- Claim: Let $(a_n)_{n=0}^{\infty}$ be a sequence of rationals then
$$\underset{n\to\infty}{\mathrm{LIM}}\, a_n = \lim_{n\to\infty} a_n$$

Proof: let $L = \underset{n\to\infty}{\mathrm{LIM}}$ $L \in \mathbb{R}$

We want to show that $(a_n)_{n=m}^{\infty}$ converges to L.

$$\forall \varepsilon > 0 \ \exists N \geq M, \text{ s.t. } \forall \ n \geq N, |a_n - L| \leq \varepsilon$$

Suppose for contradiction that $(a_n)_{n=m}^{\infty}$ doesn't converge to L.

Contrapositive:
$$\exists \varepsilon > 0, \forall N \geq m, \exists n \geq N \text{ s.t. } |a_n - L| > \varepsilon$$

Let $\varepsilon > 0$ be such a positive number

For each $N \geq m$ there is always some $n \geq N$ s.t. $|a_n - L| > \varepsilon$

There are infinitely many $n \geq N$ s.t. $L - \varepsilon > a_n > L + \varepsilon$

$\Rightarrow$ Either there are infinitely many n s.t. $a_n > L + \varepsilon$ or there are infinitely many n s.t. $a_n < L - \varepsilon_0$

(Pigeonhole principle)

We also know that $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence
$\forall \varepsilon > 0 \; \exists N$ s.t. $\forall \; n, m \geq N$

$\Rightarrow \; |a_n - a_m| \leq \varepsilon$

For $\varepsilon = \varepsilon_0/2$

$$|a_n - a_m| \leq \varepsilon_0/2 \;\; \forall \; n, m \geq N_0 \longrightarrow (3)$$

then let $n_1 \geq N_0$ be the index s.t. $a_{n_1} > L + \varepsilon_0$

From (3) $\quad a_n \geq a_{n_1} - \varepsilon_0/2$

$\boxed{a_m - \varepsilon_0/2 \leq a_n \leq a_m + \varepsilon_0/2}$

$$\lim_{n\to\infty} a_n \geq \lim_{n\to\infty} (L + \varepsilon_0/2)$$

$$L \geq L + \varepsilon_0/2 \text{ for some } \varepsilon_0 > 0. \text{ [Contradiction]}$$

Proof (Case 2): Let $n_2 \geq N_0$ be an index s.t.
$a_n < L - \varepsilon_0$

From (3) $\quad a_n \leq a_{n_2} + \varepsilon_0/2$
$\quad < L - \varepsilon_0/2$

$$L = \lim_{n\to\infty} a_n \leq L - \varepsilon_0/2$$

$\Rightarrow L \leq L - \varepsilon_0/2$ for some $\varepsilon_0 > 0$ [Contradiction]

$\Rightarrow$ $(a_n)_{n=m}^{\infty}$ is a convergent sequence converging to L or $\lim_{n\to\infty} a_n = L$

- **Bounded sequences**: A sequence $(a_n)_{n=m}^{\infty}$ is said to be bounded iff $\exists$ some $M \in \mathbb{R}$ s.t. $|a_n| \leq M$ $\forall$ $n \geq m$.

- **Lemma**: Every Cauchy sequence is bounded.

Proof: $(a_1, a_2 \ldots\ldots a_{N-1}, a_N, a_{N+1} \ldots\ldots)$

Cauchy sequence of reals.

- **Limit laws**:

Let $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ s.t.

$$x = \lim_{n\to\infty} a_n, \quad y = \lim_{n\to\infty} b_n$$

Then,

① $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$

② $\lim_{n\to\infty} (a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n)$

③ If $b_n \neq 0$ & $y \neq 0$ then

$$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$$

$(1, 2, 3, \ldots)$

Extended real number system

$$R^{+} = R \cup \{+\infty, -\infty\}$$

Define $\begin{cases} -(+\infty) = -\infty \\ -(-\infty) = \infty \end{cases}$

Supremum on extended real line R^{*}:

Let $E \subseteq R^{*}$ then

① $Sup(E)$ = Least upper bound on E if $E \subseteq R$

② If E contains $+\infty$, i.e, $+\infty \in E$ then $sup(E) = +\infty$

③ If $+\infty \notin E$ & $-\infty \in E$ then

$$sup(E) = sup(E / \{-\infty\})$$

$Inf(E) = -Sup(-E)$, where

$$-E = \{-x : x \in R^{+}\} \text{ if } E = \{x : x \in R^{*}\}$$

Supremum need not to be in the set.

eg. $\{x : x \in R, x^2 < 2\} = A$

$sup(A) = \sqrt{2}$ but $\sqrt{2} \notin A$

Let $(a_n)_{n=m}^{\infty}$ be a sequence of reals then

Def: $\sup(a_n)_{n=m}^{\infty} := \sup(E)$

$E = \{a_n \in \mathbb{R} : n \geq m\}$

$\inf(a_n)_{n=m}^{\infty} = \inf E$

- Every convergent and/or Cauchy sequence is bounded

Is converse true? NO

$\{1, -1, 1, -1, \ldots\}$ ← Bounded but not convergent

But we can impose more conditions on bounded sequence to get convergence? Yes.

- Proposition:

ⓐ Let $(a_n)_{n=m}^{\infty}$ be a sequence that is upper bounded by some $M \in \mathbb{R}$ and $a_{n+1} \geq a_n \;\forall\; n \geq m$ then $(a_n)_{n=m}^{\infty}$ is convergent. In particular,

$$\lim_{n\to\infty} a_n = \sup(a_n)_{n=m}^{\infty} \leq M$$

ⓑ Let $(b_n)_{n=m}^{\infty}$ be a sequence that is lower bounded by some $M \in \mathbb{R}$ and $a_{n+1} \leq a_n \;\forall\; n \geq m$ then $\lim_{n\to\infty} b_n = \inf(b_n)_{n=m}^{\infty} \geq M$

Proof:

ⓐ Since $(a_n)_{n=m}^{\infty}$ has an upper bound, this means $\sup(a_n)_{n=m}^{\infty}$ exists

$$L := \sup(a_n)_{n=m}^{\infty} = \sup(E), \text{ where } E = \{a_n : n \geq m\}$$

We want to show that $(a_n)_{n=m}^{\infty}$ converges to L

$\forall \varepsilon > 0, \exists N \geq m$ s.t. $\forall n \geq N$

$$|a_n - L| \leq \varepsilon$$

Let us fix some $\varepsilon > 0$ then $L - \varepsilon$ cannot be an Upper bound on E.

$\Rightarrow \exists$ some $N_0 \geq m$ for which $a_{N_0} > L - \varepsilon$

For $n \geq N_0$

$a_n \geq a_{N_0} > L - \varepsilon$ —— ①

② But L is an upper bound on E

$$a_n \leq L \quad (n \geq N_0)$$
$$< L + \varepsilon \text{ ——— ②}$$

$\Rightarrow$ ① & ② $\Rightarrow$

$$L + \varepsilon > a_n > L - \varepsilon$$

or $|a_n - L| < \varepsilon \ \forall n \geq N_0$

$\Rightarrow \forall \varepsilon > 0, \exists$ some No s.t. $\forall n > N$.

$|a_N - L| < \varepsilon \Rightarrow \lim_{n\to\infty} a_n = L$

- Upper Bounded + Increasing $\Rightarrow$ convergent
  Lower Bounded + Decreasing $\Rightarrow$ convergent

- Not convergent $\Rightarrow$ Not upper bounded OR Not Increasing

Ex:

ⓐ $\lim_{n\to\infty} x^n = 0 \quad \forall \; 0<x<1$

ⓑ $\lim_{n\to\infty} x^n = +\infty \quad \forall \; x>1$

$\Rightarrow$ ⓐ

$x^{n+1} = x^n \cdot x < x^n$

It is a decreasing sequence

Also, $x^n > 0$ (because $x>0$)

$\Rightarrow (x^n)_{n=1}^{\infty}$ is convergent

$\Rightarrow \lim_{n\to\infty} x^n := L$

$\lim_{n\to\infty} x^{n+1} = (\lim_{n\to\infty} x^n)(\lim_{n\to\infty} x)$

$\Rightarrow L = Lx$

since $x<1$, $L=0$

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ⓑ $x > 1 > 0$

$(b_n) = \left(\frac{1}{x^n}\right)_{n=1}^{\infty}$

$b_{n+1} = \frac{1}{x^{n+1}} = \frac{1}{x^n} \cdot \frac{1}{x} < b_n$ —— ①

Also since $x > 1 \Rightarrow \frac{1}{x} > 0$

$\Rightarrow \frac{1}{x^n} > 0$ —— ②

$\Rightarrow \lim_{n\to\infty} \frac{1}{x^n}$ exists

Now suppose for contradiction that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} x^n$ exists

Then $1 = x^n \cdot \frac{1}{x^n}$

$\lim_{n\to\infty} 1 = \lim_{n\to\infty}(x^n) \times \lim_{n\to\infty}\left(\frac{1}{x^n}\right)$

$\boxed{1 = 0} \Rightarrow \lim_{n\to\infty} a_n$ doesn't exist.