§ Lecture 14.1

Monday, 29 September 2025 21:23

Summation over finite sets:

Let X be a finite set of n elements and let $f : X \to \mathbb{R}$ be a function. Then $\sum_{x \in X} f(x)$ is defined as follows.

(1) select any bijection $g : \{i \in \mathbb{N} : 1 \le i \le n\} \to X$

(This exists by definition).

(2) $\sum_{x \in X} f(x) = \sum_{i=1}^{n} f(g(i))$

e.g. $X = \{a, b, c\}$ $n = 3$

Define $g(1) = a$ $g(2) = b$, $g(3) = c$

Then $\sum_{x \in X} f(x) = \sum_{i=1}^{3} f(g(i))$

$= f(a) + f(b) + f(c)$.

If we define another bijection h s.t

$h(1) = c$, $h(2) = a$, $h(3) = b$

$\Rightarrow \sum_{x \in X} f(x) = f(c) + f(a) + f(b) = \sum_{i=1}^{3} f(g(i))$

$= \sum_{i=1}^{3} f(h(i))$.

Proposition:

Let X be a finite set with n element. and let $f : X \to \mathbb{R}$ be a function. Let $g, h : \{i \in \mathbb{N} : 1 \le i \le n\} \to X$ be two bijections. Then

$$\sum_{i=1}^{n} f(g(i)) = \sum_{i=1}^{n} f(h(i))$$

Proof: Let $P(n)$ be the following property:

" For any set X of n elements and function $f : X \to \mathbb{R}$ and any two bijections g, h from $\{i \in \mathbb{N} : 1 \le i \le n\}$ to $\mathbb{R}$

$\sum_{i=1}^{n} f(g(i)) = \sum_{i=1}^{n} f\, h(i)$. "

Base case: $n = 0$

$P(0)$ is true because

$\sum_{i=1}^{0} f(g(i)) = 0 = \sum_{i=1}^{0} f(h(i))$

Assume $P(n)$ is true.

Then for $P(n+1)$, set X has $(n+1)$ elements.

$\sum_{i=1}^{n+1} f(g(i)) = \sum_{i=1}^{n} f(g(i)) + f(g(n+1))$

Since g is a bijection $g(n+1) = x \in X$.

$\Rightarrow \sum_{i=1}^{n+1} f(g(i)) = \sum_{i=1}^{n} f(g(i)) + f(x)$

If $h(n+1) = x$ we are done.

If not then there exists some index j s.t.

$h(j) = x$.

Then $\sum_{i=1}^{n+1} f(h(i)) = \sum_{i=1}^{j-1} f(h(i)) + f(x) + \sum_{i=j+1}^{n+1} f(h(i))$

$= \sum_{i=1}^{j-1} f(h(i)) + f(x) + \sum_{i=j}^{n} f(h(i+1))$

Define $\tilde{h} : \{i \in \mathbb{N} : 1 \le i \le n\} \to X - \{x\}$

$$\tilde{h}(i) = \begin{cases} h(i) & \text{for } i < j \\ h(i+1) & \text{for } i \ge j \end{cases}$$

RHS $= \sum_{i=1}^{j-1} f(\tilde{h}(i)) + \sum_{i=j}^{n} f(\tilde{h}(i)) + f(x)$

$= \sum_{i=1}^{n} f(\tilde{h}(i)) + f(x)$

Recall $g(n+1) = x$

Then g restricted to $\{i \in \mathbb{N} : 1 \le i \le n\}$ is a bijection from $\{i \in \mathbb{N} : 1 \le i \le n\} \to X - \{x\}$.

The function $\tilde{h}$ is also a bijection.

$\Rightarrow \sum_{i=1}^{n} f(\tilde{h}(i)) = \sum_{i=1}^{n} f(g(i))$

$\Rightarrow \sum_{i=1}^{n+1} f(h(i)) = \sum_{i=1}^{n} f(g(i)) + f(x)$

$= \sum_{i=1}^{n+1} f(g(i))$.

Properties: (1) $\sum_{x \in \phi} f(x) = 0$ (The only bijection is the empty map)

$\sum_{i=1}^{0} f(g(i)) = 0$.

(2) Let X be a finite set and $f : X \to \mathbb{R}$ be a function. and $g : Y \to X$ be a bijection then

$\sum_{x \in X} f(x) = \sum_{y \in Y} f(g(y))$

Proof: Let $h : \{i \in \mathbb{N} : 1 \le i \le n\} \to X$

$u : \{i \in \mathbb{N} : 1 \le i \le n\} \to Y$

be bijections then. $g \circ u$ is a bijection from $\{i \in \mathbb{N} : 1 \le i \le n\} \to X$.

$\Rightarrow \sum_{x \in X} f(x) = \sum_{i=1}^{n} f(g \circ u(i))$

$= \sum_{i=1}^{n} (f \circ g)(u(i))$

$= \sum_{y \in Y} f \circ g(y)$

$= \sum_{y \in Y} f(g(y))$.

(3) Let X, Y be finite sets with $X \cap Y = \phi$ then

$\sum_{x \in X \cup Y} f(x) = \sum_{x \in X} f(x) + \sum_{y \in Y} f(y)$

$f : X \cup Y \to \mathbb{R}$.

Proof: Let X, Y have n, m elements, respectively.

n, m could be zero too.

$g : \{1, 2, \dots n\} \to X$

$h : \{1, 2, \dots m\} \to Y$

Define a map

$k : \{1, 2, \dots, m+n\} \to X \cup Y$ by

$$k(i) = \begin{cases} g(i) & 1 \le i \le n \\ h(i-n) & n+1 \le i \le m+n \end{cases}$$

Because $X \cap Y = \phi$ k is a bijection onto $X \cup Y$.

$\sum_{z \in X \cup Y} f(z) = \sum_{i=1}^{m+n} f(k(i))$

$= \sum_{i=1}^{n} f(k(i)) + \sum_{i=n+1}^{n+m} f(k(i))$

$= \sum_{i=1}^{n} f(g(i)) + \sum_{i=n+1}^{n+m} f(h(i-n))$

$j = i - n$

$= \sum_{x \in X} f(x) + \sum_{j=1}^{m} f(h(j))$

$= \sum_{x \in X} f(x) + \sum_{y \in Y} f(y)$.

Lemma: Let X, Y be finite sets. Let $f : X \times Y \to \mathbb{R}$ be a function then

$$\sum_{x \in X} \left(\sum_{y \in Y} f(x, y) \right) = \sum_{(x,y) \in X \times Y} f(x, y).$$

Proof: Let n be the number of elements in X.

We will use induction on n.

$P(n)$: Let lemma is true for any set X with n elements any finite set Y and any function $f : X \times Y \to \mathbb{R}$

We aim to prove that $P(n)$ is true for all n.

Base case: $P(0)$, ie. X is empty

$\sum_{x \in \phi} \left(\sum_{y \in Y} f(x, y) \right) = 0$

$\phi \times Y = \phi$ why?

$\Rightarrow \sum_{(x,y) \in \phi} f(x, y) = 0$.

Assume $P(n)$ is true.

We want to prove $P(n+1)$.

Let X has $(n+1)$ elements.

Let $x_0 \in X$ then $\underbrace{X - \{x_0\}}_{= X'}$ has n elements.

$X' \cap \{x_0\} = \phi$

$X' \cup \{x_0\} = X$

$\sum_{x \in X} \left(\sum_{y \in Y} f(x, y) \right) = \sum_{x \in X'} \left(\sum_{y \in Y} f(x, y) \right) + \sum_{x \in \{x_0\}} \left(\sum_{y} f(x, y) \right)$

$= \sum_{x \in X'} \left(\sum_{y \in Y} f(x, y) \right) + \sum_{y \in Y} f(x_0, y)$

$= \sum_{(x,y) \in (X' \times Y)} f(x, y) + \sum_{y \in Y} f(x_0, y)$

$= \sum_{(x,y) \in X' \times Y} f(x, y) + \sum_{(x,y) \in \{x_0\} \times Y} f(x, y)$

$(X' \times Y) \cap \{x_0\} \times Y = \phi$

$\Rightarrow \quad = \sum_{(x,y) \in (X' \times Y) \cup (\{x_0\} \times Y)} f(x, y)$

$= \sum_{(x,y) \in X \times Y} f(x, y)$

§ Lecture 15.0

Thursday, 2 October 2025 20:57

Infinite series:

An infinite series is any expression of the form

$$\sum_{n=m}^{\infty} a_n \qquad a_n \in \mathbb{R},\ m \in \mathbb{Z}.$$

$$= a_m + a_{m+1} + \cdots$$

[may not be a real number]

Convergence of a series:

Let $\sum_{n=m}^{\infty} a_n$ be an infinite series. For $N \geq m$, let S_N be the N^{th} partial sum of this series defined as

$$S_N := \sum_{n=m}^{N} a_n \quad \in \mathbb{R}.$$

If the sequence $(S_N)_{N\geq m}$ converges to L then we say that the series is convergent and converges to L. We then write

$$L = \sum_{n=m}^{\infty} a_n \qquad \left(L = \text{sum of the infinite series } \sum_{n=m}^{\infty} a_n\right)$$

If the partial sum diverges, then the series is called divergent.

[Since limit of a convergent sequence is unique L is unique].

Ex: $\sum_{n=1}^{\infty} 2^{-n}$

$$S_N = \sum_{n=1}^{N} 2^{-n} \qquad \text{(Finite)}$$

$$2S_N = \sum_{n=1}^{N} 2^{-n+1}$$

Define $-n+1 = -m$ or $m = n-1$ $\quad N \to N-1,\ 1 \to 0$

$$2S_N = \sum_{m=0}^{N-1} 2^{-m}$$

$$= 2^{-0} + \sum_{m=1}^{N} 2^{-m} - 2^{-N}$$

$$= S_N + 1 - 2^{-N}$$

or $S_N = 1 - 2^{-N}$

$$\lim_{N\to\infty} S_N = 1 - \lim_{N\to\infty} 2^{-N} \qquad \left(\lim_{n\to\infty} x^n \to 0 \text{ for } 0<x<1.\right)$$

$$= 1.$$

Ex: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ where $|x|<1$.

$$S_N = \sum_{n=0}^{N} x^n$$

$$xS_N = \sum_{n=0}^{N} x^{n+1}$$

$n+1 = m$

$$= \sum_{m=1}^{N+1} x^m$$

$$= \sum_{m=0}^{N} x^m + x^{N+1} - 1$$

or $(1-x)S_N = 1 - x^{N+1}$

$$\Rightarrow S_N = \frac{1-x^{N+1}}{1-x}$$

$$= \frac{1}{1-x} - \frac{x^{N+1}}{1-x}$$

For $|x|<1$ $\quad \lim_{N\to\infty} x^{N+1} = 0$

$$\Rightarrow \lim_{N\to\infty} S_N = \frac{1}{1-x}$$

or $\boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \forall\ |x|<1}$

But if $|x|>1$ Divergent (Unbounded)

$|x| = 1$ Divergent (Unbounded).

Proposition: $\sum_{n=m}^{\infty} a_n$ converges iff $\forall\, \varepsilon>0,\ \exists\, N\geq m$ s.t.

$$\left|\sum_{n=p}^{q} a_n\right| \leq \varepsilon \quad \forall\ p,q \geq N.$$

Proof: Define k^{th} partial sum

$$S_k = \sum_{n=m}^{k} a_n$$

$\Rightarrow$

Assume the series converges to S. Then

$$\lim_{k\to\infty} S_k = S.$$

Then $\exists\, N \geq m$ s.t. $\forall\ k \geq N-1$

$$|S_k - S| \leq \varepsilon/2 \quad \forall\,\varepsilon>0.$$

Now for any $p,q \geq N$ (Assume $p \leq q$)

$$\left|\sum_{n=p}^{q} a_n\right| = \left|\sum_{n=m}^{p-1} a_n + \sum_{n=p}^{q} a_n - \sum_{n=m}^{p-1} a_n\right|$$

$$= \left|\sum_{n=m}^{q} a_n - \sum_{n=m}^{p-1} a_n\right|$$

$$= \left|(S_q - S) - (S_{p-1} - S)\right|$$

$$\leq |S_q - S| + |S_{p-1} - S|$$

$$\leq \varepsilon/2 + \varepsilon/2$$

$$= \varepsilon.$$

$\Leftarrow$ Assume that $\forall\,\varepsilon>0\ \exists\, N\geq m$ s.t.

$$\left|\sum_{n=p}^{q} a_n\right| < \varepsilon \quad \forall\ p,q \geq N.$$

Fix $\varepsilon>0$ and choose above N. Then

for any $s \geq r \geq N$ $\quad (s, r \geq N)$

$$|S_s - S_r| = \left|\sum_{n=m}^{s} a_n - \sum_{n=m}^{r} a_n\right|$$

$$= \left|\cancel{\sum_{n=m}^{r} a_n} + \sum_{n=r+1}^{s} a_n - \cancel{\sum_{n=m}^{r} a_n}\right|$$

$$\leq \varepsilon$$

That is $(S_k)_{k\geq m}$ is a Cauchy sequence

$\Rightarrow$ $(S_k)_{k\geq m}$ converges.

This test would require computing partial sums.

Not very handy. [Take $p=q$ above to get $|a_p| \leq \varepsilon\ \forall\,\varepsilon$]

Corollary: (Zero test)

If $\sum_{n=m}^{\infty} a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$.

(contrapositive: if $\lim_{n\to\infty} a_n \neq 0 \Rightarrow$ series is divergent).

Proof: Let $S_k = \sum_{n=m}^{k} a_n$

By assumption $S_k \to S$.

For $k \geq m+1$

$$S_k - S_{k-1} = a_k$$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} S_k - \lim_{k\to\infty} S_{k-1}$$

$$= S - S = 0.$$

Ex: $\sum_{n=m}^{\infty} a_n$ where $a_n = 1$.

$\lim_{n\to\infty} a_n = 1 \neq 0 \Rightarrow$ series is not convergent.

while sequence is!!

Absolute convergence: $\sum_{n=m}^{\infty} a_n$ is called absolutely convergent iff $\sum_{n=m}^{\infty} |a_n|$ is convergent.

Lemma: If a series is absolutely convergent then it is convergent and

conditionally convergent!

$$\left|\sum_{n=m}^{\infty} a_n\right| \leq \sum_{n=m}^{\infty} |a_n|.$$

Proof: Let $S_k = \sum_{n=m}^{k} a_n$

$$t_k = \sum_{n=m}^{k} |a_n|$$

Note that for $p+1 \leq q$

$$|S_q - S_p| = \left|\sum_{n=m}^{q} a_n - \sum_{n=m}^{p} a_n\right|$$

$$= \left|\sum_{n=p+1}^{q} a_n\right|$$

$$\leq \sum_{n=p+1}^{q} |a_n|$$

$$= t_q - t_p$$

Because $(t_k)_{k\geq m}$ converges $\Rightarrow$ (t_k) is a Cauchy sequence. So $\forall\,\varepsilon>0,\ \exists\, N\geq m$ s.t.

if $q, p \geq N$ then

$$|t_q - t_p| \leq \varepsilon.$$

$\Rightarrow$ $|S_q - S_p| \leq \varepsilon$

$\Rightarrow$ (S_k) is a Cauchy sequence. $\Rightarrow$ (S_k) is convergent.

Since $S_k \leq t_k \quad \forall\ k \geq m$

$$\Rightarrow \lim_{k\to\infty} |S_k| \leq \lim_{k\to\infty} t_k$$

or $\left|\sum_{n=m}^{\infty} a_n\right| \leq \sum_{n=m}^{\infty} |a_n|$

[Converse is not true].

§ Lecture 15.1

Thursday, 2 October 2025 22:32

Alternating series test : Let $\sum_{n=m}^{\infty} a_n$ be an infinite series with $a_n \geq 0$ and $a_{n+1} \leq a_n \;\forall\, n \geq m$.

Then $\sum_{n=m}^{\infty} (-1)^n a_n$ is convergent iff $(a_n)_{n\geq m}$ converges to zero.

<u>Proof</u>: $\Rightarrow$ From zero test if $\sum_{n=m}^{\infty} (-1)^n a_n$ is convergent

then $\lim_{n\to\infty} \left[(-1)^n a_n\right] = 0$

or $\forall\, \varepsilon > 0 \;\exists\, N \geq m$ s.t $\forall\; n \geq m$

$$|(-1)^n a_n - 0| \leq \varepsilon$$

$$\Rightarrow \quad |a_n - 0| \leq \varepsilon$$

$\Rightarrow$ $(a_n)_{n\geq m}$ converges to zero.

$\Leftarrow$ Suppose conversely that (a_n) converges to 0.

Let $S_N = \sum_{n=m}^{N} (-1)^n a_n \quad \forall\, n \geq m.$

Our goal is to show that $(S_N)_N$ converges.

Note that m is arbitrary.

Define $E_k = S_{m+2k} = \sum_{n=m}^{m+2k} (-1)^n a_n \quad k \geq 0$

$O_k = S_{m+2k+1} = \sum_{n=m}^{m+2k+1} (-1)^n a_n \quad k \geq 0$

Then $(S_N)_{N\geq m} = (S_m, S_{m+1}, S_{m+2}, S_{m+3}, \cdots)$

$= (E_0, O_0, E_1, O_1, E_2, O_2, \cdots)$

$$E_k - E_{k+1} = \sum_{n=m}^{m+2k} (-1)^n a_n - \sum_{n=m}^{m+2k+2} (-1)^n a_n$$

$$= (-1)^{m+2k+1}\left[a_{m+2k+1} - a_{m+2k+2}\right]$$

$$= (-1)^{m+1}\left[\underbrace{a_{m+2k+1} - a_{m+2k+2}}_{\geq 0}\right]$$

$$O_k - O_{k+1} = (-1)^{m+2k+2}\left[a_{m+2k+2} - a_{m+2k+3}\right]$$

$$= (-1)^m\left[\underbrace{a_{m+2k+2} - a_{m+2k+3}}_{\geq 0}\right]$$

If m is even. then	m is odd
$E_k \leq E_{k+1}$	$E_k \geq E_{k+1}$
$O_k \geq O_{k+1}$	$O_k \leq O_{k+1}$.

$(S_N)_{N\geq m} = (E_0, O_0, E_1, O_1, E_2, O_2, \cdots)$

$(E_k)_{k\geq 0}$ is increasing sequence

$(O_k)_{k\geq 0}$ is decreasing sequence.

Note that $E_k - O_k = \sum_{n=m}^{+2k} a_n (-1)^n - \sum_{n=m}^{m+2k+1} (-1)^n a_n$

$$= -(1)^{m+2k+1} a_{m+2k+1}$$

$$= (-1)^{m+1} a_{m+2k+1}$$

$$= -a_{m+2k+1}$$

$$\leq 0$$

$\Rightarrow \quad E_k \leq O_k \quad \forall\, k \geq 0$

$\Rightarrow$ (E_k) is an increasing sequence and upper bounded by any O_ℓ. $(\ell > k)$

$\Rightarrow$ (O_k) is decreasing sequence and lower bounded by an earlier E_ℓ.

$\Rightarrow$ Both (E_k) and (O_k) converges.

let $L_{even} = \lim_{k\to\infty} E_k$

$L_{odd} = \lim_{k\to\infty} O_k$

Then

$E_k - O_k = -a_{m+2k+1}$

$L_{even} - L_{odd} = -\lim_{k\to\infty} a_{m+2k+1}$

$= 0$

Call $L = L_{even} = L_{odd}$.

Given $\varepsilon > 0$ choose K large enough s.t. $\forall\, k \geq K$

$|E_k - L| \leq \varepsilon/2$

$|O_k - L| \leq \varepsilon/2$

Then $M \geq K$ s.t $\forall\, p, q \geq M$

$$|S_p - S_q| \leq |S_p - L| + |S_q - L|$$

$$\leq \varepsilon/2 + \varepsilon/2$$

$$= \varepsilon$$

$\Rightarrow$ $(S_N)_{N\geq m}$ is a Cauchy sequence and hence convergent.

Note that $E_0 \leq \cdots \leq E_k \leq O_k \leq O_{k-1} \leq \cdots \leq O_0$

$\Rightarrow$ E_0 and O_0 are global lower and upper bounds respectively. (m even).