \delta.$$

$\Rightarrow |f(x) - L| \leq \varepsilon$ is vacuously true
for all L .

so the definition is not so good.

Remark 2: some authors define limits at

limit / cluster points (like Bartle & Sherbert)

set of adherent points = ^{set of} limit points $\cup X$.

so our definition is general.

Corollary: let X be a subset of \mathbb{R} and x_0 be an

adherent point of X and $f: X \rightarrow \mathbb{R}$ be a function.

Then f can have at most one limit at x_0 in X .

Proof: Suppose f has limits L and L' at x_0 .

Since x_0 is an adherent point of X , then there exists a sequence (a_n) of elements of X that converges to x_0 .

$\Rightarrow f(a_n) \rightarrow L$
 and $f(a_n) \rightarrow L'$ (by assumption & previous lemma)

This is a contradiction to the uniqueness of limits.

$\Rightarrow f$ has a unique limit if it exists.

§ Lecture 21.1

Friday, 31 October 2025 00:54

Limit laws for functions:

Let $X \subseteq \mathbb{R}$. x_0 be an adherent point of X .

$$f: X \rightarrow \mathbb{R}$$

$$g: X \rightarrow \mathbb{R} \quad \text{be two functions.}$$

Suppose

$$\lim_{x \rightarrow x_0} f(x) = L$$

$$\lim_{x \rightarrow x_0} g(x) = M$$

Then

$$(1) \quad \lim_{x \rightarrow x_0} (f \pm g)(x) = L \pm M$$

$$(2) \quad \lim_{x \rightarrow x_0} \max(f, g)(x) = \max\{L, M\}$$

$$(3) \quad \lim_{x \rightarrow x_0} fg(x) = LM$$

$$(4) \quad \lim_{x \rightarrow x_0} cf(x) = cL$$

$$(5) \quad \text{If } g(x) \neq 0 \quad \forall x \in X \text{ and } M \neq 0 \text{ then}$$

$$\lim_{x \rightarrow x_0} \frac{f}{g}(x) = \frac{L}{M}.$$

Note that: (1) $(f \pm g)(x) = f(x) \pm g(x)$

$$(2) \quad \max(f, g)(x) = \max\{f(x), g(x)\}$$

$$(3) \quad \min(f, g)(x) = \min\{f(x), g(x)\}$$

$$(4) \quad fg(x) = f(x)g(x)$$

(composition is different)

$$(5) \quad \text{if } g(x) \neq 0$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

$$(6) \quad cf(x) = c \cdot x \cdot f(x).$$

Proof: Since x_0 is an adherent point of X

$\Rightarrow \exists (a_n)_{n \geq 0}$ such that $a_n \in X$ & $\lim_{n \rightarrow \infty} a_n = x_0$

From above lemma

$$\lim_{n \rightarrow \infty} f(a_n) = L$$

$$\lim_{n \rightarrow \infty} g(a_n) = M$$

By limit laws of sequences

$$\begin{aligned} \lim_{n \rightarrow \infty} (f+g)(a_n) &= \lim_{n \rightarrow \infty} (f(a_n) + g(a_n)) \\ &= \lim_{n \rightarrow \infty} f(a_n) + \lim_{n \rightarrow \infty} g(a_n) \\ &= L + M. \end{aligned}$$

Similarly other claims.

Some simple limits.

① $\lim_{x \rightarrow x_0} c = c$

Take $f: \mathbb{R} \rightarrow \mathbb{R}$ as $f(x) = c \quad \forall x \in \mathbb{R}$.

x_0 is an adherent point of \mathbb{R} .

Let (a_n) be a sequence of \mathbb{R} s.t.

$$\lim_{n \rightarrow \infty} a_n = x_0$$

then
$$\lim_{n \rightarrow \infty} f(a_n) = \lim_{n \rightarrow \infty} c = c.$$

$\Rightarrow \lim_{x \rightarrow x_0} c = c.$

② $\lim_{x \rightarrow x_0} x = x_0$

Let $f(x) = x$

Let $\lim_{n \rightarrow \infty} a_n = x_0$

$\Rightarrow \lim_{n \rightarrow \infty} f(a_n) = \lim_{n \rightarrow \infty} a_n = x_0$

$\Rightarrow \lim_{x \rightarrow x_0} x = x_0.$

$x \rightarrow x_0$

$$\textcircled{3} \quad \lim_{x \rightarrow x_0} x^2 = \lim_{x \rightarrow x_0} x \cdot \lim_{x \rightarrow x_0} x \\ = x_0^2.$$

$$\textcircled{4} \quad \lim_{x \rightarrow x_0} \sum_{l=0}^N a_l x^l = \sum_{l=0}^N a_l x_0^l.$$

Ex2: Signum function $\text{sgn}: \mathbb{R} \rightarrow \mathbb{R}$

$$\text{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$

$$\textcircled{a} \quad \lim_{x \rightarrow 0; x \in (0, \infty)} \text{sgn}(x) = 1$$

$$\textcircled{b} \quad \lim_{x \rightarrow 0; x \in (-\infty, 0)} \text{sgn}(x) = -1$$

$$\textcircled{c} \quad \lim_{x \rightarrow 0, x \in \mathbb{R}} \text{sgn}(x) = \text{Undefined.}$$

Proof: \textcircled{a} Let $a_n \in (0, \infty)$ form a sequence

such that $\lim_{n \rightarrow \infty} a_n = 0$. (As 0 is an adjacent point of $(0, \infty)$)

$$\text{Then } \lim_{n \rightarrow \infty} \text{sgn}(a_n) = \lim_{n \rightarrow \infty} 1 = 1 \quad \text{as } a_n > 0.$$

$$\Rightarrow \lim_{x \rightarrow 0; x \in (0, \infty)} \text{sgn}(x) = 1.$$

$$\textcircled{b} \quad \text{similarly } \lim_{x \rightarrow 0; x \in (-\infty, 0)} \text{sgn}(x) = -1$$

$$\textcircled{c} \quad \text{Take } a_n = \frac{1}{n} > 0 \in \mathbb{R}$$

$$b_n = -\frac{1}{n} < 0 \in \mathbb{R}.$$

$$\lim_{n \rightarrow \infty} a_n = 0 \quad ; \quad \lim_{n \rightarrow \infty} b_n = 0$$

$$\Downarrow$$

$$\Downarrow$$

$$\lim_{n \rightarrow \infty} \text{sgn}(a_n) = 1 \quad \lim_{n \rightarrow \infty} \text{sgn}(b_n) = -1$$

$n \rightarrow \infty$ $n \rightarrow \infty$

There is not a single number L such that

$$\lim_{x \rightarrow 0; x \in \mathbb{R}} g(x) = L$$

Example: $f: \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

$\lim_{x \rightarrow 0; x \in \mathbb{R}} f(x) =$ is not defined.

$$(a_n) = \frac{1}{n+1} \in \mathbb{Q} \quad n \geq 0$$

$$\lim_{n \rightarrow \infty} a_n = 0$$

$$\lim_{n \rightarrow \infty} f(a_n) = \lim_{n \rightarrow \infty} 1 = 1.$$

$$b_n = \frac{\sqrt{2}}{n} \notin \mathbb{Q}$$

$$\lim_{n \rightarrow \infty} b_n = 0$$

$$\lim_{n \rightarrow \infty} f(b_n) = \lim_{n \rightarrow \infty} 0 = 0.$$

\Rightarrow limit of f at 0 in \mathbb{R} is not defined.

Continuity of functions:

Let $X \subseteq \mathbb{R}$ and let $f: X \rightarrow \mathbb{R}$. Let $x_0 \in X$.

We say that f is continuous at x_0 iff

$$\lim_{x \rightarrow x_0} f(x) = f(x_0).$$

As x converges to x_0

$$f(x) \rightarrow f(x_0).$$

We say that f is continuous on X iff it is

continuous at x_0 for all $x_0 \in X$.

Otherwise discontinuous.

eg $f(x) = x \quad x \in \mathbb{R}$

Then $\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} x = x_0 = f(x_0)$.
 $\forall x_0 \in \mathbb{R}$.

on the other hand