Lecture 3 and 4 Comprehensive Practice Problems and Detailed Solutions

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Abstract

This document provides a curated collection of practice problems covering fundamental concepts in Linear Algebra. The problems cover various levels of difficulty, accompanied by detailed, step-by-step solutions and explanations to aid understanding.

Contents

1 Dot Product

The dot product (or scalar product) is an algebraic operation that takes two equal-length sequences of numbers and returns a single number. In Euclidean geometry, it relates the magnitudes of the vectors and the angle between them.

1.1 Problem 1 (Easy)

Calculate the dot product of the vectors $\mathbf{u} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix}$.

Solution: The dot product $\mathbf{u} \cdot \mathbf{v}$ is calculated by multiplying corresponding components and summing the results.

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= (1)(4) + (-2)(5) + (3)(-1) \\ &= 4 - 10 - 3 \\ &= -9 \end{aligned}$$

The dot product is -9.

1.2 Problem 2 (Easy)

Find the angle θ between the vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Solution: We use the geometric definition of the dot product: $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos(\theta)$.

Step 1: Calculate the dot product $\mathbf{a} \cdot \mathbf{b}$.

$$\mathbf{a} \cdot \mathbf{b} = (1)(1) + (0)(1) = 1$$

Step 2: Calculate the magnitudes (norms) of the vectors.

$$\begin{aligned} \|\mathbf{a}\| &= \sqrt{1^2 + 0^2} = \sqrt{1} = 1 \\ \|\mathbf{b}\| &= \sqrt{1^2 + 1^2} = \sqrt{2} \end{aligned}$$

Step 3: Use the formula to find $\cos(\theta)$.

$$\begin{aligned} \cos(\theta) &= \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} \\ &= \frac{1}{(1)(\sqrt{2})} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \end{aligned}$$

Step 4: Find the angle θ.

$$\theta = \arccos\left(\frac{\sqrt{2}}{2}\right) = 45^\circ \text{ or } \frac{\pi}{4} \text{ radians.}$$

1.3 Problem 3 (Moderate)

Determine if the following pairs of vectors are orthogonal, parallel, or neither.

(a) $\mathbf{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} -6 \\ 4 \end{pmatrix}$

(b) $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} -2 \\ -4 \\ -6 \end{pmatrix}$

Solution: Recall the conditions:

- Orthogonal: $\mathbf{u} \cdot \mathbf{v} = 0$.
- Parallel: $\mathbf{u} = k\mathbf{v}$ for some scalar k.

(a) $\mathbf{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} -6 \\ 4 \end{pmatrix}$ Calculate the dot product:

$$\mathbf{u} \cdot \mathbf{v} = (2)(-6) + (3)(4) = -12 + 12 = 0$$

Since the dot product is 0, the vectors are **orthogonal**.

(b) $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} -2 \\ -4 \\ -6 \end{pmatrix}$ Check if one is a scalar multiple of the other.

$$\mathbf{v} = \begin{pmatrix} -2 \\ -4 \\ -6 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = -2\mathbf{u}$$

Since $\mathbf{v} = -2\mathbf{u}$, the vectors are **parallel**.

1.4 Problem 4 (Difficult)

Prove the Law of Cosines using the dot product. That is, for a triangle with sides of length a, b, c, and the angle C opposite side c, prove that $c^2 = a^2 + b^2 - 2ab\cos(C)$.

Solution: Let's represent the sides of the triangle as vectors. Place the vertex corresponding to angle C at the origin. Let $\mathbf{a}$ and $\mathbf{b}$ be the vectors corresponding to the sides adjacent to C. The lengths are $\|\mathbf{a}\| = a$ and $\|\mathbf{b}\| = b$.

The third side of the triangle, opposite to C, can be represented by the vector $\mathbf{c} = \mathbf{b} - \mathbf{a}$. The length of this side is $\|\mathbf{c}\| = c$.

Step 1: Express c^2 using the vector definition. $c^2 = \|\mathbf{c}\|^2 = \|\mathbf{b} - \mathbf{a}\|^2$.

Step 2: Use the property that the squared norm of a vector is the dot product of the vector with itself.

$$\|\mathbf{b} - \mathbf{a}\|^2 = (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a})$$

Step 3: Expand the dot product using distributive properties.

$$(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) = \mathbf{b} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{a}$$

Step 4: Use the commutative property of the dot product ($\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$) and the definition of the norm.

$$= \|\mathbf{b}\|^2 - 2(\mathbf{a} \cdot \mathbf{b}) + \|\mathbf{a}\|^2$$

Step 5: Substitute the geometric definition of the dot product. The angle between vectors $\mathbf{a}$ and $\mathbf{b}$ is C.

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos(C)$$

Step 6: Substitute this back into the equation from Step 4.

$$\|\mathbf{b} - \mathbf{a}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\|\mathbf{b}\|\cos(C)$$

Step 7: Replace the vector norms with the side lengths a, b, c.

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

This completes the proof of the Law of Cosines.

2 Inner Product

An inner product is a generalization of the dot product. In a vector space V, an inner product is a function $\langle \cdot, \cdot \rangle$ that satisfies linearity, symmetry (or conjugate symmetry), and positive-definiteness.

2.1 Problem 5 (Easy)

Let $V = \mathbb{R}^2$. Verify that the standard Euclidean dot product, defined as $\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_2 v_2$, satisfies the positive-definiteness axiom of an inner product.

Solution: The positive-definiteness axiom states that for any vector $\mathbf{u} \in V$: 1. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$. 2. $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$.

Step 1: Calculate $\langle \mathbf{u}, \mathbf{u} \rangle$.

$$\langle \mathbf{u}, \mathbf{u} \rangle = u_1 u_1 + u_2 u_2 = u_1^2 + u_2^2$$

Step 2: Verify the first condition. Since u_1 and u_2 are real numbers, their squares u_1^2 and u_2^2 are always non-negative (≥ 0). The sum of non-negative numbers is also non-negative. Thus, $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$.

Step 3: Verify the second condition. If $\mathbf{u} = \mathbf{0}$, then $u_1 = 0$ and $u_2 = 0$. $\langle \mathbf{u}, \mathbf{u} \rangle = 0^2 + 0^2 = 0$. If $\langle \mathbf{u}, \mathbf{u} \rangle = 0$, then $u_1^2 + u_2^2 = 0$. Since $u_1^2 \geq 0$ and $u_2^2 \geq 0$, the only way their sum can be zero is if both $u_1^2 = 0$ and $u_2^2 = 0$. This implies $u_1 = 0$ and $u_2 = 0$, so $\mathbf{u} = \mathbf{0}$.

The standard Euclidean dot product satisfies the positive-definiteness axiom.

2.2 Problem 6 (Moderate)

In $\mathbb{R}^2$, a weighted inner product is defined as $\langle \mathbf{u}, \mathbf{v} \rangle_w = 2u_1 v_1 + 5u_2 v_2$. Calculate the inner product of $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ using this weighted inner product.

Solution: We use the definition provided: $\langle \mathbf{u}, \mathbf{v} \rangle_w = 2u_1 v_1 + 5u_2 v_2$. Here $u_1 = 1, u_2 = 1$ and $v_1 = 3, v_2 = -1$.

$$\begin{aligned} \langle \mathbf{u}, \mathbf{v} \rangle_w &= 2(1)(3) + 5(1)(-1) \\ &= 6 - 5 = 1 \end{aligned}$$

The weighted inner product is 1. Note that this differs from the standard dot product, which would be $3 - 1 = 2$.

3 Positive Definite Matrix

A symmetric $n \times n$ real matrix A is positive definite if the scalar $\mathbf{x}^T A\mathbf{x}$ is strictly positive for every non-zero column vector $\mathbf{x} \in \mathbb{R}^n$.

3.1 Problem 7 (Easy)

Determine if the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ is positive definite using Sylvester's criterion.

Solution: Sylvester's criterion states that a symmetric matrix is positive definite if and only if all its leading principal minors are strictly positive.

Step 1: Check if the matrix is symmetric. $A^T = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} = A$. The matrix is symmetric.

Step 2: Calculate the leading principal minors (determinants of the upper-left submatrices).

First leading principal minor (D_1):

$$D_1 = \det(2) = 2$$

$D_1 > 0$.

Second leading principal minor (D_2):

$$D_2 = \det(A) = (2)(3) - (1)(1) = 6 - 1 = 5$$

$D_2 > 0$.

Step 3: Conclusion. Since all leading principal minors are positive, the matrix A is **positive definite**.

3.2 Problem 8 (Easy)

Determine if the matrix $B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ is positive definite.

Solution: We use Sylvester's criterion.

Step 1: Check symmetry. B is symmetric.

Step 2: Calculate leading principal minors. $D_1 = \det(1) = 1$. ($D_1 > 0$). $D_2 = \det(B) = (1)(1) - (2)(2) = 1 - 4 = -3$.

Step 3: Conclusion. Since $D_2 = -3$ is not positive, the matrix B is **not positive definite**. (It is indefinite).

3.3 Problem 9 (Moderate)

Find the range of values for the parameter k such that the matrix $C = \begin{pmatrix} 4 & 2 & 2 \\ 2 & k & 1 \\ 2 & 1 & 3 \end{pmatrix}$ is positive definite.

Solution: We use Sylvester's criterion. The matrix C is symmetric. We need all leading principal minors D_1, D_2, D_3 to be positive.

Step 1: Calculate D_1.

$$D_1 = \det(4) = 4$$

$D_1 > 0$ is satisfied.

Step 2: Calculate D_2 and find the condition on k. D_2 is the determinant of the upper-left 2×2 submatrix $\begin{pmatrix} 4 & 2 \\ 2 & k \end{pmatrix}$.

$$D_2 = (4)(k) - (2)(2) = 4k - 4$$

We require $D_2 > 0$: $4k - 4 > 0 \implies 4k > 4 \implies k > 1$.

Step 3: Calculate D_3 (the determinant of C) and find the condition on k.

$$\begin{aligned} D_3 = \det(C) &= 4\det\begin{pmatrix} k & 1 \\ 1 & 3 \end{pmatrix} - 2\det\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} + 2\det\begin{pmatrix} 2 & k \\ 2 & 1 \end{pmatrix} \\ &= 4(3k-1) - 2(6-2) + 2(2-2k) \\ &= 12k - 4 - 2(4) + 4 - 4k \\ &= 12k - 4 - 8 + 4 - 4k \\ &= 8k - 8 \end{aligned}$$

We require $D_3 > 0$: $8k - 8 > 0 \implies 8k > 8 \implies k > 1$.

Step 4: Combine the conditions. We require $k > 1$ (from D_2) AND $k > 1$ (from D_3). The range of values for k that makes the matrix C positive definite is $k > 1$.

3.4 Problem 10 (Difficult)

Prove that a symmetric matrix A is positive definite if and only if all its eigenvalues are strictly positive.

Solution: This is a biconditional statement, so we must prove both directions.

Part 1: (If A is positive definite, then its eigenvalues are positive.)

Assume A is a symmetric positive definite matrix. Let λ be an eigenvalue of A and $\mathbf{v}$ be a corresponding non-zero eigenvector. By definition, $A\mathbf{v} = \lambda\mathbf{v}$.

Step 1: Multiply both sides by $\mathbf{v}^T$ on the left.

$$\mathbf{v}^T A\mathbf{v} = \mathbf{v}^T(\lambda\mathbf{v}) = \lambda(\mathbf{v}^T\mathbf{v})$$

Step 2: Analyze the terms. Since A is positive definite, by definition, $\mathbf{v}^T A\mathbf{v} > 0$ (because $\mathbf{v} \neq \mathbf{0}$). Also, $\mathbf{v}^T\mathbf{v} = \|\mathbf{v}\|^2$. Since $\mathbf{v} \neq \mathbf{0}$, $\|\mathbf{v}\|^2 > 0$.

Step 3: Solve for λ.

$$\lambda = \frac{\mathbf{v}^T A\mathbf{v}}{\|\mathbf{v}\|^2}$$

Since the numerator is positive and the denominator is positive, λ must be positive ($\lambda > 0$).

Part 2: (If all eigenvalues of a symmetric matrix A are positive, then A is positive definite.)

Assume A is a symmetric matrix and all its eigenvalues $\lambda_1, \dots, \lambda_n$ are positive.

Step 1: Use the Spectral Theorem. Since A is symmetric, it is orthogonally diagonalizable. There exists an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A, say $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$.

Step 2: Express any non-zero vector $\mathbf{x} \in \mathbb{R}^n$ in this basis.

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

Since $\mathbf{x} \neq \mathbf{0}$, at least one c_i must be non-zero.

Step 3: Calculate $\mathbf{x}^T A\mathbf{x}$. First calculate $A\mathbf{x}$:

$$\begin{aligned} A\mathbf{x} &= A(c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n) \\ &= c_1 A\mathbf{v}_1 + \cdots + c_n A\mathbf{v}_n \\ &= c_1\lambda_1\mathbf{v}_1 + \cdots + c_n\lambda_n\mathbf{v}_n \end{aligned}$$

Now calculate $\mathbf{x}^T A\mathbf{x}$:

$$\mathbf{x}^T A\mathbf{x} = (c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n)^T(c_1\lambda_1\mathbf{v}_1 + \cdots + c_n\lambda_n\mathbf{v}_n)$$

Step 4: Expand the product and use orthogonality. The basis $\{\mathbf{v}_i\}$ is orthonormal, meaning $\mathbf{v}_i^T\mathbf{v}_j = 0$ if $i \neq j$, and $\mathbf{v}_i^T\mathbf{v}_i = 1$. When we expand the product, all cross terms vanish.

$$\begin{aligned} \mathbf{x}^T A\mathbf{x} &= (c_1\mathbf{v}_1^T)(c_1\lambda_1\mathbf{v}_1) + \cdots + (c_n\mathbf{v}_n^T)(c_n\lambda_n\mathbf{v}_n) \\ &= \lambda_1 c_1^2(\mathbf{v}_1^T\mathbf{v}_1) + \cdots + \lambda_n c_n^2(\mathbf{v}_n^T\mathbf{v}_n) \\ &= \lambda_1 c_1^2 + \lambda_2 c_2^2 + \cdots + \lambda_n c_n^2 \end{aligned}$$

Step 5: Analyze the result. We assumed all $\lambda_i > 0$. Also, $c_i^2 \geq 0$. Since at least one c_i is non-zero, at least one $c_i^2 > 0$. Therefore, the sum $\lambda_1 c_1^2 + \cdots + \lambda_n c_n^2$ must be strictly positive. $\mathbf{x}^T A \mathbf{x} > 0$. Thus, A is positive definite.

4 Gram-Schmidt Orthogonalization

The Gram-Schmidt process is a method for converting a basis of a subspace into an orthogonal (or orthonormal) basis for the same subspace.

4.1 Problem 11 (Easy)

Use the Gram-Schmidt process to orthogonalize the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ where $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Solution: We want to find an orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2\}$.

Step 1: Set the first vector.

$$\mathbf{u}_1 = \mathbf{v}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

Step 2: Calculate the second vector using the formula $\mathbf{u}_2 = \mathbf{v}_2 - \text{proj}_{\mathbf{u}_1}\mathbf{v}_2$. The projection formula is $\text{proj}_{\mathbf{u}_1}\mathbf{v}_2 = \frac{\mathbf{v}_2 \cdot \mathbf{u}_1}{\|\mathbf{u}_1\|^2}\mathbf{u}_1$.

Calculate the dot product and the norm squared.

$$\begin{aligned} \mathbf{v}_2 \cdot \mathbf{u}_1 &= (1)(3) + (0)(4) = 3 \\ \|\mathbf{u}_1\|^2 &= 3^2 + 4^2 = 9 + 16 = 25 \end{aligned}$$

Calculate the projection.

$$\text{proj}_{\mathbf{u}_1}\mathbf{v}_2 = \frac{3}{25}\mathbf{u}_1 = \frac{3}{25}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 9/25 \\ 12/25 \end{pmatrix}$$

Calculate $\mathbf{u}_2$.

$$\begin{aligned} \mathbf{u}_2 &= \mathbf{v}_2 - \text{proj}_{\mathbf{u}_1}\mathbf{v}_2 \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 9/25 \\ 12/25 \end{pmatrix} \\ &= \begin{pmatrix} 25/25 - 9/25 \\ 0 - 12/25 \end{pmatrix} = \begin{pmatrix} 16/25 \\ -12/25 \end{pmatrix} \end{aligned}$$

The orthogonal basis is $\left\{ \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 16/25 \\ -12/25 \end{pmatrix} \right\}$.

4.2 Problem 12 (Easy)

Find an orthogonal basis for the subspace spanned by $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

Solution: We apply the Gram-Schmidt process to find the orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2\}$.

Step 1: Set $\mathbf{u}_1$.

$$\mathbf{u}_1 = \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}. \quad (\|\mathbf{u}_1\|^2 = 1^2 + 1^2 + 0^2 = 2)$$

Step 2: Calculate $\mathbf{u}_2$. $\mathbf{u}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{u}_1}{\|\mathbf{u}_1\|^2}\mathbf{u}_1$.

$$\mathbf{v}_2 \cdot \mathbf{u}_1 = (1)(1) + (0)(1) + (1)(0) = 1$$

$$\begin{aligned} \mathbf{u}_2 &= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 - 1/2 \\ 0 - 1/2 \\ 1 - 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix} \end{aligned}$$

The orthogonal basis is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix} \right\}$.

4.3 Problem 13 (Moderate)

Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by the vectors: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$.

Solution: First, we find an orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ using Gram-Schmidt. Then we normalize them.

Step 1: Find $\mathbf{u}_1$.

$$\mathbf{u}_1 = \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \quad (\|\mathbf{u}_1\|^2 = 4)$$

Step 2: Find $\mathbf{u}_2$. $\mathbf{u}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{u}_1}{\|\mathbf{u}_1\|^2}\mathbf{u}_1$.

$$\mathbf{v}_2 \cdot \mathbf{u}_1 = 0 + 1 + 1 + 1 = 3$$

$$\mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{3}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{pmatrix}$$

For easier calculation, let's use the scaled vector $\mathbf{u}_2' = 4\mathbf{u}_2 = \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix}$. $(\|\mathbf{u}_2'\|^2 = 9 + 1 + 1 + 1 = 12)$.

Step 3: Find $\mathbf{u}_3$. $\mathbf{u}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_1}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_2'}{\|\mathbf{u}_2'\|^2}\mathbf{u}_2'$.

$$\mathbf{v}_3 \cdot \mathbf{u}_1 = 0 + 0 + 1 + 1 = 2$$
$$\mathbf{v}_3 \cdot \mathbf{u}_2' = 0 + 0 + 1 + 1 = 2$$

$$\begin{aligned}
\mathbf{u}_3 &= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{12}\begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 - 1/2 - (-3/6) \\ 0 - 1/2 - 1/6 \\ 1 - 1/2 - 1/6 \\ 1 - 1/2 - 1/6 \end{pmatrix} = \begin{pmatrix} 0 \\ -4/6 \\ 2/6 \\ 2/6 \end{pmatrix} = \begin{pmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{pmatrix}
\end{aligned}$$

Let's use the scaled vector $\mathbf{u}_3' = 3\mathbf{u}_3 = \begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix}$. ($\|\mathbf{u}_3'\|^2 = 0 + 4 + 1 + 1 = 6$).

Step 4: Normalize the orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2', \mathbf{u}_3'\}$.

$$\mathbf{q}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \frac{1}{\sqrt{4}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$$

$$\mathbf{q}_2 = \frac{\mathbf{u}_2'}{\|\mathbf{u}_2'\|} = \frac{1}{\sqrt{12}} \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2\sqrt{3}} \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathbf{q}_3 = \frac{\mathbf{u}_3'}{\|\mathbf{u}_3'\|} = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix}$$

The orthonormal basis is $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$.

5 Eigenvectors and Eigenvalues

Given a square matrix A, a non-zero vector $\mathbf{v}$ is an eigenvector of A if $A\mathbf{v} = \lambda\mathbf{v}$ for some scalar λ. The scalar λ is called the eigenvalue.

5.1 Problem 14 (Easy)

Find the eigenvalues of the matrix $A = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 5 & 0 \\ -2 & 4 & -1 \end{pmatrix}$.

Solution: The matrix A is a lower triangular matrix.

The eigenvalues of any triangular matrix (upper triangular, lower triangular, or diagonal) are the entries on the main diagonal.

The main diagonal entries are 3, 5, and -1. Therefore, the eigenvalues are $\lambda_1 = 3, \lambda_2 = 5, \lambda_3 = -1$.

5.2 Problem 15 (Easy)

Verify if the vector $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of the matrix $B = \begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix}$. If so, what is the corresponding eigenvalue?

Solution: To verify if $\mathbf{v}$ is an eigenvector of B, we need to check if the product $B\mathbf{v}$ is a scalar multiple of $\mathbf{v}$ (i.e., $B\mathbf{v} = \lambda\mathbf{v}$).

Step 1: Calculate $B\mathbf{v}$.

$$\begin{aligned} B\mathbf{v} &= \begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 3(1) + 1(2) \\ 4(1) + 3(2) \end{pmatrix} \\ &= \begin{pmatrix} 3 + 2 \\ 4 + 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix} \end{aligned}$$

Step 2: Check if the result is a scalar multiple of $\mathbf{v}$.

$$\begin{pmatrix} 5 \\ 10 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 5\mathbf{v}$$

Step 3: Conclusion. Yes, $\mathbf{v}$ is an eigenvector of B, and the corresponding eigenvalue is $\lambda = 5$.

5.3 Problem 16 (Moderate)

Find the eigenvalues and the corresponding eigenvectors of the matrix $C = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$.

Solution: Step 1: Find the eigenvalues by solving the characteristic equation $\det(C - \lambda I) = 0$.

$$C - \lambda I = \begin{pmatrix} 1 - \lambda & 4 \\ 2 & 3 - \lambda \end{pmatrix}$$

$$\begin{aligned} \det(C - \lambda I) &= (1 - \lambda)(3 - \lambda) - (4)(2) \\ &= (3 - 4\lambda + \lambda^2) - 8 \\ &= \lambda^2 - 4\lambda - 5 \end{aligned}$$

Set the characteristic polynomial to zero: $\lambda^2 - 4\lambda - 5 = 0$. Factor the quadratic equation: $(\lambda - 5)(\lambda + 1) = 0$. The eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = -1$.

Step 2: Find the eigenvector corresponding to $\lambda_1 = 5$. We need to solve the system $(C - 5I)\mathbf{v} = \mathbf{0}$.

$$C - 5I = \begin{pmatrix} 1-5 & 4 \\ 2 & 3-5 \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix}$$

We solve the system:

$$\begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Both equations simplify to $v_1 = v_2$. Let $v_2 = t$. Then $v_1 = t$. A basis for the eigenspace corresponding to $\lambda_1 = 5$ is $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$.

Step 3: Find the eigenvector corresponding to $\lambda_2 = -1$. We solve the system $(C + I)\mathbf{v} = \mathbf{0}$.

$$C + I = \begin{pmatrix} 1+1 & 4 \\ 2 & 3+1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix}$$

We solve the system:

$$\begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives the equation $2v_1 + 4v_2 = 0$, which simplifies to $v_1 = -2v_2$. Let $v_2 = t$. Then $v_1 = -2t$. A basis for the eigenspace corresponding to $\lambda_2 = -1$ is $\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$.

5.4 Problem 17 (Difficult)

Find the eigenvalues and the corresponding eigenspaces of the matrix $D = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. Determine if the matrix is diagonalizable.

Solution: Step 1: Find the eigenvalues. $\det(D - \lambda I) = 0$.

$$D - \lambda I = \begin{pmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{pmatrix}$$

Calculate the determinant:

$$\begin{aligned}
\det(D - \lambda I) &= (2-\lambda)\det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} - 1\det\begin{pmatrix} 1 & 1 \\ 1 & 2-\lambda \end{pmatrix} + 1\det\begin{pmatrix} 1 & 2-\lambda \\ 1 & 1 \end{pmatrix} \\
&= (2-\lambda)[(2-\lambda)^2 - 1] - [(2-\lambda) - 1] + [1 - (2-\lambda)] \\
&= (2-\lambda)(\lambda^2 - 4\lambda + 3) - (1-\lambda) + (\lambda - 1) \\
&= (2-\lambda)(\lambda-1)(\lambda-3) + 2(\lambda-1)
\end{aligned}$$

Factor out $(\lambda - 1)$:

$$\begin{aligned}
&= (\lambda-1)[(2-\lambda)(\lambda-3) + 2] \\
&= (\lambda-1)[(-\lambda^2 + 5\lambda - 6) + 2] \\
&= (\lambda-1)[-\lambda^2 + 5\lambda - 4] \\
&= -(\lambda-1)[\lambda^2 - 5\lambda + 4] \\
&= -(\lambda-1)(\lambda-1)(\lambda-4) = -(\lambda-1)^2(\lambda-4)
\end{aligned}$$

The eigenvalues are $\lambda_1 = 4$ (algebraic multiplicity 1) and $\lambda_2 = 1$ (algebraic multiplicity 2).

Step 2: Find the eigenvector for $\lambda_1 = 4$. Solve $(D - 4I)\mathbf{v} = \mathbf{0}$.

$$D - 4I = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

We use Gaussian elimination on the augmented matrix:

$$\left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The equations are $v_1 - v_3 = 0$ and $v_2 - v_3 = 0$. Let $v_3 = t$. Then $v_1 = t, v_2 = t$. The eigenspace E_4 is spanned by $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$. (Geometric multiplicity 1).

Step 3: Find the eigenvectors for $\lambda_2 = 1$. Solve $(D - I)\mathbf{v} = \mathbf{0}$.

$$D - I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

We solve the system:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The only equation is $v_1 + v_2 + v_3 = 0$. We have two free variables. Let $v_2 = s$ and $v_3 = t$. Then $v_1 = -s - t$. The eigenvectors are of the form:

$$\mathbf{v} = \begin{pmatrix} -s - t \\ s \\ t \end{pmatrix} = s \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

The eigenspace E_1 is spanned by $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$. (Geometric multiplicity 2).

Step 4: Determine diagonalizability. Since the geometric multiplicity equals the algebraic multiplicity for all eigenvalues (1=1 for $\lambda = 4$, and 2=2 for $\lambda = 1$), the matrix D is diagonalizable.

6 Determinants

The determinant is a scalar value that is a function of the entries of a square matrix. It characterizes properties such as invertibility.

6.1 Problem 18 (Easy)

Calculate the determinant of the matrix $A = \begin{pmatrix} 5 & -2 \\ 3 & 4 \end{pmatrix}$.

Solution: The determinant of a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by $ad - bc$.

$$\begin{aligned} \det(A) &= (5)(4) - (-2)(3) \\ &= 20 - (-6) \\ &= 26 \end{aligned}$$

6.2 Problem 19 (Easy)

Calculate the determinant of the matrix $B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$.

Solution: The matrix B is an upper triangular matrix. The determinant of a triangular matrix is the product of the entries on the main diagonal.

$$\det(B) = (1)(4)(6) = 24$$

6.3 Problem 20 (Moderate)

Calculate the determinant of the matrix $C = \begin{pmatrix} 3 & 1 & 0 \\ 2 & -1 & 5 \\ 4 & 2 & 3 \end{pmatrix}$ using cofactor expansion along the first row.

Solution: The cofactor expansion along the first row is given by: $\det(C) = c_{11}C_{11} + c_{12}C_{12} + c_{13}C_{13}$. The cofactor C_{ij} is $(-1)^{i+j}M_{ij}$, where M_{ij} is the minor.

Step 1: Calculate the first term $c_{11}C_{11}$. $c_{11} = 3$. $M_{11} = \det \begin{pmatrix} -1 & 5 \\ 2 & 3 \end{pmatrix} = (-1)(3) - (5)(2) = -3 - 10 = -13$. $C_{11} = (-1)^{1+1}(-13) = -13$. $c_{11}C_{11} = 3(-13) = -39$.

Step 2: Calculate the second term $c_{12}C_{12}$. $c_{12} = 1$. $M_{12} = \det \begin{pmatrix} 2 & 5 \\ 4 & 3 \end{pmatrix} = (2)(3) - (5)(4) = 6 - 20 = -14$. $C_{12} = (-1)^{1+2}(-14) = (-1)(-14) = 14$. $c_{12}C_{12} = 1(14) = 14$.

Step 3: Calculate the third term $c_{13}C_{13}$. $c_{13} = 0$. The term $c_{13}C_{13} = 0$.

Step 4: Sum the terms.

$$\det(C) = -39 + 14 + 0 = -25$$

6.4 Problem 21 (Difficult)

Use properties of determinants (e.g., row reduction) to calculate the determinant of the 4×4 matrix $D = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 3 & 7 & 0 & 4 \\ -1 & -2 & 1 & 5 \\ 2 & 4 & 1 & 6 \end{pmatrix}$.

Solution: We will use Gaussian elimination to transform the matrix into an upper triangular form. We must keep track of how the row operations affect the determinant.

- Adding a multiple of one row to another row does not change the determinant.
- Swapping two rows multiplies the determinant by -1.

Step 1: Eliminate entries below the first pivot.

$$D = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 3 & 7 & 0 & 4 \\ -1 & -2 & 1 & 5 \\ 2 & 4 & 1 & 6 \end{pmatrix} \xrightarrow[R_4 \leftarrow R_4 - 2R_1]{R_2 \leftarrow R_2 - 3R_1, R_3 \leftarrow R_3 + R_1} D_1 \qquad = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 3 & 0 \end{pmatrix}$$

These operations do not change the determinant. $\det(D) = \det(D_1)$.

Step 2: We need a non-zero pivot in the third row, third column. We swap R_3 and R_4.

$$D_1 = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 3 & 0 \end{pmatrix} \xrightarrow{R_3 \leftrightarrow R_4} D_2 \qquad = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix}$$

Swapping rows multiplies the determinant by -1. $\det(D_1) = -1 \cdot \det(D_2)$. So, $\det(D) = -\det(D_2)$.

Step 3: Calculate the determinant of the triangular matrix D_2. The determinant is the product of the diagonal entries.

$$\det(D_2) = (1)(1)(3)(8) = 24$$

Step 4: Calculate the determinant of D.

$$\det(D) = -\det(D_2) = -24$$

The determinant is -24.