Practice Set

Cholesky, Eigenvalues, Diagonalization, and SVD

Saurabh

Contents

1 Cholesky Decomposition

Question 1 (Easy: 2×2 Calculation)

Determine if the matrix $A = \begin{pmatrix} 4 & 2 \\ 2 & 5 \end{pmatrix}$ is symmetric positive definite. If so, find its Cholesky decomposition $A = LL^T$, where L is lower triangular.

Detailed Solution:
Step 1: Check Positive Definiteness A matrix is positive definite if all leading principal minors are positive.

- 1st Minor: $\det(4) = 4 > 0$.
- 2nd Minor: $\det(A) = (4)(5) - (2)(2) = 20 - 4 = 16 > 0$.

Since all minors are positive and A is symmetric ($A^T = A$), the decomposition exists.

Step 2: Calculate L Let $L = \begin{pmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{pmatrix}$. We equate $LL^T = A$:

$$\begin{pmatrix} l_{11}^2 & l_{11}l_{21} \\ l_{11}l_{21} & l_{21}^2 + l_{22}^2 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 5 \end{pmatrix}$$

1. **Solve for l_{11}:**

$$l_{11}^2 = 4 \implies l_{11} = \sqrt{4} = 2$$

2. **Solve for l_{21}:**

$$l_{11}l_{21} = 2 \implies 2(l_{21}) = 2 \implies l_{21} = 1$$

3. **Solve for l_{22}:**

$$l_{21}^2 + l_{22}^2 = 5 \implies 1^2 + l_{22}^2 = 5 \implies l_{22}^2 = 4 \implies l_{22} = 2$$

Final Matrix:

$$L = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$$

Question 2 (Moderate: 3×3 Calculation)

Find the Cholesky factor L for the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 8 & 10 \\ 3 & 10 & 22 \end{pmatrix}$.

Detailed Solution:
We assume $L = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}$. We solve column by column.

Column 1:

- $l_{11} = \sqrt{a_{11}} = \sqrt{1} = 1$.
- $l_{21} = \frac{a_{21}}{l_{11}} = \frac{2}{1} = 2$.
- $l_{31} = \frac{a_{31}}{l_{11}} = \frac{3}{1} = 3$.

Column 2:

- Diagonal entry (l_{22}):

$$l_{22} = \sqrt{a_{22} - l_{21}^2} = \sqrt{8 - 2^2} = \sqrt{4} = 2$$

- Below diagonal (l_{32}):

$$l_{32} = \frac{1}{l_{22}}(a_{32} - l_{31}l_{21}) = \frac{1}{2}(10 - (3)(2)) = \frac{1}{2}(4) = 2$$

Column 3:

- Diagonal entry (l_{33}):

$$l_{33} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2} = \sqrt{22 - 3^2 - 2^2}$$

$$l_{33} = \sqrt{22 - 9 - 4} = \sqrt{9} = 3$$

Final Matrix:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 2 & 3 \end{pmatrix}$$

2 Eigen Decomposition and Diagonalization

Question 3 (Easy: Conceptual Check)

Let $A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$. Explain step-by-step why this matrix is **not** diagonalizable.

Detailed Solution:
Step 1: Find Eigenvalues (Algebraic Multiplicity) Since A is upper triangular, the eigenvalues are the diagonal entries.

$$\lambda_1 = 3, \quad \lambda_2 = 3$$

The eigenvalue $\lambda = 3$ appears twice, so its **Algebraic Multiplicity is 2**.

Step 2: Find Eigenvectors (Geometric Multiplicity) We find the null space of $(A - 3I)$:

$$A - 3I = \begin{pmatrix} 3-3 & 1 \\ 0 & 3-3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

We solve $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The first row gives the equation: $0x + 1y = 0 \implies y = 0$. The variable x is free. Let $x = 1$. The eigenvector is $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Since there is only one free variable, the **Geometric Multiplicity is 1**.

Conclusion: For a matrix to be diagonalizable, Geometric Multiplicity must equal Algebraic Multiplicity for all eigenvalues. Here, $1 \neq 2$. Therefore, A is defective and not diagonalizable.

Question 4 (Moderate: 2×2 Matrix Powers)

Given $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Use diagonalization to find an expression for A^{10}.

Detailed Solution:
Step 1: Find Eigenvalues

$$\det(A - \lambda I) = (1-\lambda)(1-\lambda) - 4 = 0$$

$$(1-\lambda)^2 = 4 \implies 1 - \lambda = \pm 2$$

$$\lambda_1 = -1, \quad \lambda_2 = 3$$

Step 2: Find Eigenvectors

- For $\lambda_1 = -1$: solve $(A + I)v = 0$.

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \implies 2x + 2y = 0 \implies x = -y$$

  Let $y = 1 \implies x = -1$. $v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

- For $\lambda_2 = 3$: solve $(A - 3I)v = 0$.

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \implies -2x + 2y = 0 \implies x = y$$

Let $y = 1 \implies x = 1$. $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Step 3: Construct P and D

$$D = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}, \quad P = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

Find P^{-1} using formula $\frac{1}{ad-bc}$:

$$\det(P) = (-1)(1) - (1)(1) = -2$$

$$P^{-1} = \frac{1}{-2} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

Step 4: Compute A^{10}

$$A^{10} = PD^{10}P^{-1} = P \begin{pmatrix} (-1)^{10} & 0 \\ 0 & 3^{10} \end{pmatrix} P^{-1}$$

$$= \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3^{10} \end{pmatrix} \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} -1 & 3^{10} \\ 1 & 3^{10} \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 + 3^{10} & -1 + 3^{10} \\ -1 + 3^{10} & 1 + 3^{10} \end{pmatrix}$$

Question 5 (Tough: Reverse Engineering)

A 2×2 matrix A has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$. The corresponding eigenvectors are $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Find the matrix A.

Detailed Solution:
We rely on the definition $A = PDP^{-1}$.

Step 1: Construct D and P

$$D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{(Eigenvalues on diagonal)}$$

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{(Eigenvectors as columns)}$$

Step 2: Find P^{-1}

$$\det(P) = (1)(1) - (1)(0) = 1$$

$$P^{-1} = \frac{1}{1} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

Step 3: Multiply Matrices

$$A = P(DP^{-1}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \left[\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \right]$$

Multiply the bracketed term first:

$$\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 0 & -1 \end{pmatrix}$$

Now multiply by P:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2+0 & -2-1 \\ 0+0 & 0-1 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ 0 & -1 \end{pmatrix}$$

Question 6 (Tough: 3×3 Spectral Decomposition)

Find the eigenvalues of the symmetric matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

Detailed Solution:
Step 1: Characteristic Equation

$$\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix}$$

Expand along row 1:

$$-\lambda\left((-\lambda)(-\lambda) - 1\right) - 1\left(1(-\lambda) - 1\right) + 1\left(1 - (-\lambda)\right) = 0$$

$$-\lambda(\lambda^2 - 1) - (-\lambda - 1) + (1 + \lambda) = 0$$

$$-\lambda^3 + \lambda + \lambda + 1 + 1 + \lambda = 0$$

$$-\lambda^3 + 3\lambda + 2 = 0 \implies \lambda^3 - 3\lambda - 2 = 0$$

Step 2: Solve for Roots Test integer factors of -2: $\pm 1, \pm 2$. Try $\lambda = -1$: $(-1)^3 - 3(-1) - 2 = -1 + 3 - 2 = 0$. So $(\lambda + 1)$ is a factor. Perform polynomial division or inspection:

$$(\lambda + 1)(\lambda^2 - \lambda - 2) = 0$$

$$(\lambda + 1)(\lambda - 2)(\lambda + 1) = 0$$

Eigenvalues: $\lambda_1 = 2, \lambda_2 = -1, \lambda_3 = -1$.

Question 7 (Extra: 3×3 Eigen Decomposition)

Find the eigenvalues and eigenvectors for the matrix $A = \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$.

Detailed Solution:

Step 1: Find Eigenvalues Expand determinant along Column 2 (since it has two zeros):

$$\det(A - \lambda I) = (1-\lambda)\det\begin{pmatrix} 4-\lambda & 1 \\ -2 & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda)[(4-\lambda)(1-\lambda) - (-2)(1)]$$

$$= (1-\lambda)[4 - 5\lambda + \lambda^2 + 2]$$

$$= (1-\lambda)(\lambda^2 - 5\lambda + 6) = (1-\lambda)(\lambda-2)(\lambda-3)$$

Eigenvalues: $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$.

Step 2: Find Eigenvectors

- **For $\lambda = 1$:** Solve $(A - I)v = 0$

$$\begin{pmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

  Row 2 implies $-2x = 0 \implies x = 0$. Row 1 implies $3(0) + z = 0 \implies z = 0$. y is free. Let $y = 1$. $v_1 = (0, 1, 0)^T$.

- **For $\lambda = 2$:** Solve $(A - 2I)v = 0$

$$\begin{pmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

  Row 1: $2x + z = 0 \implies z = -2x$. Row 2: $-2x - y = 0 \implies y = -2x$. Let $x = 1$, then $y = -2, z = -2$. $v_2 = (1, -2, -2)^T$.

- **For $\lambda = 3$:** Solve $(A - 3I)v = 0$

$$\begin{pmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

  Row 1: $x + z = 0 \implies z = -x$. Row 2: $-2x - 2y = 0 \implies y = -x$. Let $x = 1$, then $y = -1, z = -1$. $v_3 = (1, -1, -1)^T$.

3 Singular Value Decomposition (SVD)

Question 8 (Easy: Square Diagonal Matrix)

Find the SVD of $A = \begin{pmatrix} 3 & 0 \\ 0 & -5 \end{pmatrix}$. Note: SVD requires positive diagonal entries in Σ.

Detailed Solution:
We want $A = U\Sigma V^T$. **Step 1: Determine** Σ Singular values are the absolute values of the eigenvalues of a diagonal matrix. $\sigma_1 = |-5| = 5$, $\sigma_2 = |3| = 3$. We order them largest to smallest: $\Sigma = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$.

Step 2: Determine U **and** V Since we swapped the order (5 came from position 2, 3 came from position 1), we use a permutation matrix for V. $V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Now we adjust U to ensure $AV = U\Sigma$.

$$Av_1 = A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \end{pmatrix} = -1 \cdot (5)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \implies u_1 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$Av_2 = A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} = 1 \cdot (3)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies u_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \Sigma = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}, V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Question 9 (Moderate: 3×2 Tall Matrix)

Find the SVD of $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$.

Detailed Solution:
Step 1: Find V **and** Σ **using** A^TA

$$A^TA = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

Eigenvalues: Trace is 4, Determinant is 0. $\lambda_1 = 4, \lambda_2 = 0$. Singular values: $\sigma_1 = \sqrt{4} = 2, \sigma_2 = 0$.

$$\Sigma = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Eigenvectors of A^TA: For $\lambda = 4$: $v_1 = \frac{1}{\sqrt{2}}(1, 1)^T$. For $\lambda = 0$: $v_2 = \frac{1}{\sqrt{2}}(1, -1)^T$.

$$V = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

Step 2: Find U using $u_i = \frac{1}{\sigma_i} A v_i$

$$u_1 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2/\sqrt{2} \\ 0 \\ 2/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$$

To find u_2 and u_3, we extend the basis to $\mathbb{R}^3$. We need vectors orthogonal to u_1. By inspection: $u_2 = (0, 1, 0)^T$ is clearly orthogonal. $u_3 = (1/\sqrt{2}, 0, -1/\sqrt{2})^T$ is orthogonal to both.

$$U = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}$$

Question 10 (Moderate: Symmetric Square SVD)

Find the SVD of the symmetric matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Detailed Solution:
Step 1: Eigenvalues vs Singular Values The eigenvalues of A are $\lambda_1 = 1, \lambda_2 = -1$. The singular values are $\sigma_1 = |1| = 1, \sigma_2 = |-1| = 1$. Thus $\Sigma = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Step 2: Construct Decomposition Since A is symmetric orthogonal, $A^T A = I$. We can pick any orthonormal basis for V. Let's choose the standard basis:

$$V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then compute U:

$$U = AV\Sigma^{-1} = AII = A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Check: $U\Sigma V^T = AII = A$. Correct.

Question 11 (Tough: 2×3 Wide Matrix)

Find the SVD of $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

Detailed Solution:
Since A is wide, it is easier to find U first using AA^T. **Step 1: Find U and Σ using AA^T**

$$AA^T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

Eigenvalues are 2 and 1. Singular values: $\sigma_1 = \sqrt{2}, \sigma_2 = 1$.

$$\Sigma = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Eigenvectors of AA^T are just the standard basis (since it's diagonal): $u_1 = (1, 0)^T$, $u_2 =$

$(0, 1)^T$. So $U = I_2$.

Step 2: Find V using $v_i = \frac{1}{\sigma_i} A^T u_i$

$$v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$$

$$v_2 = \frac{1}{1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

We need v_3 to complete the basis for $\mathbb{R}^3$. It must be orthogonal to v_1 and v_2. By inspection: $v_3 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$.

$$V = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}$$