§ Lecture 26.0

Saturday, 15 November 2025 00:19

Local maxima and minima:

Let $f: X \to \mathbb{R}$ be a function and let $x_0 \in X$. We say that f attains a local maxima at x_0 iff $\exists\ \delta > 0$ s.t

$$f(x_0) \geq f(x) \qquad \forall\ x \in X \cap (x_0 - \delta, x_0 + \delta)$$

f attains a local minima at x_0 iff $\exists\ \delta > 0$ s.t.

$$f(x_0) \leq f(x) \quad \forall\ x \in X \cap (x_0 - \delta, x_0 + \delta).$$

Theorem (Local extrema are stationary):

Let $a < b$ and $f: (a,b) \to \mathbb{R}$ be a function. If

(i) $x_0 \in (a,b)$

(ii) f is differiable at x_0

(iii) f attains either local maxima or local minima at x_0

Then $\quad f'(x_0) = 0.$

Proof: Given

$$\lim_{x \to x_0;\, x \in X - \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

or

$\forall\ \varepsilon > 0\ \exists\ \delta_1 > 0$ s.t whenever $|x - x_0| \leq \delta_1$

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| \leq \varepsilon. \quad \to (1)$$

Assume f attains a local maxima at x_0.
(local minima similarly)

Then $\exists\ \delta_2 > 0$ s.t.

$$f(x_0) \geq f(x) \quad \forall\ x \in X \cap (x_0 - \delta_2, x_0 + \delta_2) \quad \to (2)$$

Case 1: Let $f'(x_0) > 0$.

Apply eqn (1) for $\varepsilon = f'(x_0)/2$

Then $\exists \delta_1 > 0$ s.t. whenever $|x - x_0| \leq \delta_1$ we have

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| \leq \frac{f'(x_0)}{2}$$

or $$\frac{f(x) - f(x_0)}{x - x_0} \geq \frac{f'(x_0)}{2} \longrightarrow (3)$$

Now consider a point x s.t.

(i) $|x - x_0| < \min\{\delta_1, \delta_2\}$

(ii) $x > x_0$

Then from (3) $f(x) - f(x_0) \geq 0$

or $f(x) \geq f(x_0)$

But this contradicts the fact that f achieves local maximum at $x = x_0$.

$f'(x_0) \leq 0$

Case 2: $f'(x_0) < 0$

Apply eqn (1) for $\varepsilon = -\frac{f'(x_0)}{2}$

Then

$\exists \delta_1 > 0$ s.t. whenever $|x - x_0| \leq \delta_1$ we have

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| \leq -\frac{f'(x_0)}{2}$$

or $$\frac{f(x) - f(x_0)}{(x - x_0)} \leq \frac{f'(x_0)}{2}$$

Now choose a point x s.t.

(i) $|x - x_0| \leq \min\{\delta_1, \delta_2\}$

(ii) $x < x_0$

For this point

$f(x) - f(x_0) \geq 0$

or $f(x) \geq f(x_0)$

This is a contradiction.

Thus $f'(x_0) = 0$.

The theorem doesn't work if function is defined on close interval.

$f : [0, 2] \to \mathbb{R}$

$f(x) = x$

The maximum is achieved at $x_0 = 2$.
The minimum " " $x_0 = 0$

while $f'(0) = 1 = f'(2)$.

<u>Rolle's theorem</u>: Let $a<b$. Let $g:[a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$ and differentiable on (a,b). Suppose $g(a) = g(b)$, then $\exists\, x \in (a,b)$ s.t.

$g'(x) = 0$.

<u>Proof</u>: Since $g:[a,b] \to \mathbb{R}$ is continuous on $[a,b]$, from extreme value theorem $\exists\, x_{min} \in [a,b]$ and $x_{max} \in [a,b]$

s.t. $g(x_{min}) \leq g(x)$
$g(x_{max}) \geq g(x) \quad \forall\, x \in [a,b]$

① If $g(x_{max}) = g(x_{min})$

Then $g(x_{min}) \leq g(x) \leq g(x_{max}) \quad \forall\, x \in [a,b]$
$= g(x_{min})$

$\Rightarrow \quad g(x) = g(x_{min}) = g(x_{max}) \quad \forall\, x \in [a,b]$

Thus g is a constant function and $g'(x) = 0 \;\forall\, x \in (a,b)$.

② Assume $g(x_{max}) > g(x_{min})$

We claim that at least x_{min} or $x_{max} \in (a,b)$.

For contradiction assume $\{x_{min}, x_{max}\} \subset \{a,b\}$.

Then $g(x_{max}) \in \{g(a), g(b)\}$
$g(x_{min}) \in \{g(a), g(b)\}$

But $g(a) = g(b) \Rightarrow g(x_{min}) = g(x_{max})$

This can't happen as $g(x_{min}) < g(x_{max})$.

Therefore at least x_{min} or x_{max} lies in (a,b).

Let $x_{max} \in (a,b)$

Since g is differentiable on (a,b), and function achieves maximum at $x_{max} \in (a,b)$ then

$g'(x_{max}) = 0.$

Similarly $g'(x_{min}) = 0$ if $x_{min} \in (a,b)$.

Thus there exists a point $x \in (a,b)$ with $g'(x) = 0$.

Theorem (Mean value theorem): Let $a<b$. Let the function $f: [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). Then $\exists\ x \in (a,b)$ s.t.

$$f(x) = \frac{f(b)-f(a)}{(b-a)}$$

Proof: Define $L = \dfrac{f(b)-f(a)}{b-a}$

Consider the function $g(x) = f(x) - Lx \quad x \in [a,b]$

g is continuous and differentiable on $[a,b]$ and (a,b), respectively.

$$g(b) - g(a) = f(b) - f(a) - L(b-a)$$
$$= 0$$

From Roll's theorem $\exists\ x_0 \in (a,b)$ s.t.

$$g'(x_0) = 0$$

or $\quad f'(x_0) - L = 0$

or $\quad f'(x_0) = \dfrac{f(b)-f(a)}{b-a}$

This completes the proof.

Cauchy mean value theorem: Let $a<b$. Let $g, f: [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Let $g(a) \neq g(b)$. Then $\exists\ c \in (a,b)$

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof: Define $\phi : [a,b] \to \mathbb{R}$

$$\phi(t) = \big(f(b) - f(a)\big)\big(g(t) - g(a)\big) - \big(f(t) - f(a)\big)\big(g(b) - g(a)\big)$$

This is a continuous f^n on $[a,b]$ and differentiable on (a,b).

$\phi(a) = 0 = \phi(b)$

From Rolle's theorem $\exists\, c \in (a, b)$

$\phi'(c) = 0$

$\Rightarrow \phi'(x) = (f(b) - f(a))\, g'(x) - f'(x)(g(b) - g(a))$

$\Rightarrow \phi'(c) = 0 \Rightarrow \dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f'(c)}{g'(c)}.$

§ Lecture 26.1

Sunday, 16 November 2025 10:47

L'Hôpital's rule: [loh-pee-TAHL]

(I): let $X \subseteq \mathbb{R}$. let $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ be two functions and let $x_0 \in X$ be a limit point of X.

let $f(x_0) = g(x_0) = 0$ and f, g are differentiable at x_0. and $g'(x_0) \neq 0$. Then $\exists \delta > 0$ s.t.

(i) $g(x) \neq 0 \quad \forall x \in \underbrace{(X - \{x_0\}) \cap (x_0 - \delta, x_0 + \delta)}_{Y}$

(ii) $$\lim_{x \to x_0, x \in Y} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}.$$

Proof: For $x \in X - \{x_0\}$ define

$$p(x) = \frac{f(x) - f(x_0)}{x - x_0} = \frac{f(x)}{x - x_0}$$

$$q(x) = \frac{g(x) - g(x_0)}{x - x_0} = \frac{g(x)}{x - x_0}$$

Then $$\lim_{x \to x_0; x \in X - \{x_0\}} p(x) = f'(x_0)$$

$$\lim_{x \to x_0; x \in X - \{x_0\}} q(x) = g'(x_0) \neq 0$$

Then $\forall \eta > 0 \ \exists \delta_1 > 0$ s.t. $\forall x \in X$ with $|x - x_0| \leq \delta_1$

$$|p(x) - f'(x_0)| \leq \eta \longrightarrow ①$$

$\forall \eta > 0 \ \exists \delta_2 > 0$ s.t. $\forall x \in X$ with $|x - x_0| \leq \delta_2$

$$|q(x) - g'(x_0)| \leq \eta. \longrightarrow ②$$

Fix some $\varepsilon > 0$. We need to find $\delta > 0$ s.t. $\forall$ $|x - x_0| \leq \delta$ with $g(x) \neq 0$ and

$$\left| \frac{f(x)}{g(x)} - \frac{f'(x_0)}{g'(x_0)} \right| \leq \varepsilon.$$

let $\delta = \min\{\delta_1, \delta_2\}$

Then $\forall \eta > 0 \ \exists \delta > 0$ s.t. with $|x - x_0| \leq \delta$ we have

$$|p(x) - f'(x_0)| \leq \eta \quad \rightarrow ③$$

$$|q(x) - g'(x_0)| \leq \eta \quad \rightarrow ④$$

Triangle inequality

$$||b| - |a|| \leq |a-b| \leq |a| + |b|$$

or $\quad |b| \geq |a| - |a-b|$

$$b = q(x) \; ; \; a = g'(x_0)$$

$$|q(x)| \geq |g'(x_0)| - |q(x) - g'(x_0)|$$

$$\geq |g'(x_0)| - \eta$$

For $\eta \leq \frac{|g'(x_0)|}{2}$

$$|q(x)| \geq \frac{|g'(x_0)|}{2} \neq 0.$$

Thus x with $|x - x_0| \leq \delta \quad q(x) \neq 0$.

For such x,

$$\frac{f(x)}{g(x)} = \frac{p(x)(x-x_0)}{q(x)(x-x_0)} = \frac{p(x)}{q(x)}$$

$$\left| \frac{f(x)}{g(x)} - \frac{f'(x_0)}{g'(x_0)} \right| = \left| \frac{p(x)}{q(x)} - \frac{f'(x_0)}{g'(x_0)} \right|$$

$$= \left| \frac{p(x)\, g'(x_0) - q(x)\, f'(x_0)}{q(x)\, g'(x_0)} \right|$$

$$= \left| \frac{g'(x_0)\left(p(x) - f'(x_0)\right) + \left(g'(x_0) - q(x)\right) f'(x_0)}{q(x)\, g'(x_0)} \right|$$

$$\leq \frac{|g'(x_0)|\,|p(x) - f'(x_0)| + |f'(x_0)|\,|q(x) - g'(x_0)|}{|q(x)|\,|g'(x_0)|}$$

Using $|q(x)| \geq \frac{|g'(x_0)|}{2}$; $|p(x) - f'(x_0)| \leq \eta$

$|q(x) - g'(x_0)| \leq \eta$.

Thus $\left| \frac{f(x)}{g(x)} - \frac{f'(x_0)}{g'(x_0)} \right| \leq \frac{2\left(|g'(x_0)| + |f'(x_0)|\right)\eta}{|g'(x_0)|^2}$

Taking $\eta = \min\left\{ \frac{|g'(x_0)|}{2}, \frac{|g'(x_0)|^2 \varepsilon}{2\left(|g'(x_0)| + |f'(x_0)|\right)} \right\}$

$$\Rightarrow \left| \frac{f(x)}{g(x)} - \frac{f'(x_0)}{g'(x_0)} \right| \leq \varepsilon$$

$$\Rightarrow \lim_{x \to x_0;\, x \in X - \{x_0\} \cap (x_0 - \delta, x_0 + \delta)} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}$$

(II) Let $a<b$. Let $f, g : [a,b] \to \mathbb{R}$ be differentiable on (a,b). Suppose that

① $f(a) = g(a) = 0$

② $g'(x) \neq 0 \quad \forall\, x \in (a,b)$

③ $\displaystyle \lim_{x \to a;\, x \in (a,b]} \frac{f'(x)}{g'(x)} = L$

Then

ⓐ $g(x) \neq 0 \quad \forall\, x \in (a,b]$

ⓑ $\displaystyle \lim_{x \to a;\, x \in (a,b]} \frac{f(x)}{g(x)} = L.$

Proof: ⓐ $g(x) \neq 0 \quad \forall\, x \in (a,b]$

For contradiction assume that $\exists\, x \in (a,b]$ s.t.

$$g(x) = 0$$

Then on interval $[a,x]$ g is continuous and differentiable on (a,x) and $g(a) = 0 = g(x)$
↳ given

then from Rolle's theorem $\exists\, c \in (a,x)$ s.t.

$$g'(c) = 0$$

But this contradicts $g'(x) \neq 0 \;\forall\, x \in (a,b)$.

$\Rightarrow g(x) \neq 0 \quad \forall\, x \in (a,b]$.

② Fix any $x \in (a,b]$. Since f, g are continuous on $[a,x]$ and differentiable on (a,x) then there exists $c \in (a,x)$ s.t.

$$\frac{f(x) - f(a)}{} = \frac{f'(c)}{g'(c)}$$

$f(a) - g(a) \qquad g'(c)$

But $f(a) = 0 = g(a)$

$$\Rightarrow \quad \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} \longrightarrow (2)$$

(3) Given $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L$

Then $\forall \varepsilon > 0 \;\; \exists \delta > 0$ s.t. $\forall y \in (a, a+\delta]$

$$\left| \frac{f'(y)}{g'(y)} - L \right| \leq \varepsilon. \longrightarrow (3)$$

Take any $x \in (a, a+\delta]$. By (1) $\exists c \in (a, x)$.

$$a < c < x \leq a+\delta.$$

Thus (3) applies to c.

$$\left| \frac{f'(c)}{g'(c)} - L \right| \leq \varepsilon$$

or $\left| \frac{f(x)}{g(x)} - L \right| \leq \varepsilon$

$$\Rightarrow \lim_{x \to a^+} \frac{f(x)}{g(x)} = L.$$

L'Hôpital rule II:

Let $a < b$. Let $f, g : [a,b] \to \mathbb{R}$ be two functions that are differentiable on $[a,b]$. Suppose

(i) $f(a) = g(a) = 0$.

(ii) $g'(x) \neq 0 \;\; \forall x \in [a,b]$

(iii) $\lim_{x \to a;\, x \in (a,b]} \frac{f'(x)}{g'(x)} = L$.

Then

(a) $g(x) \neq 0 \;\; \forall x \in (a,b]$

(b) $\lim_{x \to a,\, x \in (a,b]} \frac{f(x)}{g(x)} = L$.

Proof: (a) $g(x) \neq 0$

suppose $\exists x \in (a,b]$ s.t. $g(x) = 0$.

Given that $g(a)=0$. Thus on interval

$[a,x]$ g is continuous and on interval (a,x)

differentiable and $g(a)=0=g(a)$.

Then $\exists\, c \in (a,x) \subset (a,b]$ s.t.

$$g'(c)=0$$

This contradict premise (ii)

$\Rightarrow$ $g(x) \neq 0 \quad \forall\, x \in (a,b]$.

(ii) Need to show that $\lim\limits_{x\to a;\, x\in(a,b]} \frac{f(x)}{g(x)} = L$.

This is equivalent to showing

$\lim\limits_{n\to\infty} \frac{f(x_n)}{g(x_n)} = L$ for any sequence (x_n) with

$x_n \in (a,b]$ that converges to a.

Let $x_n \in (a,b]$ and $\lim\limits_{n\to\infty} x_n = a$.

Consider another function

$$h_n : [a, x_n] \to \mathbb{R}$$

and $h_n(x) = f(x)\, g(x_n) - f(x_n)\, g(x)$

Since f, g are differentiable on (a,b)

h_n is differentiable on (a, x_n) and

continuous on $[a, x_n]$

$h_n(a) = 0$ as $f(a)=0=g(a)$

$h_n(x_n) = 0$

From Rolle's theorem $\exists\, y_n \in (a, x_n)$ s.t.

$$h_n'(y_n) = 0$$

or $\quad \dfrac{f(x_n)}{g(x_n)} = \dfrac{f'(y_n)}{g'(y_n)}$

since $\quad y_n \in (a, x_n)$

$\Rightarrow \quad a < y_n < x_n$

or $\quad a \leq \lim\limits_{n \to \infty} y_n \leq a$

or $\quad \lim\limits_{n \to \infty} y_n = a$

since f', g' are continuous

$$\lim_{n \to \infty} \frac{f'(y_n)}{g'(y_n)} = L = \lim_{n \to \infty} \frac{f(x_n)}{g(x_n)}.$$

or $\quad \lim\limits_{x \to a;\, x \in (a, b]} \dfrac{f(x)}{g(x)} = L.$

Similarly one can derive for left limit.

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.}$$