BITS Pilani
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Mathematical Foundations

MFDS & MFML Team

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Webinar#1

Agenda

Problems on

- **Solution to the system of linear equations**
- **Vector space**
- **Subspace**
- **Linearly Independent and Dependent**
- **Row Space and Null Space**
- **Basis and Dimension**
- **Inner product**

Some Important Definitions

Elementary row transformation

Example: A= $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

Sl. No.	Elementary row transformation	Notation	Resultant of a matrix A
1	Interchange of first and second row	$R_1 \leftrightarrow R_2$	$\begin{bmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$
2	Multiplication of third row by a constant k	kR_3	$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ kc_1 & kc_2 & kc_3 \end{bmatrix}$
3	Addition to second row k times the first row	$R_2 \rightarrow kR_1 + R_2$	$\begin{bmatrix} a_1 & a_2 & a_3 \\ (ka_1 + b_1) & (ka_2 + b_2) & (ka_3 + b_3) \\ c_1 & c_2 & c_3 \end{bmatrix}$

Rank of a matrix:
The rank of a matrix A in its echelon form is equal to the number of non zero rows. It is denoted by $\rho(A)$.

Steps to find the rank of a matrix

Step 1. In order to reduce the given matrix to a row echelon form we must prefer to have the leading entry (first entry in the first row) non zero, much preferably 1.

Step 2. In case this entry is zero, we can interchange with any suitable row to meet the requirement.

Step 3. We then focus on the leading non zero entry (starting from the first row) to make all the elements in that column zero. However, the transformation has to be performed for the entire row.

Step 4. Row echelon form will be achieved first and we can instantly write down the rank, which is the number of non-zero rows.

System of linear equations and Consistency

Consider a system of '*m*' linear equations in '*n*' unknowns as follows

$$a_{11}\,x_1 + a_{12}\,x_2 + \dots\dots\dots + a_{1n}\,x_n = b_1$$

$$a_{21}\,x_1 + a_{22}\,x_2 + \dots\dots\dots + a_{2n}\,x_n = b_2$$

$$\dots\dots\dots \qquad\qquad \dots\dots\dots$$

$$\dots\dots\dots \qquad\qquad \dots\dots\dots$$

$$a_{m1}x_1 + a_{m2}\,x_2 + \dots\dots\dots\dots + a_{mn}\,x_n = b_m$$

Where a_{ij}'s and b_i's are constants.

- If $b_1, b_2, \dots\dots\dots, b_m$ are all zero, the system is said to be homogeneous. Otherwise, it is said to be Non-homogeneous.

- The set of values $x_1, x_2, \dots, x_n$ which satisfy all the equations simultaneously is called a solution of the system of equations.

- A system of linear equations is said to be **consistent** if it possess a solution. Otherwise, it is said to be **inconsistent**.

Vector Space

Let F be a field, V be a non-empty set for every ordered pair on which two operations (vector addition and scalar multiplication) are defined. If the listed axioms are satisfied for every u, v, w in V and scalars c and d in F, then V is called a vector space over the field.

- If for all u,v in V then u + v is a vector in V (closure under addition).
- If for all u,v in V then u + v = v + u (Commutative property of addition).
- If for all u,v,w in V then(u+v)+w = u+ (v+w) (Associative property of addition).
- If for all u in V , there is a zero vector 0 in V such that, we have (u + 0) = u =(0+u) (Additive identity).
- If for all u in V, there is a vector in V denoted by −u such that u + (−u) = 0=(-u)+u (Additive inverse).
- If for all c in F and u in V then cu is in V (closure under scalar multiplication).
- If for all c in F and u in V then c(u + v) = cu + cv (Distributive property of scalar multiplication).
- If for all c,d in F and u in V then (c + d)u = cu + du (Distributive property of scalar multiplication).
- If for all c,d in F and u in V then c(du) = (cd)u (Associate property of scalar multiplication).
- If for all u in V and 1 in F then 1(u) = u (Scalar identity property).

Subspace

A non-empty subset W of a vector space V over a field F is a subspace of V if and only if the following conditions are satisfied:
Identity Element: The zero vector (0) of the vector space V is in W.
Closure under Addition: For all $\alpha,\beta \in W$, $\alpha+\beta \in W$.
Closure under Scalar Multiplication: For all $c \in F$ and $\alpha \in W$, $c\cdot\alpha \in$.

Example: Let V be the set of all ordered pairs (x,y) where x and y are real numbers. Let $u=(x_1,y_1)$ and $v=(x_2,y_2)$ be two elements in V. Define the addition $u+v=(x_1,y_1)+(x_2,y_2)=(2x_1-3x_2,\ y_1-y_2)$ and the scalar multiplication as $k(x_1,y_1)=((kx_1/3),\ (ky_1/3))$. Show that V is not a vector space.

Solution:

- For u,v in R^2 ,

$$u+v \quad =(x_1,y_1)+(x_2,y_2) = (2x_1-3x_2,\ y_1-y_2) \neq (2x_2-3x_1,\ y_2-y_1)$$

$$\neq (x_2,y_2)+(x_1,y_1) \qquad \neq v+u.$$

which does not satisfy the commutative law.

- For u,v,w in R^2,

Consider (u+v)+w $=\{(x_1,y_1)+(x_2,y_2)\}+(x_3+y_3)$

$=(2x_1-3x_2, y_1-y_2)+(x_3+y_3)$

$=(2(2x_1-3x_2)-3x_3 , y_1-y_2-y_3)$

$=(4x_1-6x_2-3x_3 , y_1-y_2-y_3)$

Now consider u+ (v+w) $= (x_1,y_1)+\{(x_2,y_2)+(x_3+y_3)\}$

$= (x_1,y_1)+(2x_2-3x_3, y_2-y_3)$

$= (2x_1-3(2x_2-3x_3), y_1-(y_2-y_3))$

$= (2x_1-6x_2-9x_3 , y_1-y_2+y_3)$

Thus, $(4x_1-6x_2-3x_3 , y_1-y_2-y_3) \neq (2x_1-6x_2-9x_3 , y_1-y_2+y_3)$.
which does not satisfy the Associative law

- For u in R^2 1in F,

  we have $1.u= 1.(x_1,y_1) = ((x_1/3),(y_1/3)) \neq (x_1,y_1) \neq u$

Thus, $V_2(R)$ is not vector space.

Basis set for a Vector Space

Linear Combination :

Let V (F) be a vector space over the field F and $u_1, u_2, \dots, u_n$ be any n vectors of V. the vector of the form $\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n = \alpha$ where $\alpha_1, \alpha_2, \dots, \alpha_n$ in F is called a linear combination of the vectors $u_1, u_2, \dots, u_n$.

Linear Span :

Let V (F) be a vector space and let $S = \{u_1, u_2, \dots, u_n\}$ be a non-empty subset of V. The linear span of S is the set defined by $L(S) = \{\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n : \alpha_i \in F, 1 \leq i \leq n\}$.

Linearly Dependent Vectors:

A set $\{v_1, v_2, \dots, v_n\}$ of vectors of a vector space V(F) is said to be linearly dependent if there exist scalars $c_1, c_2, \dots, c_n$ in F, not all zeros, such that

$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$.

Linearly Independent Vectors:

A set $\{v_1, v_2, \dots, v_n\}$ of vectors of a vector space V(F) is said to be linearly independent if and only if $c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$ implies $c_1 = 0, c_2 = 0, \dots, c_n = 0$.

Basis of a vector space :

A non-empty subset B of a vector space V is called a basis of V if

- B is a linearly independent set, and
- L(B) = V, i.e., every vector in V can be expressed as a linear combination of the elements of B.

Dimension :

The dimension of a finite dimensional vector space V over F is the number of elements in any basis of V and is denoted by d(V).

Problem 1

$$\mathbf{A} = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

(A) Using elementary row operations, write the matrix in its row echelon form

(B) Let $\mathbf{V}$ be a vector subspace spanned by the columns of matrix $\mathbf{A}$. Find the basis and dimension of $\mathbf{V}$.

(C) Let $\mathbf{V}$ be a vector subspace spanned by vectors $\mathbf{x}$, such that $\mathbf{Ax} = 0$. Find the basis and dimension of $\mathbf{V}$.

(D) Give the set of linearly independent rows of $\mathbf{A}$. What is the number of vectors in this set?

Problem 1 –Solution

(A)

$$\mathbf{A} = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ -3 & 6 & -1 & 1 & -7 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftarrow R_2 + 3 * R_1, R_3 \leftarrow R_3 - 2 * R_1, R_2 \leftarrow R_2/5, R_3 \leftarrow R_3 - R_2, R_1 \leftarrow R_1 - 2 * R_2}$$

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

NOTE: REF is not unique.

(B) column number 1 and column number 3 have pivot.
Basis of column space is:

$$\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$$

Rank of col space $=$ dimension of $\mathbf{V} = 2$

Problem 1 – Solution

(C) From REF, the null space is:

$$\begin{bmatrix} x \\ r \\ y \\ s \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} r + \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} t$$

basis =

$$\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Rank of null space = dimension of $\mathbf{V} = 3$

(D) From REF, row 1 and row 2 form basis of row space

Rank of row space = 2

Problem 2

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(A) Suppose $\boldsymbol{A} = [C_1, C_2, C_3, C_4]$ where $\boldsymbol{C}_m \in \mathbb{R}^4, m = 1, 2, 3, 4$ are columns of $\boldsymbol{A}$. It is known that rank$(\boldsymbol{A}) = 2$ and $\boldsymbol{C}_2 = 3\boldsymbol{C}_1$ and $\boldsymbol{C}_4 = 2\boldsymbol{C}_1 + 3\boldsymbol{C}_3$. If a particular solution of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is $[1, 0, 1, 0]^T$, then

(i) Find the general solution of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.

(ii) Find $\boldsymbol{b}$ if RREF$(\boldsymbol{A}) = \boldsymbol{A}$.

(B) Consider $M = \{\boldsymbol{A} \in \mathbb{R}^{2\times 2} \mid \boldsymbol{A} = -\boldsymbol{A}^T\}$.

(i) Prove that M is subspace of vector space V, where V is set of all 2×2 real matrices.

(ii) Prove or disprove $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for M defined in (i).

(iii) Prove or disprove that $\left\{ \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 5 & -5 \\ 5 & 5 \end{bmatrix}, \begin{bmatrix} 0 & -3 \\ 3 & 1 \end{bmatrix} \right\}$ is a linearly independent set.

Problem 2 – Solution

(A) Now $\boldsymbol{A} = [\boldsymbol{C}_1, \boldsymbol{C}_2, \boldsymbol{C}_3, \boldsymbol{C}_4]$ where $\boldsymbol{C}_m \in \mathbb{R}^4, m = 1, 2, 3, 4$ are columns of $\boldsymbol{A}$. It is known that $\text{rank}(\boldsymbol{A}) = 2$

(i) $\boldsymbol{C}_2 = 3\boldsymbol{C}_1 \Rightarrow 3\boldsymbol{C}_1 - \boldsymbol{C}_2 + 0\boldsymbol{C}_3 + 0\boldsymbol{C}_4 = \mathbf{0}$

$$\Rightarrow [\boldsymbol{C}_1, \boldsymbol{C}_2, \boldsymbol{C}_3, \boldsymbol{C}_4] \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$\boldsymbol{C}_4 = 2\boldsymbol{C}_1 + 3\boldsymbol{C}_3 \Rightarrow 2\boldsymbol{C}_1 + 0\boldsymbol{C}_2 + 3\boldsymbol{C}_3 - 1\boldsymbol{C}_4 = \mathbf{0}$

$$\Rightarrow [\boldsymbol{C}_1, \boldsymbol{C}_2, \boldsymbol{C}_3, \boldsymbol{C}_4] \begin{bmatrix} 2 \\ 0 \\ 3 \\ -1 \end{bmatrix}$$

Thus, the general solution of $\boldsymbol{Ax} = \boldsymbol{b}$ is given by

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 \\ 0 \\ 3 \\ -1 \end{bmatrix} \mid \forall \lambda_1, \lambda_2 \in \mathbb{R} \right\}$$

Problem 2 - Solution

(ii) Now $\boldsymbol{A} = [\boldsymbol{C}_1, \boldsymbol{C}_2, \boldsymbol{C}_3, \boldsymbol{C}_4] = [\boldsymbol{C}_1, 2\boldsymbol{C}_1, \boldsymbol{C}_3, 2\boldsymbol{C}_1 + 3\boldsymbol{C}_3]$ and $\text{rank}(\boldsymbol{A}) = 2$.

Therefore $\boldsymbol{C}_1, \boldsymbol{C}_3$ are pivot columns and $\text{RREF}(\boldsymbol{A}) = \boldsymbol{A}$

$\Rightarrow \boldsymbol{C}_1 = [1, 0, 0, 0]^T$ and $\boldsymbol{C}_3 = [0, 1, 0, 0]^T$

$\Rightarrow \boldsymbol{C}_2 = [3, 0, 0, 0]^T$ and $\boldsymbol{C}_4 = [2, 3, 0, 0]^T$

$$\Rightarrow \boldsymbol{A} = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore $\boldsymbol{b} = \boldsymbol{A}[1, 0, 1, 0]^T = [1, 1, 0, 0]^T$

Problem 2 - Solution

(B) (i) Clearly the zero matrix of order 2, $\mathbf{0} \in M \Rightarrow M \neq \phi$.
Let $\boldsymbol{A}, \boldsymbol{B} \in M, k \in \mathbb{R}$
$\Rightarrow \boldsymbol{A} = -\boldsymbol{A}^T, \boldsymbol{B} = -\boldsymbol{B}^T$
$\Rightarrow \boldsymbol{A} + \boldsymbol{B} = -\boldsymbol{A}^T - \boldsymbol{B}^T = -(\boldsymbol{A} + \boldsymbol{B})^T$
$\Rightarrow \boldsymbol{A} + \boldsymbol{B} \in M$.
$k\boldsymbol{A} = k(-\boldsymbol{A}^T) = (-k)\boldsymbol{A}^T = -(k\boldsymbol{A})^T$
So, M is a subspace.

(ii) Clearly $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ does not belong to M and hence cannot be a basis of M.
(Kindly award full marks for any valid reason for disproving the claim.)

(iii) Consider $a \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} + b \begin{bmatrix} 5 & -5 \\ 5 & 5 \end{bmatrix} + c \begin{bmatrix} 0 & -3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\begin{aligned} \Rightarrow a + 5b &= 0 \\ -a - 5b - 3c &= 0 \\ a + 5b + 3c &= 0 \\ 5b + c &= 0 \end{aligned}$$

$\Rightarrow a = b = c = 0$. Hence it a linearly independent set.

Problem 3

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A function $f : \mathbb{R}^3 \Rightarrow \mathbb{R}^3$ is given below:

$$\mathbf{f}([\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}]^T) = [\mathbf{2} * \mathbf{x_1} + \mathbf{x_2}, \mathbf{3} * \mathbf{x_1} + \mathbf{7} * \mathbf{x_2} + \mathbf{3} * \mathbf{x_3}, \mathbf{3} * \mathbf{x_1} + \mathbf{x_2} + \mathbf{x_3}]^T$$

(A) Using elementary row operations, write the matrix in its row echelon form

(B) Let $\mathbf{V}$ be a vector subspace spanned by the columns of matrix $\mathbf{A}$. Give a set of vectors that form the basis for the vector space $\mathbf{V}$. What is the number of vectors in this basis set?

(C) Let $\mathbf{V}$ be a vector subspace spanned by vectors $\mathbf{x}$, such that $\mathbf{Ax} = 0$. Give a set of vectors that form the basis for the vector space $\mathbf{V}$. What is the number of vectors in this basis set?

(D) Give the set of linearly independent rows of $\mathbf{A}$. What is the number of vectors in this set?

Problem 3 Solution

(A)

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 7 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftarrow R_1/2, R_2 \leftarrow R_2 - 3 * R_1, R_3 \leftarrow R_3 - 3 * R_1,} \xrightarrow{R_2 \leftarrow R_2/(11/2), R_3 \leftarrow R_3/(-1/2), R_3 \leftarrow R_3 - R_2}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{6}{11} \\ 0 & 0 & 1 \end{bmatrix}$$

NOTE: REF is not unique.

(B) All columns have pivot.

All columns form the basis of column space

$$\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$$

Rank of col space = 3

(C) From REF, the null space is: $\{\mathbf{0}\}$
Rank of null space = 0

(D) Since each column has a pivot, A has 3 linearly independent rows.

The set of Linearly Independent rows is given below.

There are three elements

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \\ 3 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

Norms

- A norm on a vector space is a function $\|.\| : V \to \mathbb{R}$, $\boldsymbol{x} \to \|\boldsymbol{x}\|$ which assigns to each vector $\boldsymbol{x}$ a length $\|\boldsymbol{x}\|$ such that for all $\lambda \in \mathbb{R}$ and $\boldsymbol{x}, \boldsymbol{y} \in V$ the following properties hold:
  - Absolutely homogeneous: $\|\lambda \boldsymbol{x}\| = |\lambda| \|\boldsymbol{x}\|$
  - Triangle inequality: $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$
  - Positive definite: $\|\boldsymbol{x}\| \geq 0$ and $\|\boldsymbol{x}\| = 0 \implies \boldsymbol{x} = 0$
- Manhattan norm : $\|\boldsymbol{x}\| = \sum_{i=1}^{i=n} |x_i|$
- Euclidean norm : $\|\boldsymbol{x}\|_2 = \sqrt{\sum_{i=1}^{i=n} x_i^2}.$

Inner Product

- Dot product in $\mathbb{R}^n$ is given by $\boldsymbol{x}^T\boldsymbol{y} = \sum_{i=1}^{i=n} x_i y_i$
- A bilinear mapping Ω is a mapping with two arguments and is linear in both arguments: Let V be a vector space such that $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in V$, and let $\lambda, \psi \in \mathbb{R}$. Then we have $\Omega(\lambda\boldsymbol{x} + \psi\boldsymbol{y}, \boldsymbol{z}) = \lambda\Omega(\boldsymbol{x}, \boldsymbol{z}) + \psi\Omega(\boldsymbol{y}, \boldsymbol{z})$, and $\Omega(\boldsymbol{x}, \lambda\boldsymbol{y} + \psi\boldsymbol{z}) = \lambda\Omega(\boldsymbol{x}, \boldsymbol{y}) + \psi\Omega(\boldsymbol{x}, \boldsymbol{z})$.
- Let V be a vector space and $\Omega : V \times V \to \mathbb{R}$ be a bilinear mapping that takes two vectors as arguments and returns a real number. Then Ω is called symmetric if $\Omega(\boldsymbol{x}, \boldsymbol{y}) = \Omega(\boldsymbol{y}, \boldsymbol{x})$. Also Ω is called positive-definite if $\forall x \in V \setminus \{0\},\ \ \Omega(\boldsymbol{x}, \boldsymbol{x}) > 0$ and $\Omega(\mathbf{0}, \mathbf{0}) = 0$.

Problem 4

Consider an inner product space with an inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^{\mathbf{T}} \mathbf{A} \mathbf{y}$ defined with the help of matrix $\mathbf{A}$ defined below.

$$\mathbf{A} = \begin{bmatrix} 5.5 & -1.5 \\ -1.5 & 5.5 \end{bmatrix}$$

Consider two vectors $\mathbf{a} = [1 \quad 5]^T$ and $\mathbf{b} = [2 \quad 7]^T$ in the inner product space.

(a) Find the distance $d(\mathbf{a}, \mathbf{b}) = \|a - b\|$ between vectors $\mathbf{a}$ and $\mathbf{b}$ in the above inner product space where $\| \cdot \|$ is the norm induced by the inner product.

(b) Find the angle between vectors $\mathbf{a}$ and $\mathbf{b}$ in the above inner product space.

Problem 4 – Solution

(a)

$$(\mathbf{d}(\mathbf{a},\mathbf{b}))^{\mathbf{2}} = \begin{bmatrix}1-2 & 5-7\end{bmatrix}\begin{bmatrix}5.5 & -1.5\\ -1.5 & 5.5\end{bmatrix}\begin{bmatrix}1-2\\ 5-7\end{bmatrix} = \mathbf{21.5}$$

$$d(\mathbf{a},\mathbf{b}) = 4.63$$

(b)

$$\cos(\theta) = \frac{\mathbf{a^T A b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \frac{\mathbf{a^T A b}}{\sqrt{\mathbf{a^T A a}}\sqrt{\mathbf{b^T A b}}}$$

$$\cos(\theta) = \frac{178}{\sqrt{128}\sqrt{249.5}} = 0.005573$$

$$\theta = 89.6 \text{ deg}$$

THANK YOU