Lecture 2: The Structure of Space

From Groups to Basis Vectors

MFML Companion Guide

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1 Introduction: The Stage for Machine Learning

In our previous explorations, we solved systems of linear equations. Now, we zoom out. We stop looking at individual equations and start looking at the *universe* where these mathematical objects live. This universe is called a **Vector Space**.

Why do we care? In Machine Learning, everything is a vector:

- **Input:** An image is a vector of pixel intensities. A sentence is a vector of word embeddings.
- **Model:** Neural networks process these inputs using linear transformations (matrices) that exist within these spaces.

- **Output:** A prediction (like a house price or a class probability) is a vector in an output space.

To understand how models learn, we must understand the structure of the space they operate in. We need to know: *What are the valid moves? When are two pieces of data "redundant"? How do we define a coordinate system?*

This guide answers those questions.

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2 The Rules of the Game: Groups and Vector Spaces

Before we define a vector space, we need to understand a simpler structure: a **Group**.

2.1 Groups: The Foundation

A Group is a set of elements combined with an operation (like addition) that follows strict rules. Think of the integers $\mathbb{Z}$ with addition $(+)$. They form a group because:

1. **Closure:** If you add two integers, you get an integer. (You don't suddenly get a fraction).
2. **Associativity:** $(1 + 2) + 3 = 1 + (2 + 3)$. The grouping doesn't matter.
3. **Identity:** There is a "do nothing" element (0). $5 + 0 = 5$.
4. **Inverse:** Every element has an opposite. $5 + (-5) = 0$.

If the order doesn't matter (i.e., $a + b = b + a$), it is an **Abelian Group**.

2.2 Vector Spaces: Adding "Scaling"

A Vector Space V is an upgrade to a Group. It has **two** operations:

1. **Vector Addition $(+)$:** Combining two vectors (Inner operation).
2. **Scalar Multiplication $(\cdot)$:** Stretching a vector by a real number (Outer operation).

The Intuition of a Vector Space

Imagine a vector space as an infinite drawing board starting at an origin point.

- **Addition** allows you to chain movements: "Go along vector u, then vector v."
- **Scaling** allows you to extend or shrink movements: "Go twice as far along u."

If you can perform these two actions and **never fall off the board**, you are in a valid vector space.

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3 Subspaces: Spaces within Spaces

Often, we are interested in a smaller section of a larger vector space. For example, a plane slicing through the 3D room you are sitting in. This is a **Subspace**.

3.1 The Subspace Test

A subset U of a vector space V is a subspace if it satisfies three simple conditions:

1. **Non-empty:** It contains the zero vector $\mathbf{0}$.
2. **Closed under Addition:** If $u, v \in U$, then $u + v \in U$.
3. **Closed under Scaling:** If $u \in U$ and $\lambda \in \mathbb{R}$, then $\lambda u \in U$.

Example 1: The Line vs. The Square

Let $V = \mathbb{R}^2$ (the standard 2D plane).
Case A: A line passing through the origin. Does this line form a subspace?

- If you take two vectors on the line and add them, the result is still on the line. (Closed).
- If you stretch a vector on the line, it stays on the line. (Closed).
- **Verdict:** Yes, it is a subspace.

Case B: A square region around the origin (e.g., $-1 \le x \le 1, -1 \le y \le 1$). Is this filled square a subspace?

- Take a vector $u = [1, 0]^T$ which is inside the square.
- Scale it by $\lambda = 5$. The result is $[5, 0]^T$.
- This new vector is **outside** the square.
- **Verdict:** No. It fails closure under scaling.

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4 Linear Combinations and Span

4.1 Linear Combination

A linear combination is the result of mixing vectors together with scaling.

$$v = c_1x_1 + c_2x_2 + \cdots + c_kx_k$$

If you can write v this way, we say v is a "linear combination" of the x's.

4.2 Span

The **Span** of a set of vectors is the collection of *all possible* linear combinations they can create.

$$\text{Span}(\{x_1, \ldots, x_k\}) = \{c_1x_1 + \cdots + c_kx_k \mid c_i \in \mathbb{R}\}$$

Span as Reachable Destinations

Think of the vectors $\{x_1, x_2\}$ as available modes of transportation (e.g., a train going North and a bus going East). The **Span** is the set of all locations on the map you can reach using only these two modes.

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5 Linear Independence: The Art of Efficiency

This is a critical concept for Machine Learning and Data Science. We want to know if our data features are redundant.

5.1 Definition

A set of vectors $\{x_1, \ldots, x_k\}$ is **Linearly Independent** if no vector in the set can be written as a linear combination of the others.

Formally, they are independent if the equation:

$$c_1 x_1 + c_2 x_2 + \cdots + c_k x_k = \mathbf{0}$$

has **only the trivial solution** ($c_1 = c_2 = \cdots = 0$).

If you can find non-zero c's that satisfy this, the vectors are **Linearly Dependent**.

5.2 Checking Independence using Echelon Form

We don't need to guess. We can use Gaussian Elimination.

The Method: 1. Stack the vectors as **columns** of a matrix A. 2. Perform row operations to reach **Row Echelon Form (REF)**. 3. Check the pivot columns.

- If **every column has a pivot**, the vectors are Independent.
- If there are **free variables** (columns without pivots), the vectors are Dependent.

Example 2: A Narrated Independence Check

Determine if the set $\{v_1, v_2, v_3\}$ is linearly independent in $\mathbb{R}^4$:

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ 1 \\ 4 \\ 2 \end{bmatrix}$$

Step 1: Setup the Augmented Matrix for $Ax = 0$ We want to solve $c_1v_1 + c_2v_2 + c_3v_3 = \mathbf{0}$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & 1 & 4 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right]$$

Step 2: Gaussian Elimination

- $R_2 \to R_2 - 2R_1$
- $R_3 \to R_3 - 3R_1$
- $R_4 \to R_4 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -1 & -3 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Notice the last row is all zeros (redundancy detected!). Let's continue.

- $R_3 \to R_3 - 2R_2$

$$\sim \left[\begin{array}{ccc|c} \mathbf{1} & 1 & 2 & 0 \\ 0 & \mathbf{-1} & -3 & 0 \\ 0 & 0 & \mathbf{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Step 3: Analyze Pivots Look at the columns of the coefficient matrix (left of the bar).

- Column 1 has a pivot ($\mathbf{1}$).
- Column 2 has a pivot ($\mathbf{-1}$).
- Column 3 has a pivot ($\mathbf{4}$).

Since **every column has a pivot**, there are no free variables. The only solution is $c_1 = 0, c_2 = 0, c_3 = 0$.
Conclusion: The vectors are **Linearly Independent**.

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6 Basis and Dimension: The Skeleton of the Space

If we have a vector space, we want the most efficient way to describe it.

6.1 Basis

A **Basis** is the "Goldilocks" set of vectors for a space:

1. It must **Span** the space (it's enough to build everything).
2. It must be **Linearly Independent** (no extra/wasteful vectors).

6.2 Dimension

The **Dimension** of a vector space is simply the number of vectors in its basis.

- $\mathbb{R}^3$ has dimension 3.
- The space of 2×2 matrices (M_{22}) has dimension 4.

Why "Dimension" Matters

If you know the dimension of a space is n:

- Any set with **more** than n vectors must be Dependent.
- Any set with **fewer** than n vectors cannot Span the space.
- Any set with exactly n linearly independent vectors is automatically a Basis.

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7 Practice Problems: Test Your Understanding

7.1 Problem 1: Finding Coefficients

Write vector $b = \begin{bmatrix} 4 \\ 3 \\ 1 \\ 2 \end{bmatrix}$ as a linear combination of:

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \end{bmatrix}, v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Hint: Set up the augmented matrix $[v_1 \ v_2 \ v_3 \ v_4 \mid b]$ and solve for the scalars using Gauss Elimination. (See Slide 175-181 for a similar walkthrough).

7.2 Problem 2: Is it a Basis?

Consider the set $S = \{(2,3,5),(5,7,9),(1,11,1)\}$ in $\mathbb{R}^3$.

1. **Count:** There are 3 vectors. Since the dimension of $\mathbb{R}^3$ is 3, S is a candidate for a basis.
2. **Check Independence:** Create a matrix with these vectors as columns. Calculate the determinant or use row reduction.
3. If the matrix reduces to the Identity matrix (or has 3 pivots), it is a basis.

Detailed solution logic: If you set up the matrix:

$$A = \begin{bmatrix} 2 & 5 & 1 \\ 3 & 7 & 11 \\ 5 & 9 & 1 \end{bmatrix}$$

Row reducing this matrix leads to:

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Because we have 3 pivots, the vectors are independent. Since we have 3 independent vectors in a 3D space, they automatically span the space. **Yes, it is a basis.**

Summary: Vector spaces give us the rules for addition and scaling. Subspaces are valid "slices" of these spaces. A Basis is the most efficient set of vectors (independent and spanning) that defines the space, and the number of vectors in the basis gives us the Dimension.