Practice Set

Vector Spaces, Subspaces, Independence, and Basis

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1 Vector Spaces and Subspaces

Question 1 (Easy: Subspace Concept)

Let $V = \mathbb{R}^2$. Consider the subset $U = \{(x, y) \in \mathbb{R}^2 \mid y = 2x + 1\}$. Is U a subspace of V? Justify your answer using the subspace criteria.

Detailed Solution:
To be a subspace, U must satisfy three conditions:

1. Contains the zero vector $\mathbf{0}$.
2. Closed under vector addition.
3. Closed under scalar multiplication.

Step 1: Check for the zero vector. The zero vector in $\mathbb{R}^2$ is $(0, 0)$. Does $(0, 0)$ satisfy the condition $y = 2x + 1$? Substitute $x = 0, y = 0$:

$$0 = 2(0) + 1 \implies 0 = 1 \quad \text{(False)}$$

Since $\mathbf{0} \notin U$, U is **not** a subspace.

(Note: This represents a line that does not pass through the origin. As mentioned in the lecture, shifted lines are not subspaces).

Question 2 (Moderate: Matrix Subspace)

Let $V = M_{2\times 2}$ be the vector space of all 2×2 matrices. Let W be the set of all symmetric 2×2 matrices (where $A^T = A$). Is W a subspace of V?

Detailed Solution:
The generic form of a symmetric 2×2 matrix is $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$.

1. Zero Vector: The zero matrix $\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is symmetric $(0 = 0)$. So $\mathbf{0} \in W$.

2. Closure under Addition: Let $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ and $B = \begin{pmatrix} d & e \\ e & f \end{pmatrix}$ be in W.

$$A + B = \begin{pmatrix} a+d & b+e \\ b+e & c+f \end{pmatrix}$$

The off-diagonal elements are equal $(b + e = b + e)$. Thus, $A + B$ is symmetric.

3. Closure under Scalar Multiplication: Let $k \in \mathbb{R}$.

$$kA = \begin{pmatrix} ka & kb \\ kb & kc \end{pmatrix}$$

The off-diagonal elements are equal $(kb = kb)$. Thus, kA is symmetric.

Conclusion: Since all conditions are met, W is a subspace of $M_{2\times 2}$.

2 Linear Combinations and Span

Question 3 (Easy: Checking Linear Combination)

Determine if the vector $b = \begin{pmatrix} 4 \\ 3 \\ 12 \end{pmatrix}$ is a linear combination of vectors $v_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$.

Detailed Solution:
We want to find scalars c_1, c_2 such that $c_1 v_1 + c_2 v_2 = b$. Augmented Matrix:

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 2 & 4 & 12 \end{pmatrix}$$

Row Reduction: $R_3 \to R_3 - 2R_1$:

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 4 \end{pmatrix}$$

Look at the last row: $0c_1 + 0c_2 = 4 \implies 0 = 4$. This is a contradiction. The system is inconsistent.

Answer: No, b is not a linear combination of v_1 and v_2.

Question 4 (Moderate: Determining Span Parameter)

For what value of k is the vector $v = \begin{pmatrix} 1 \\ -2 \\ k \end{pmatrix}$ in the span of $\left\{ \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} \right\}$?

Detailed Solution:
We set up the system $c_1 v_1 + c_2 v_2 = v$:

$$\begin{pmatrix} 3 & 2 & 1 \\ 0 & -1 & -2 \\ 2 & 5 & k \end{pmatrix}$$

Step 1: Eliminate R_3 using R_1. Operation: $R_3 \to 3R_3 - 2R_1$.

$$\begin{pmatrix} 3 & 2 & 1 \\ 0 & -1 & -2 \\ 0 & 15-4 & 3k-2 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ 0 & -1 & -2 \\ 0 & 11 & 3k-2 \end{pmatrix}$$

Step 2: Eliminate R_3 using R_2. Operation: $R_3 \to R_3 + 11R_2$.

$$\begin{pmatrix} 3 & 2 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & (3k-2)+11(-2) \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 3k-24 \end{pmatrix}$$

Step 3: Condition for Consistency. For the system to have a solution, the last row must not be a contradiction (like $0 = 5$). Since the coefficient of variables is 0, the constant term must also be 0.

$$3k - 24 = 0 \implies 3k = 24 \implies k = 8$$

Answer: $k = 8$.

3 Linear Independence

Question 5 (Easy: Inspection)

Determine by inspection (without calculation) whether the set of vectors is linearly independent:

$$S = \left\{ \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix} \right\}$$

Detailed Solution:
The set S contains the zero vector $\mathbf{0} = (0,0,0)^T$. According to the lecture (Slide 431), **any set containing the zero vector is linearly dependent**. (Reason: $1 \cdot \mathbf{0} + 0 \cdot v_1 + 0 \cdot v_3 = \mathbf{0}$ is a non-trivial linear combination).

Answer: Linearly Dependent.

Question 6 (Moderate: Independence in $\mathbb{R}^3$)

Determine if the following vectors are linearly independent. If dependent, find the dependency relation.

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, v_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, v_3 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

Detailed Solution:
We solve $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$. Augmented matrix:

$$\begin{pmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{pmatrix}$$

Gaussian Elimination: $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$:

$$\begin{pmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -6 & 0 \end{pmatrix}$$

$R_3 \to R_3 - 2R_2$:

$$\begin{pmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We have a row of zeros, implying a free variable (c_3). Equation from R_2: $-3c_2 - 3c_3 = 0 \implies c_2 = -c_3$. Equation from R_1: $c_1 + 4c_2 + 2c_3 = 0 \implies c_1 + 4(-c_3) + 2c_3 = 0 \implies c_1 = 2c_3$.

Let $c_3 = 1$, then $c_2 = -1, c_1 = 2$. Check: $2v_1 - 1v_2 + 1v_3 = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Answer: Vectors are Linearly Dependent. Relation: $2v_1 - v_2 + v_3 = 0$.

Question 7 (Tough: Matrix Space Independence)

Are the matrices $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $C = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ linearly independent in $M_{2\times 2}$?

Detailed Solution:
We check $c_1A + c_2B + c_3C = \mathbf{0}$.

$$c_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

This yields 4 equations (one per matrix position): 1. $c_1 + 0 + c_3 = 0 \implies c_1 + c_3 = 0$ 2. $0 + c_2 + c_3 = 0 \implies c_2 + c_3 = 0$ 3. $0 + c_2 + c_3 = 0$ (Same as above) 4. $0 + 0 + 0 = 0$ (Trivial)

From (1), $c_1 = -c_3$. From (2), $c_2 = -c_3$. This system has infinitely many solutions (one free variable c_3). If we pick $c_3 = 1$, then $c_1 = -1, c_2 = -1$. Relation: $-A - B + C = 0 \implies C = A + B$.

Answer: Linearly Dependent.

4 Basis and Dimension

Question 8 (Easy: Dimension Check)

Find the dimension of the subspace of $\mathbb{R}^3$ spanned by the vectors:

$$v_1 = (1, 0, 0), \quad v_2 = (0, 1, 0), \quad v_3 = (1, 1, 0)$$

Detailed Solution:
We need to find the number of linearly independent vectors in this set. We observe that $v_3 = v_1 + v_2$. Thus, v_3 is redundant. v_1 and v_2 are clearly independent (one has x component only, the other y). The basis is $\{v_1, v_2\}$. The dimension is the count of basis vectors.

Answer: Dimension = 2.

Question 9 (Moderate: Finding a Basis)

Find a basis for the subspace of $\mathbb{R}^3$ spanned by $S = \{(1, 1, 2), (2, 2, 4), (2, -1, 5)\}$.

Detailed Solution:
Step 1: Check for independence. Notice that $v_2 = 2 \cdot v_1$. $(2, 2, 4) = 2(1, 1, 2)$. So, v_2 is dependent on v_1 and can be removed without changing the span.

Step 2: Check remaining vectors. We are left with $v_1 = (1, 1, 2)$ and $v_3 = (2, -1, 5)$. Are they multiples of each other? $2/1 = 2$, but $-1/1 \neq 2$. No. They are linearly independent.

Step 3: Form Basis. The basis consists of the linearly independent vectors remaining in the set.

Answer: Basis = $\{(1, 1, 2), (2, -1, 5)\}$.

Question 10 (Tough: Extending to a Basis)

The vector space $V = M_{2\times 2}$ has dimension 4. The set $S = \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$ is a linearly independent subset of V. Add two matrices to S to form a basis for V.

Detailed Solution:
The standard basis for $M_{2\times 2}$ is:

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The set S already contains E_{11} and E_{22}. To span the entire space $M_{2\times 2}$, we need to be able to generate entries in the top-right (a_{12}) and bottom-left (a_{21}) positions. Currently, any linear combination of S looks like $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$. We simply add the missing standard basis vectors.

Answer: Add $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.