Birla Institute of Technology & Science, Pilani
Work Integrated Learning Programs Division
Second Semester 2024-2025

Mid-Semester Test Solution
EC-2 Makeup

Course No. : SSCLZC416
Course Title : Mathematical Foundations for Data Science
Nature of Exam : Closed Book
Weightage : 30%
Duration : 2 Hours
Date of Exam : 12-07-2025, AN

No. of Pages = 2
No. of Questions = 6

Note to Students:
1. Please follow all the *Instructions to Candidates* given on the cover page of the answer book.
2. All parts of a question should be answered consecutively. Each answer should start from a fresh page.
3. Assumptions made if any, should be stated clearly at the beginning of your answer.
4. Notation used here are as per the text book.

Q1. Let $A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & -3 & 4 \\ 0 & 2 & -2 & 3 \\ 1 & -1 & -4 & 2 \end{bmatrix}$

(a) Find the Echelon form of A. [3M]

Ans: $A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & -3 & 4 \\ 0 & 2 & -2 & 3 \\ 1 & -1 & -4 & 2 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 3 \\ 0 & -1 & -3 & 0 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & -4 & 0 \end{bmatrix} [2M]$

$\approx \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ [1M]

(b) Using your answer in (a) find the determinant of A. [1M]
Ans: Determinant of A = (-1)*[1*1*(-4)*3] =12 [1M]

(c) Suppose $b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, what can you conclude about the solution of the linear system of equations AX = b(whether no solution or unique solution or infinite number of solutions) [1M]

Ans: Since in the reduced row echelon form of A, all columns have pivot elements, the system Ax = b has a unique solution. [1M]

Q2. Consider the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$.

(a) Prove that A is a symmetric positive definite matrix. [3M]

Ans: Let X= $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be any non-zero vector in R^3.

$$AX = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_2 - x_3 \\ -x_2 + 2x_3 \end{bmatrix} \text{ [1M]}$$

$$X^T(AX) = [x_1, x_2, x_3] \begin{bmatrix} x_1 \\ 2x_2 - x_3 \\ -x_2 + 2x_3 \end{bmatrix} = {x_1}^2 + {x_2}^2 + {x_3}^2 + (x_2 - x_3)^2 > 0 \text{(since X is non-zero)[2M]}$$

(b) Using this matrix A, find an inner product on R^3. [1M]

$$< X,\ Y > = X^T(AY) = [x_1, x_2, x_3] \begin{bmatrix} y_1 \\ 2y_2 - y_3 \\ -y_2 + 2y_3 \end{bmatrix} = x_1y_1 + 2x_2y_2 - x_2y_3 - x_3y_2 + 2x_3y_3 \text{ [1M]}$$

(c) Using the inner product defined in (b) find the angle between the vectors $\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ [1M]

$$< \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} > = 3 + 0 + 1 - 0 - 4 = 0.$$

Since the inner product is 0, the vectors are orthogonal. [1M]

Q3. Consider the matrices $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(a) Find the ranks of A and B. [1M]

Since A is in reduced row echelon form and there are two non-zero rows, rank of A is 2. [0.5M]

Since the reduced row echelon form of $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, rank of B is 1. [0.5M]

(b) Are the column vectors of A linearly independent? Why? [1M]

Since one of the column vectors is 0 vector, the column vectors are linearly dependent. [1M]

(c) Find bases and dimensions of the subspaces $\{\boldsymbol{x} \in R^3 : A\boldsymbol{x} = 0\}$ and $\{\boldsymbol{x} \in R^3 : B\boldsymbol{x} = 0\}$. [3M]

$A\boldsymbol{x} = 0$ implies $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. This implies $x_1 = 0$ and $x_3 = 0$. [0.5M]

The Basis is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ [0.5M] and the dimension is 1. [0.5M]

$B\boldsymbol{x} = 0$ implies $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. This implies $x_2 = 0$. [0.5M]

The Basis is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ $[0.5M]$ and the dimension is 2. [0.5M]

Q4. Suppose $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$ and $A = U\Sigma V^T$ is the singular value decomposition of A. Find the matrices Σ and V. [no need to find U]. [5M]

Ans: $A^T A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$ [0.5M]

The characteristic equation of $A^T A$ is $\lambda^2 - 7\lambda + 6 = 0$. [0.5M]

Solving this equation, we get the roots are 6 and 1.

So, the eigen values of $A^T A$ are 6 and 1. [0.5M]

The matrix $\Sigma = \begin{bmatrix} \sqrt{6} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ [1M]

The eigen vector corresponding to 6:

Solving $\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ we get $x_1 = 2x_2$.

Hence one eigen vector is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. After normalizing, $v_1 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$ [1M]

The eigen vector corresponding to 1:

Solving $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ we get $2x_1 = -x_2$.

Hence one eigen vector is $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$. After normalizing, $v_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} \end{bmatrix}$ [1M]

Hence the matrix $V = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{bmatrix}$. [0.5M]

Q5. Consider the function $f(x, y) = e^x(\cos y + \sin y)$.

(a) Find **∇f**(x,y) at $(0, \frac{\pi}{2})$. [2M]

Ans: $\nabla f(x, y) = \begin{bmatrix} e^x(\cos y + \sin y) \\ e^x(\cos y - \sin y) \end{bmatrix}$ [1M]

$\nabla f\left(0, \frac{\pi}{2}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ [1M]

(b) Suppose if you further consider x, y as functions of r and s given by x(r,s) = 2r + 3s, y(r, s) = r – s. Using the chain rule find $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial s}$. [3M]

Ans: $\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$ [0.5M]

$= [e^x(\cos y + \sin y)]2 + [e^x(\cos y - \sin y)]1 = 3\, e^x \cos y + e^x \sin y$

$= 3e^{(2r+2s)} \cos(r - s) + e^{(2r+3s)} \sin(r - s)$ [1M]

$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$ [0.5M]

$= [e^x(\cos y + \sin y)]3 + [e^x(\cos y - \sin y)](-1) = 2\, e^x \cos y + 4e^x \sin y$

$= 2e^{(2r+2s)} \cos(r - s) + 4e^{(2r+3s)} \sin(r - s)$ [1M]

Q6. Consider the function $f(x, y) = x^3 + y^3. -3x - 12y - 2$.

(a) Find the points of local maximum and local minimum (if any) of the function f(x, y). [3M]

$\nabla f(x, y) = \begin{bmatrix} 3x^2 - 3 \\ 3y^2 - 12 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ implies $x = \pm 1, y = \pm 2$.

The points where the function achieves local maximum or minimum are (1, 2), (-1, -2), (1, -2) and (-1, 2). [1M]

Hessian of f(x, y) = $\begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6x & 0 \\ 0 & 6y \end{bmatrix} = 36xy$ [0.5M]

Hessian of f(x, y) at (1, 2) = 72 > 0

Hessian of f(x, y) at (-1, -2) = 72 > 0

Hessian of f(x, y) at (1, -2) = -72 < 0

Hessian of f(x, y) at (-1, 2) = -72 < 0 [0.5M]

So at (1 ,2) and (-1, -2) f(x, y) attains local minimum.

At (1, -2) and (-1, 2) f(x, y) attains local maximum. [1M]

(b) On the line y = 0(x-axis) find the points of local maximum and local minimum of f(x, y). [2M]

On the line y = 0(x-axis), g(x) = f(x, 0) = $x^3 - 3x - 2$

$\frac{dg}{dx} = 3x^2 - 3 = 0$ implies $x = \pm 1$.

The points where the function f(x,y) achieves local maximum or minimum on x-axis are (1, 0), (-1, 0) [1M]

$\frac{d^2g}{dx^2} = 6x$. $\frac{d^2g}{dx^2}$ at x =1 = 6 and $\frac{d^2g}{dx^2}$ at x = -1 = -6.

So on the x-axis f(x,y) attains maximum at (-1, 0) and minimum at (1, 0). [1M]
