Bayes' Theorem & Conditional Probability

Practice Problems (Randomized Order)

March 9, 2025

1 Introduction

Key Concepts:

- $P(A \mid B)$ is the probability that A occurs given B has occurred.
- **Bayes' Theorem:**
$$P(A \mid B) = \frac{P(B \mid A)\, P(A)}{P(B)}.$$
- **Law of Total Probability:** If $A_1, \ldots, A_n$ partition all possibilities,
$$P(B) = \sum_i P(B \mid A_i)\, P(A_i).$$

2 Practice Problems

Problem 1: 3-Point Preference and Handedness

60% of players are right-handed. Right-handers prefer 3-point shots 55%. Left-handers 70%. If a random player is found to prefer 3-pointers, what is the probability they are left-handed?

Solution

Notation: L= "left-handed," R= "right-handed," T= "3-pointer preference."

$$P(R) = 0.60, \quad P(L) = 0.40, \quad P(T \mid R) = 0.55, \quad P(T \mid L) = 0.70.$$

$$P(T) = P(T \mid R)\,P(R) + P(T \mid L)\,P(L) = 0.55 \times 0.60 + 0.70 \times 0.40 = 0.33 + 0.28 = 0.61.$$

So $P(T) = 0.61$.
By **Bayes' Theorem**:

$$P(L \mid T) = \frac{P(T \mid L)\,P(L)}{P(T)} = \frac{0.70 \times 0.40}{0.61} \approx 0.459.$$

So about 45.9%.
Interpretation: Even though left-handers are the minority (40%), they prefer 3s more (70%). After seeing a 3-pointer preference, there's a relatively higher chance that the player is left-handed.

Problem 2: Conditional Normal Times

A person's commute time T is normally distributed with mean 30 mins and standard deviation 4 mins. We observe that on a certain day, $T < 30$. What is $P(25 \leq T \leq 30 \mid T < 30)$?

Solution

Plan: We want the probability that T is between 25 and 30, given $T < 30$. We'll standardize and use the ratio from conditional probability.

$$P(25 \leq T \leq 30 \mid T < 30) = \frac{P(25 \leq T \leq 30)}{P(T < 30)}.$$

Given $T \sim N(30, 4^2)$:
1. $P(25 \leq T \leq 30)$:
Standardize $Z = \frac{T-30}{4}$. Then

$$\frac{25-30}{4} = -1.25, \quad \frac{30-30}{4} = 0.$$

Hence

$$P(25 \leq T \leq 30) = P(-1.25 \leq Z \leq 0).$$

From normal tables,

$$P(Z < 0) = 0.5, \quad P(Z < -1.25) \approx 0.1056.$$

So $P(-1.25 \leq Z \leq 0) = 0.5 - 0.1056 = 0.3944$.
2. $P(T < 30) = P(Z < 0) = 0.5$.
3. Conditional Probability:

$$\frac{0.3944}{0.5} = 0.7888.$$

So about 78.88%.
Interpretation: Once we know the commute didn't exceed 30 minutes, there's nearly a 79% chance it was in the 25–30 range.

Problem 3: Fair vs. Biased Die

We have two dice, chosen at random:

- Die 1 (fair): $P(\text{roll} = 6) = 1/6$,
- Die 2 (biased): $P(\text{roll} = 6) = 1/3$.

Each die is equally likely to be chosen. The result is a 6. What is the probability we used the biased die?

Solution

Let B= "biased die," 6= "roll is six."

$$P(B) = 0.5,\ P(\neg B) = 0.5,\ P(6 \mid B) = \tfrac{1}{3},\ P(6 \mid \neg B) = \tfrac{1}{6}.$$

$$P(6) = 0.5 \cdot \tfrac{1}{6} + 0.5 \cdot \tfrac{1}{3} = \tfrac{1}{12} + \tfrac{2}{12} = \tfrac{3}{12} = \tfrac{1}{4}.$$

By Bayes:

$$P(B \mid 6) = \frac{(\frac{1}{3}) \times 0.5}{\frac{1}{4}} = \frac{\frac{1}{6}}{\frac{1}{4}} = \tfrac{2}{3}.$$

So about 66.67%.
Interpretation: Seeing a 6 is more likely if the die is biased, so two-thirds is the final probability.

Problem 4: Automated Detection of a STOP Sign

An autonomous car tries to detect a **STOP** sign under two different conditions:

- **Daytime** (60% of the time): If a sign is truly STOP, the system correctly labels it 80% of the time; if not STOP, it incorrectly says STOP 3% of the time.
- **Nighttime** (40% of the time): If a sign is truly STOP, the system correctly labels it 70% of the time; if not STOP, it incorrectly says STOP 12%.

Also, 45% of signs encountered are actually STOP. If the car reports STOP, what is the probability the sign really is STOP?

Solution

Plan: We combine both (1) day vs. night, and (2) whether the sign is actually STOP or not, and (3) the reported outcome.
Notation:

$$S = \text{"sign is STOP"}, \quad \hat{S} = \text{"car reports STOP"},$$

$$D = \text{daytime}, \quad N = \text{nighttime}.$$

Given:

$$P(D) = 0.60,\ P(N) = 0.40, \quad P(S) = 0.45,\ P(\neg S) = 0.55.$$

$$P(\hat{S} \mid S, D) = 0.80, \quad P(\hat{S} \mid \neg S, D) = 0.03,$$

$$P(\hat{S} \mid S, N) = 0.70, \quad P(\hat{S} \mid \neg S, N) = 0.12.$$

1. Overall detection if sign is STOP:

$$P(\hat{S} \mid S) = 0.60 \times 0.80 + 0.40 \times 0.70 = 0.48 + 0.28 = 0.76.$$

2. False alarm if sign not STOP:

$$P(\hat{S} \mid \neg S) = 0.60 \times 0.03 + 0.40 \times 0.12 = 0.018 + 0.048 = 0.066.$$

3. Probability that the car reports STOP:

$$P(\hat{S}) = P(\hat{S} \mid S)\,P(S) + P(\hat{S} \mid \neg S)\,P(\neg S) = 0.76 \times 0.45 + 0.066 \times 0.55 = 0.342 + 0.0363 = 0.3783.$$

4. Bayes: Probability it's truly STOP if car says STOP.

$$P(S \mid \hat{S}) = \frac{(0.76) \times (0.45)}{0.3783} = \frac{0.342}{0.3783} \approx 0.904.$$

So about 90.4%.
Interpretation: Good system accuracy plus a fairly high chance the sign is STOP to begin with yields a strong final probability of 90%.

Problem 5: Flight Delay vs. Weather

A flight is delayed 10% on sunny days, 30% on stormy days. Storm chance is 20%. If flight is delayed, what is the probability it was stormy?

Solution

Notation: D= "delayed," S= "stormy day."

$$P(S) = 0.20,\ P(\neg S) = 0.80,\ P(D \mid S) = 0.30,\ P(D \mid \neg S) = 0.10.$$

$$P(D) = 0.20 \times 0.30 + 0.80 \times 0.10 = 0.06 + 0.08 = 0.14.$$

Then

$$P(S \mid D) = \frac{P(D \mid S)\, P(S)}{P(D)} = \frac{(0.30)(0.20)}{0.14} = \frac{0.06}{0.14} \approx 0.4286.$$

So about 42.86%.
Interpretation: Storms cause more delays, but happen only 20% of the time. That yields a final probability of about 43% that a delay is due to stormy weather.

Problem 6: Two Boxes, Black Ball

Box1: 6W,4B. Box2: 3W,7B. Each chosen 50–50. A ball is drawn black. Probability it's from Box2?

Solution

Setup: Let B= "black draw," 2= "Box2."

$$P(B \mid 1) = 0.4,\ P(B \mid 2) = 0.7,\ P(1) = P(2) = 0.5.$$

Hence

$$P(B) = 0.5 \cdot 0.4 + 0.5 \cdot 0.7 = 0.2 + 0.35 = 0.55.$$

By Bayes:

$$P(2 \mid B) = \frac{(0.7) \times 0.5}{0.55} = \frac{0.35}{0.55} \approx 0.6364.$$

Problem 7: 3% Prevalence, 10% False Positive

Disease prevalence is 3%. A test has 95% true positive and 10% false positive rates. If a person tests positive, what is the probability they're not diseased?

Solution

Notation: D= diseased, $+$= test positive.

$$P(D) = 0.03, \quad P(\neg D) = 0.97, \quad P(+ \mid D) = 0.95, \quad P(+ \mid \neg D) = 0.10.$$

Compute:

$$P(+) = 0.03 \cdot 0.95 + 0.97 \cdot 0.10 = 0.0285 + 0.097 = 0.1255.$$

We want $P(\neg D \mid +)$:

$$\frac{(0.10) \times (0.97)}{0.1255} \approx 0.7733.$$

So 77.33%.

Interpretation: A test is decently accurate but the disease is rare, so many positives are false. Over three-fourths of positives are healthy.

Problem 8: Checkout Error Rates

A store has three checkout counters:

- **Counter A**: 45% of transactions, 2% error rate
- **Counter B**: 30% of transactions, 3% error rate
- **Counter C**: 25% of transactions, 4% error rate

A transaction is chosen at random and found to have an error. What is the probability it came from **Counter C**?

Solution

Plan: We'll use the law of total probability for the error and then Bayes' Theorem.
Define events:

$$E = \text{"transaction has an error"}, \quad C = \text{"from Counter C"}.$$

We also have A = "from A," B = "from B." The data:

$$P(A) = 0.45,\ P(B) = 0.30,\ P(C) = 0.25$$
$$P(E \mid A) = 0.02,\ P(E \mid B) = 0.03,\ P(E \mid C) = 0.04.$$

Step 1: Probability of an error (law of total probability).

$$P(E) = 0.45 \times 0.02 + 0.30 \times 0.03 + 0.25 \times 0.04 = 0.009 + 0.009 + 0.01 = 0.028.$$

Hence there's a 2.8% chance any random transaction is erroneous.
Step 2: Probability that it came from C if we know it's erroneous (Bayes').

$$P(C \mid E) = \frac{P(E \mid C)\,P(C)}{P(E)} = \frac{(0.04) \times (0.25)}{0.028} = \frac{0.01}{0.028} \approx 0.3571.$$

So about 35.71%.

Problem 9: Disease Testing (1% Prevalence)

A disease has 1% prevalence. The test has 98% true positive rate and 3% false positive rate. A random person tests positive. Probability that person actually has the disease?

Solution

Notation: D= disease, $+$= positive test.

$$P(D) = 0.01, \quad P(\neg D) = 0.99, \quad P(+ \mid D) = 0.98, \quad P(+ \mid \neg D) = 0.03.$$
$$P(+) = 0.01 \times 0.98 + 0.99 \times 0.03 = 0.0098 + 0.0297 = 0.0395.$$

Hence

$$P(D \mid +) = \frac{(0.98)(0.01)}{0.0395} \approx 0.2481.$$

So about 24.81%.
Interpretation: Even with a good test, the disease is so rare that many positives are false. The final chance you truly have it is only about 25%.

Problem 10: Three Urns, Red Draw

Urn A: 2 red, 8 blue. Urn B: 5 red, 5 blue. Urn C: 8 red, 2 blue. We pick an urn at random (1/3 each). A single ball is drawn and it is red. Find $P(\text{Urn B} \mid \text{Red})$.

Solution

Notation: $R=$ "ball is red," $B=$ "Urn B."

$$P(R \mid A) = 0.2, \quad P(R \mid B) = 0.5, \quad P(R \mid C) = 0.8.$$

All urns equally likely: $P(A) = P(B) = P(C) = \frac{1}{3}$.
Total $P(R)$:

$$P(R) = \tfrac{1}{3} \cdot 0.2 + \tfrac{1}{3} \cdot 0.5 + \tfrac{1}{3} \cdot 0.8 = 0.5.$$

Bayes: Probability Urn B given red ball.

$$P(B \mid R) = \frac{P(R \mid B)\, P(B)}{P(R)} = \frac{(0.5) \times (\frac{1}{3})}{0.5} = 0.3333.$$

So 33.33%.

3 Summary

In these problems, we see how **Bayes' Theorem** and the **law of total probability** let us combine partial information—like an observed test result, an outcome (red ball or black ball), or a time condition (under 30 minutes)—to figure out the updated probability of some underlying cause or scenario.

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