Practice Problems: Z-Scores and Normal Probability

Problem 1: Score Below a Threshold

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A test is normally distributed with a mean of 70 and a standard deviation of 10.

(a) What is the z-score for a score of 60?

(b) What is the probability that a randomly chosen score is less than 60?

Solution

Solution:

(a) The z-score is given by

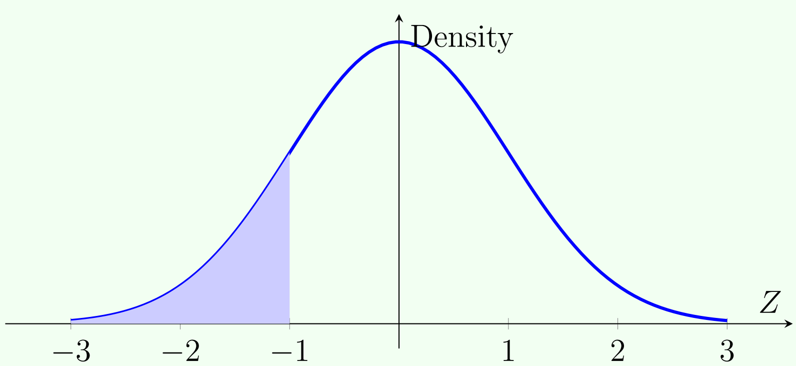
$$z = \frac{x - \mu}{\sigma} = \frac{60 - 70}{10} = -1.0.$$

(b) From the standard normal table,

$$P(Z < -1.0) \approx 0.1587.$$

Thus, about 15.87% of scores are below 60.

Illustration: The area to the left of $z = -1.0$ is shaded in the standard normal curve below.



Problem 2: Score Between Two Values

Problem 2: Score Between 80 and 90

Assume test scores are normally distributed with $\mu = 80$ and $\sigma = 8$.

(a) Find the z-scores for scores of 80 and 90.

(b) Determine the probability that a score is between 80 and 90.

Solution

Solution:

(a) For $x = 80$:

$$z = \frac{80 - 80}{8} = 0.$$

For $x = 90$:

$$z = \frac{90 - 80}{8} = 1.25.$$

(b) The probability that a score is between 80 and 90 is:

$$P(80 < X < 90) = P(0 < Z < 1.25).$$

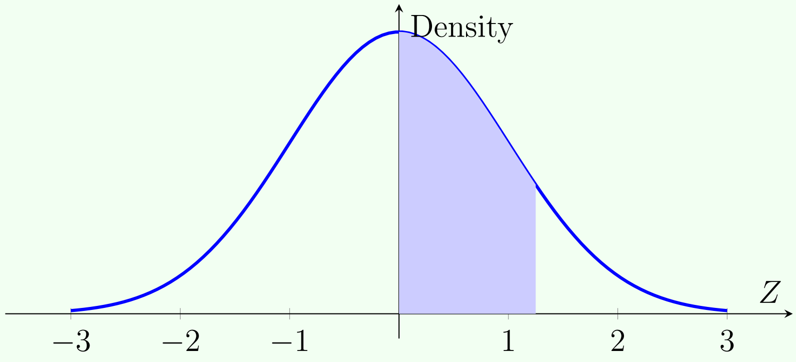
From the standard normal table,

$$P(Z < 1.25) \approx 0.8944, \quad P(Z < 0) = 0.5.$$

Thus,

$$P(0 < Z < 1.25) = 0.8944 - 0.5 = 0.3944.$$

So approximately 39.44% of scores fall between 80 and 90.
Plot: The following plot shades the area between $Z = 0$ and $Z = 1.25$.



Problem 3: Shaded Tail Probability

Problem 3: Probability Above 650

For a standardized exam with $\mu = 500$ and $\sigma = 100$, a score of 650 corresponds to a z-score of 1.5.

Find the probability that a score is greater than 650. Illustrate the corresponding area on a standard normal curve.

Solution

Solution:

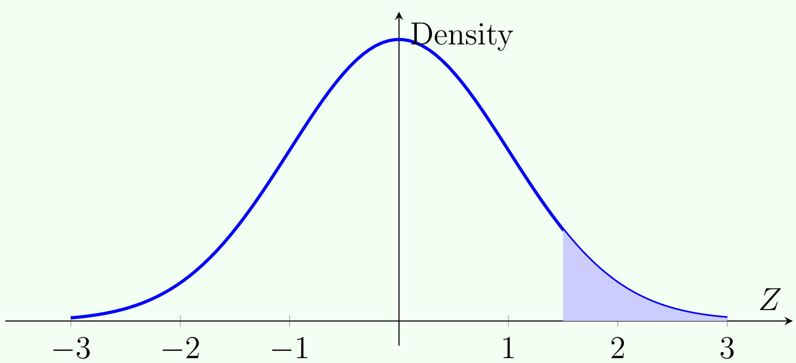
Step 1: Calculate the z-score:

$$z = \frac{650 - 500}{100} = 1.5.$$

Step 2: From the standard normal table,

$$P(Z < 1.5) \approx 0.9332 \quad \Rightarrow \quad P(Z > 1.5) = 1 - 0.9332 = 0.0668.$$

So, about 6.68% of scores are above 650.
Plot: The area to the right of $z = 1.5$ is shaded.



Problem 4: 90th Percentile Calculation

Problem 4: 90th Percentile

Assume test scores are normally distributed with $\mu = 75$ and $\sigma = 12$.

(a) Find the z-score corresponding to the 90th percentile.

(b) Determine the test score that represents the 90th percentile.

Solution

Solution:

(a) The 90th percentile corresponds approximately to $z = 1.28$.

(b) Using the formula:

$$x = \mu + z\sigma = 75 + 1.28(12) = 75 + 15.36 = 90.36.$$

Thus, the 90th percentile score is about 90.4.

Problem 5: Raw Scores to Z-Scores and Finding the Median

Problem 5: Converting Raw Data to Z-Scores

A small dataset of test scores is given: $\{60, 65, 70, 75, 80\}$.

(a) Calculate the mean and standard deviation.

(b) Convert each score into a z-score.

(c) Find the median of the z-scores.

Solution

Solution:

(a) **Mean:**

$$\text{Mean} = \frac{60 + 65 + 70 + 75 + 80}{5} = \frac{350}{5} = 70.$$

Standard Deviation: Compute squared differences:

$$(60-70)^2 = 100, \quad (65-70)^2 = 25, \quad (70-70)^2 = 0, \quad (75-70)^2 = 25, \quad (80-70)^2 = 100.$$

Sum $= 100 + 25 + 0 + 25 + 100 = 250$, Variance $= 250/5 = 50$, so $\sigma \approx \sqrt{50} \approx 7.07$.

(b) **Z-scores:**

$$z = \frac{x - 70}{7.07}.$$

Thus:

- For 60: $z \approx \frac{-10}{7.07} \approx -1.414$.
- For 65: $z \approx -0.707$.
- For 70: $z = 0$.
- For 75: $z \approx 0.707$.
- For 80: $z \approx 1.414$.

(c) **Median of the z-scores:** The sorted z-scores are {-1.414, -0.707, 0, 0.707, 1.414}. The median is the middle value, **0**.

Problem 6: Comparing Two Groups Using Z-Scores

Problem 6: Comparing Group Performances

Two classes take the same exam.

- Group A: $\mu = 65$, $\sigma = 8$, a student scores 78.
- Group B: $\mu = 75$, $\sigma = 10$, a student scores 85.

(a) Calculate the z-score for each student.

(b) Which student performed better relative to their class?

Solution

Solution:

For Group A:

$$z_A = \frac{78 - 65}{8} = \frac{13}{8} \approx 1.625.$$

For Group B:

$$z_B = \frac{85 - 75}{10} = \frac{10}{10} = 1.0.$$

Since a higher z-score means the score is further above the class mean, the student in **Group A** performed better relative to their class.

Problem 7: Tail Probability from a Normal Distribution

Problem 7: Probability Above a Threshold

In a standardized exam, scores are normally distributed with $\mu = 500$ and $\sigma = 100$.

(a) Calculate the z-score for a score of 650.

(b) Find the probability that a score is greater than 650.

Solution

Solution:

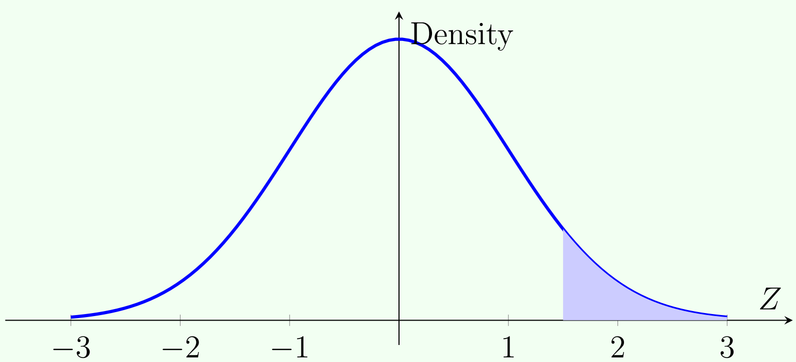
(a)

$$z = \frac{650 - 500}{100} = 1.5.$$

(b) Using the standard normal table, $P(Z < 1.5) \approx 0.9332$, so:

$$P(X > 650) = 1 - 0.9332 = 0.0668.$$

Plot: The area to the right of $z = 1.5$ is shaded below.



Thus, about 6.68% of scores exceed 650.

Problem 8: Middle 50% of the Distribution

Problem 8: Probability Between the 25th and 75th Percentiles

For test scores with $\mu = 500$ and $\sigma = 100$, find the probability that a score lies between the 25th and 75th percentiles.

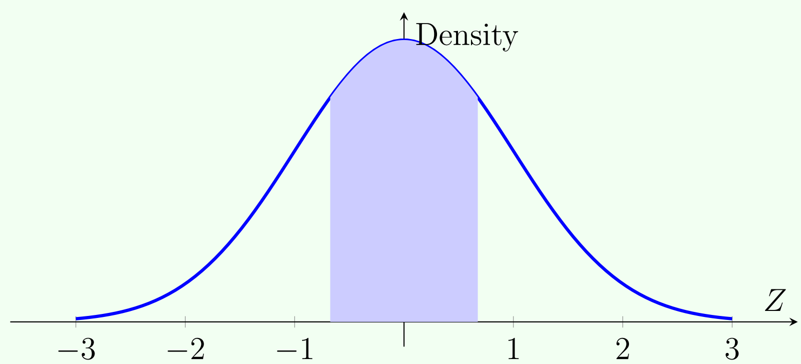
Solution

Solution:

For the standard normal distribution, the 25th percentile is approximately $z = -0.6745$ and the 75th percentile is $z = 0.6745$. Then,

$$P(-0.6745 < Z < 0.6745) \approx 0.75 - 0.25 = 0.50.$$

So about 50% of scores lie in the middle 50% of the distribution.
Plot: The shaded area between $z = -0.6745$ and $z = 0.6745$ is shown below.



Problem 9: 97.5th Percentile Calculation

Problem 9: 97.5th Percentile

For a distribution with $\mu = 100$ and $\sigma = 15$, find the test score that represents the 97.5th percentile.

Solution

Solution:

The 97.5th percentile corresponds approximately to $z = 1.96$. Thus,

$$x = \mu + z\sigma = 100 + 1.96(15) = 100 + 29.4 = 129.4.$$

So the 97.5th percentile is about 129.4.

Problem 10: Quality Control in Package Weights

Problem 10: Package Weight Probability

In a quality control test, package weights are normally distributed with a mean of 10 kg and a standard deviation of 0.5 kg. Find the probability that a package weighs between 9.5 and 10.3 kg.

Solution

Solution:

Calculate the z-scores:

$$z_{9.5} = \frac{9.5 - 10}{0.5} = -1, \quad z_{10.3} = \frac{10.3 - 10}{0.5} = 0.6.$$

From the standard normal table:

$$P(Z < 0.6) \approx 0.7257, \quad P(Z < -1) \approx 0.1587.$$

Thus, the probability is:

$$P(9.5 < X < 10.3) = 0.7257 - 0.1587 = 0.5670.$$

So, about 56.70% of packages weigh between 9.5 and 10.3 kg.