Birla Institute of Technology & Science, Pilani
Work Integrated Learning Programmes Division
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Mid-Semester Test
EC-2 Regular

Course No.	: AIMLCZC416
Course Title	: Mathematical Foundations for Machine Learning
Nature of Exam	: Closed Book
Weightage	: 30%
Duration	: 2 Hours
Date of Exam	: 28-06-2025, AN

No. of Pages = 2
No. of Questions = 6

Note to Students:
1. Please follow all the *Instructions to Candidates* given on the cover page of the answer book.
2. All parts of a question should be answered consecutively. Each answer should start from a fresh page.
3. Assumptions made if any, should be stated clearly at the beginning of your answer.
4. Notation used here are as per the text book.

Q1. Consider the matrices $A = \begin{bmatrix} 1 & 3 & -1 & 4 \\ 2 & 6 & -2 & 8 \\ 3 & 9 & -3 & 12 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(a) Find the Echelon form of A and hence find the rank of A. [2M]

$$\begin{bmatrix} 1 & 3 & -1 & 4 \\ 2 & 6 & -2 & 8 \\ 3 & 9 & -3 & 12 \end{bmatrix} \approx \begin{bmatrix} 1 & 3 & -1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

R2 ⟵ -2R1 + R2

R3 ⟵ -3R1 + R3

[1M]

Since in the row echelon form there are only one non zero row, rank of A = 1. [1M]

(b) Find all the solutions of AX = b. [2M]

The augmented matrix = $\begin{bmatrix} 1 & 3 & -1 & 4 & 1 \\ 2 & 6 & -2 & 8 & 2 \\ 3 & 9 & -3 & 12 & 3 \end{bmatrix} \approx \begin{bmatrix} 1 & 3 & -1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ [1M]

Converting back to equation we have

$x_1 + 3x_2 - x_3 + 4x_4$=1. Since x_2, x_3, x_4 are free variables, the solutions are

$$\left\{ \begin{bmatrix} 1 - 3r + s - 4t \\ r \\ s \\ t \end{bmatrix} : r, s, t \in R \right\}.$$ [1M]

(c) Suppose if the matrix $b = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ what can you conclude about the linear system of equations AX = b, Where A is the same matrix given in (a). [1M]

The Echelon form of the augmented matrix = $\begin{bmatrix} 1 & 3 & -1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$

Since the last row implies 0 = -1, the system has no solution. [1M]

Q2. Consider the matrix $A = \begin{bmatrix} m & 1 & 1 \\ 1 & m & 1 \\ 1 & 1 & m \end{bmatrix}$, where m ≥ 2, a positive integer.

(a) Prove that A is a positive definite matrix. [3M]

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be any non-zero vector in R^3.

Then $x^T Ax = [x_1\ x_2\ x_3]\left(\begin{bmatrix} m & 1 & 1 \\ 1 & m & 1 \\ 1 & 1 & m \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)[1M] = [x_1\ x_2\ x_3]\begin{bmatrix} mx_1 + x_2 + x_3 \\ x_1 + mx_2 + x_3 \\ x_1 + x_2 + mx_3 \end{bmatrix} =$

$$= m(x_1{}^2 + x_2{}^2 + x_3{}^2) + 2x_1x_2 + 2x_1x_3 + 2x_2x_3$$
$$= (x_1 + x_2)^2 + (x_1 + x_3)^2 + (x_2 + x_3)^2 + (m-2)(x_1{}^2 + x_2{}^2 + x_3{}^2)[1M]$$

Since at least one $x_i \neq 0$ and $m \geq 2,\ x^T Ax > 0.$ [1M]

(b) Using this matrix A, find an inner product on R^3. [1M]

Suppose $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ are any two vectors in R^3.

$$< x, y >_A = x^T Ay = [x_1\ x_2\ x_3]\left(\begin{bmatrix} m & 1 & 1 \\ 1 & m & 1 \\ 1 & 1 & m \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right)$$
$$= [x_1\ x_2\ x_3]\begin{bmatrix} my_1 + y_2 + y_3 \\ y_1 + my_2 + y_3 \\ y_1 + y_2 + my_3 \end{bmatrix}$$
$$= m(x_1y_1 + x_2y_2 + x_3y_3) + x_1y_2 + x_1y_3 + x_2y_1 + x_2y_3 + x_3y_1 + x_3y_2$$

(c) Using the inner product defined in (b) find the norm of the vector $\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$. [1M]

$$\left\|\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}\right\| = [1 -1 -2]\left(\begin{bmatrix} m & 1 & 1 \\ 1 & m & 1 \\ 1 & 1 & m \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}\right) = [1 -1 -2]\begin{bmatrix} m-3 \\ -m-1 \\ -2m \end{bmatrix}$$
$$= 6m - 2 \quad [1M]$$

Q3. (a) Is the set $\left\{\begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix}\right\}$ a basis for R^4. [3M]

Forming a matrix with columns as given vectors: $\begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix}$

$$\approx \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix} R1 \leftrightarrow R2 \approx \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix} R3 \leftarrow -2R1 + R3$$

$$\approx \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & -6 & 4 \end{bmatrix} R4 \leftarrow -3R2 + R4 \approx \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} R4 \leftarrow -3R3 + R4\ [1M]$$

Since each column in the echelon form has pivot element the system of equations

$$a\begin{bmatrix}0\\1\\2\\0\end{bmatrix}+b\begin{bmatrix}1\\0\\0\\3\end{bmatrix}+c\begin{bmatrix}2\\1\\0\\0\end{bmatrix}+d\begin{bmatrix}0\\0\\0\\4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$ has only a trivial solution. Hence the given vectors are linearly independent. [1M]

Since the dimension of R^4 is 4 and the given set consists of 4 elements which are linearly independent, the given set is a basis of R^4. [1M]

(b) Write $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ as a linear combination of the elements given in (a). [2M]

The augmented matrix $=\begin{bmatrix}0&1&2&0&1\\1&0&1&0&0\\2&0&0&0&0\\0&3&0&4&0\end{bmatrix} \approx \begin{bmatrix}1&0&1&0&0\\0&1&2&0&1\\2&0&0&0&0\\0&3&0&4&0\end{bmatrix} R1 \leftrightarrow R2$

$\approx \begin{bmatrix}1&0&1&0&0\\0&1&2&0&1\\0&0&-2&0&0\\0&3&0&4&0\end{bmatrix} R3 \leftarrow -2R1+R3$

$\approx \begin{bmatrix}1&0&1&0&0\\0&1&2&0&1\\0&0&-2&0&0\\0&0&-6&4&-3\end{bmatrix} R4 \leftarrow -3R2+R4 \approx \begin{bmatrix}1&0&1&0&0\\0&1&2&0&1\\0&0&-2&0&0\\0&0&0&4&-3\end{bmatrix} R4 \leftarrow -3R3+R4$

[1M]

Converting back into equations we get

a + c = 0

b – 2c = 1

-2c = 0

4 d = -3 implies d = -3/4

c = 0 implies b =1 and a = 0.

Therefore $\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = 0\begin{bmatrix}0\\1\\2\\0\end{bmatrix}+1\begin{bmatrix}1\\0\\0\\3\end{bmatrix}+0\begin{bmatrix}2\\1\\0\\0\end{bmatrix}+(-\frac{3}{4})\begin{bmatrix}0\\0\\0\\4\end{bmatrix}$ [1M]

Note: If they guess a, b, c, and d without going through the steps, give full 2 marks.

Q4. (a) Suppose $A=\begin{bmatrix}-1&0\\0&-1\\2&-2\end{bmatrix}$. Find the diagonal matrix D and the invertible matrix P

Such that $(A^TA)P = PD$ [4M]

$A^TA=\begin{bmatrix}-1&0&2\\0&-1&-2\end{bmatrix}\begin{bmatrix}-1&0\\0&-1\\2&-2\end{bmatrix}=\begin{bmatrix}5&-4\\-4&5\end{bmatrix}$

The characteristic equation is $(5-\lambda)^2 = 16$

The eigen values are 1, 9. [1M]

The eigen vector corresponding to 1 is obtained by solving the equations

$\begin{bmatrix}4&-4\\-4&4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$. ie $4x-4y=0$. One solution is $\begin{bmatrix}1\\1\end{bmatrix}$. [1/2M]

The eigen vector corresponding to 9 is obtained by solving the equations

$\begin{bmatrix}-4&-4\\-4&-4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$. ie $-4x-4y=0$. One solution is $\begin{bmatrix}1\\-1\end{bmatrix}$. [1/2M]

Hence the matrix $D=\begin{bmatrix}1&0\\0&9\end{bmatrix}$ and the matrix $P=\begin{bmatrix}1&1\\1&-1\end{bmatrix}$.

(b) Why for any mxn matrix A, A^TA is always diagonalizable [1M].

$$(A^TA)^T = A^T(A^T)^T = A^TA.$$

Hence A^TA is symmetric and we know that every symmetric matrix is diagonalizable. [1M]

Q5. Consider the function $f(x,y,z) = x^2 + y^2 + z^2 + 3xyz$.

(a) Find $\nabla\mathbf{f}(x,y,z)$ at (1, -1, 2). [2M]

$$\nabla f(x,y,z) = \begin{bmatrix} 2x+3yz \\ 2y+3xz \\ 2z+3xy \end{bmatrix} \text{ [1M]}$$

$$\nabla f(1,-1,2) = \begin{bmatrix} -4 \\ 4 \\ 1 \end{bmatrix} \text{ [1M]}$$

(b) Is the vector $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ orthogonal to $\nabla\mathbf{f}(x,y,z)$ at (1, -1, 2). Why? [1M]

The inner product $< \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 4 \\ 1 \end{bmatrix} > = -4-4 = -8 \neq 0.$

Hence these vectors are not orthogonal. [1M]

(c) Suppose if you further consider x, y, z as functions of t given by $x(t) = t$, $y(t) = t^2$, $z(t) = t^3$. Using the chain rule find $\frac{df}{dt}$. [2M]

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \text{ [1M]}$$

$= (2x+3yz)1 + (2y+3xz)(2t) + (2z+3xy)3\,t^2$

$= (2t + 3t^5) + (2t^2 + 3t^4)(2t) + (2t^3 + 3t^3)\, 3t^2$

$= 24t^5 + 4t^3 + 2t$ [1M]

Q6. (a) Find the Taylor polynomial of degree 3 of the function $f(x) = e^{x^2}$ about the point 1. [3M]

$f(x) = e^{x^2}$

$f^{(1)}(x) = 2xe^{x^2}$

$f^{(2)}(x) = 2x2xe^{x^2} + 2e^{x^2} = 4x^2e^{x^2} + 2e^{x^2}$

$f^{(3)}(x) = 8xe^{x^2} + 4x^22xe^{x^2} + 4xe^{x^2} = 8x^3e^{x^2} + 12xe^{x^2}$ [1M]

$f(1) = e$

$f^{(1)}(1) = 2e$

$f^{(2)}(1) = 6e$

$f^{(3)}(1) = 20e$ [1M]

Taylor polynomial of degree 3 of the function $f(x) = e^{x^2}$ about the point 1 is

$$T_3(x) = e + 2e(x-1) + \frac{6e}{2}(x-1)^2 + \frac{20e}{6}(x-1)^3$$

$$= e + 2e(x-1) + 3e(x-1)^2 + \frac{10e}{3}(x-1)^3 \text{ [1M]}$$

(b) Find the Hessian matrix of the function $f(x, y) = xy$. [2M]

Hessian matrix of the function $f(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}$ [1/2 M]

$f_x = y,\ f_y = x.$ [1/2 M]

$f_{xx} = 0$ $f_{yy} = 0$ and $f_{xy} = 1$

Hessian matrix of the function $f(x,y) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ [1M]
