Maths and Statistics for AI and Data Science

Practical Assessment – 1

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Question 1

1. a) Function $f(x) = 2x + 3$, $g(x) = x^3$. Find $f \circ g$ and inverse of $f \circ g$.

   b) Find the derivative of the functions $f(x) = \sin x \ln \cos 2x + 6e^x$

   c) Find the critical points for function ,$f(x) = -x^3 + 6x^2 + 15x + 4, x \in \mathbf{R}$, and state if the critical points are minima or maxima

   Provide appropriate justification and explanation to all your answers, detailing the methods used.

Solution:

a) Find $f \circ g$ and inverse of $f \circ g$

a.1) Find $f \circ g$

The $f \circ g$ denotes a composite function (Hass, 2019), meaning a combination of functions $f(x)$ and $g(x)$. To evaluate a composite function, below definition is applied:

$$f \circ g(x) = f(g(x)) \tag{A}$$

We have,

$$f(x) = 2x + 3 \tag{1}$$

$$g(x) = x^3 \tag{2}$$

Applying the definition (A) to equations (1) and (2) , we get:

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ f(x^3) &= 2x^3 + 3\end{aligned} \tag{3}$$

a.2) Inverse of $f \circ g$ i.e. $[(f \circ g)(x)]^{-1}$

The inverse of a function f^{-1} , if $f(a) = b$ is defined by (Hass, 2019),

$$f^{-1}(b) = a \tag{B}$$

Using equation (3) and solving for x in terms of y

$$\begin{aligned}f(x^3) &= 2x^3 + 3 \\ y &= 2x^3 + 3\end{aligned}$$

Interchanging y and x to solve for y,

$$\begin{aligned} x &= 2y^3 + 3 \\ 2y^3 &= x - 3 \\ y^3 &= \frac{x}{2} - \frac{3}{2} \\ y &= \sqrt[3]{\frac{x-3}{2}} \end{aligned}$$

Rationalizing the denominator by multiplying the numerator and the denominator by $\sqrt[3]{2^2}$, we get,

$$\begin{aligned} y &= \frac{\sqrt[3]{x-3}}{\sqrt[3]{2}} \\ y &= \frac{\sqrt[3]{x-3}}{\sqrt[3]{2}} * \frac{\sqrt[3]{2^2}}{\sqrt[3]{2^2}} = \frac{\sqrt[3]{x-3}}{\sqrt[3]{2}} * \frac{\sqrt[3]{4}}{\sqrt[3]{4}} \\ y &= \frac{\sqrt[3]{4(x-3)}}{2} \end{aligned}$$

Replacing y with f^{-1} as per definition (B),

$$f^{-1}(x) = \frac{\sqrt[3]{4(x-3)}}{2} \tag{4}$$

Therefore from equations (3) and (4), we have,

$$\boxed{(f \circ g)(x) = 2x^3 + 3}$$

$$\boxed{[(f \circ g)(x)]^{-1} = \frac{\sqrt[3]{4(x-3)}}{2}}$$

b) Find the derivative of the given function

We have the below function:

$$f(x) = \sin x \ln \cos 2x + 6e^x$$

The sum rule of derivatives is given by,

$$\frac{dx}{dy}(u+v) = \frac{du}{dx} + \frac{dv}{dx} \tag{C}$$

Applying this rule to the given function as below,

$$\begin{aligned} f'(x) &= \frac{d}{dx}[\sin x \ln \cos 2x + 6e^x] \\ f'(x) &= \frac{d}{dx} \sin x \ln \cos 2x + \frac{d}{dx} 6e^x \end{aligned}$$

For ease of computing, splitting the terms on right hand side as:

$$f'(x) = f'(l) + f'(r)$$

where,

$$f'(l) = \frac{d}{dx} \sin x \ln \cos 2x \tag{5}$$

$$f'(r) = \frac{d}{dx} 6e^x \tag{6}$$

b.1) Let's solve $f'(l)$ first:

The product rule of derivatives is given by,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \tag{D}$$

Applying this rule to $f'(l)$ from equation (5), we get,

$$f'(l) = \sin x \frac{d}{dx} \ln \cos 2x + \ln \cos 2x \frac{d}{dx} \sin x$$

The chain rule, in Leibniz's notation (Hass, 2019), if $y = f(u)$ and $u = g(x)$ is given by,

$$\frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dx} \tag{E}$$

This rule is used to simplify the composite function $\ln \cos 2x$ as below,

$$f'(l) = \sin x [\frac{1}{\cos 2x} \frac{d}{dx} \cos 2x] + \ln \cos 2x * \cos x$$

The cosine derivative rule and trignometric rule is given by,

$$\frac{d}{dx}(\cos x) = -\sin x \tag{F}$$
$$\tan x = \frac{\sin x}{\cos x}$$

Applying both of these rules to $f'(l)$ as below,

$$\begin{aligned} f'(l) &= \sin x [\frac{1}{\cos 2x} - \sin 2x * 2] + \cos x \ln \cos 2x \\ &= -2 \sin x [\frac{\sin 2x}{\cos 2x}] + \cos x \ln \cos 2x \end{aligned}$$

$$f'(l) = \cos x \ln \cos 2x - 2 \sin x \tan 2x \tag{7}$$

b.2) Now, solving $f'(r)$:

Referring to equation (6),

$$f'(r) = \frac{d}{dx}6e^x$$

The derivative constant rule is given by,

$$\frac{d}{dx}cu = c\frac{du}{dx} \tag{G}$$

Applying this rule to $f'(r)$, we get

$$f'(r) = 6\frac{d}{dx}e^x$$

The derivative rule of an exponential function is given by,

$$\frac{d}{dx}e^u = e^u\frac{du}{dx} \tag{H}$$

Applying this rule to $f'(r)$, we get

$$f'(r) = 6e^x\frac{d}{dx}x$$

Applying the power rule, i.e.

$$\frac{d}{dx}x^n = x^{n-1} \tag{I}$$

The $f'(r)$ becomes,

$$\begin{aligned} f'(r) &= 6e^x \times x^{1-1} \\ &= 6e^x \times x^0 \\ &= 6e^x \end{aligned}$$

$$f'(r) = 6e^x \tag{8}$$

Combining equations (7) and (8), we get the final derivative as below:

$$f'(x) = f'(l) + f'(r)$$

$$\boxed{f'(x) = \cos x \ln \cos 2x - 2\sin x \tan 2x + 6e^x}$$

c) Find the critical points for function $f(x) = -x^3 + 6x^2 + 15x + 4, x \in R$ and state if the critical points are minima or maxima.

As per definition (Hass, 2019), "An interior point of the domain of a function f where f' is zero or undefined is a critical point of f." Critical points are the points on the graph where the function's concativity is amended i.e. changed from increasing to decreasing concativity or vice a versa.

Following this, the steps we incorporate to determine the critical points of the given function $f(x)$ are:

1. Determine the derivative $f'(x)$ of the given function
2. Equate $f'(x)$ to 0 to calculate critical points

Step 1: Find $f'(x)$

We have,

$$f(x) = -x^3 + 6x^2 + 15x + 4$$

$$f'(x) = \frac{d}{dx}[-x^3 + 6x^2 + 15x + 4]$$

Applying sum rule as given by (C) and derivative of constant function from (G), we get,

$$f'(x) = \frac{d}{dx}(-x^3) + 6\frac{d}{dx}(x^2) + 15\frac{d}{dx}x + \frac{d}{dx}(4)$$

Further, applying power rule of derivatives from (I),

$$f'(x) = 3 \times (-x)^{3-1} + 6 \times 2(x^{2-1}) + 15 \times x^{1-1} + 0$$
$$f'(x) = -3x^2 + 12x + 15$$

Step 2: Find critical points

As per the definition, to find critical points, we equate the derivative to 0

$$f'(x) = 0$$

$$-3x^2 + 12x + 15 = 0$$

Dividing both sides by 3, we get,

$$-x^2 + 4x + 5 = 0$$
$$(-1)(x^2 - 5x + x - 5 = 0$$
$$(-1)(x - 5)(x + 1) = 0$$

Solving for x, by setting

$$x - 5 = 0, \quad x = 5$$
$$x + 1 = 0, \quad x = -1$$

Evaluating function $f(x)$ at each of these values of x:

i) $x = 5$

$$\begin{aligned} f(5) &= -5^3 + 6 \times 5^2 + 15 \times 5 + 4 \\ &= -125 + 150 + 75 + 4 \\ &= 104 \end{aligned}$$

ii) $x = -1$

$$\begin{aligned} f(-1) &= -(-1^3) + 6 \times (-1^2) + 15 \times -1 + 4 \\ &= 1 + 6 - 15 + 4 \\ &= -4 \end{aligned}$$

The critical points are:

$(5, 104)$ and $(-1, -4)$

Question 2

2. Function $f(x, y) = x^2y^3 + y^2 + 2x(y + 1)$ Suppose the initial values are given $x_0 = 1, y_0 = 0$, and the learning rate is set to 0.01

a) Perform one iteration of the gradient descent algorithm

b) Find the Hessian matrix of $f(x, y)$

Provide appropriate justification and explanation to all your answers, detailing the methods used.

Solution:

a) Gradient Descent algorithm Iteration

The mathematical formula for calculating iteration step size in Gradient Descent algorithm is (UoL,2025):

$$\rho_{n+1} = \rho_n - \eta \nabla f(\rho_n) \tag{A}$$

The given function is,

$$f(x, y) = x^2y^3 + y^2 + 2x(y + 1)$$

Substituting the values $x_0 = 1$ and $y_0 = 0$ in the function, we get,

$$\begin{aligned} f(x_0, y_0) &= x_0^2 y_0^3 + y_0^2 + 2x_0(y_0 + 1) \\ &= (1^2) \times 0 + 0 + 2 \times (1)(0 + 1) \\ &= 2 \end{aligned}$$

Hence, the starting point of optimization is,

$$f(x_0, y_0) = 2 \tag{9}$$

As per formula (A), the first iteration can be calculated with below equation:

$$\rho_{x_1,y_1} = \rho_{x_0,y_0} - \eta \nabla f(\rho_{x_0,y_0}) \tag{10}$$

Partial derivatives of functions with two variables (dependent or independent) are normal derivatives, with respect to only one variable at a time (Hass, 2019).
Based on this, below calculation represents the derivatives of $f(x, y)$ w.r.t. x i.e. f_x and w.r.t. y i.e. f_y:

$$\nabla f = \left[\begin{array}{cc} \underbrace{\dfrac{\partial f}{\partial x}}_{f_x} & \underbrace{\dfrac{\partial f}{\partial y}}_{f_y} \end{array} \right] = \left[\begin{array}{cc} \frac{\partial}{\partial x}[x^2y^3 + y^2 + 2x(y + 1)] & \frac{\partial}{\partial y}[x^2y^3 + y^2 + 2x(y + 1)] \end{array} \right]$$

Applying constant and power derivative rules (as mentioned in (G) and (I) earlier) and calculating partial derivatives of $f(x, y)$ as below:

$$\nabla f = \Big[2xy^3 + 2(y + 1) \quad 3x^2y^2 + 2y + 2x\Big] \tag{11}$$

Substituting values from equations (9), (11) and learning rate η in equation (10), we get,

$$\begin{aligned} \rho_{x_1,y_1} &= \rho_{x_0,y_0} - \eta \nabla f(\rho_{x_0,y_0}) \\ &= (1, 0) - (0.01) \times [2 \cdot 1 \cdot 0 + 2(0 + 1), 3 \cdot (1^3) \cdot 0 + 2 \cdot 0 + 2 \cdot 1] \\ &= (1, 0) - 0.01(2, 2) \\ &= (1, 0) - (0.02, 0.02) \\ &= (0.98, -0.02) \end{aligned}$$

$$\rho_{x_1,y_1} = (0.98, -0.02) \tag{12}$$

Computing $f(0.98, -0.02)$ as below to get value of the function:

$$\begin{aligned} f(x_1, y_1) &= x_1^2 y_1^3 + y_1^2 + 2x_1(y_1 + 1) \\ &= (0.98^2)(-0.02^3) + (-0.02^2) + 2 \cdot (0.98)(-0.02 + 1) \\ &= 1.92 \end{aligned}$$

b) Hessian Matrix calculation

Hessian matrix is a matrix of second order partial derivativatives which is used to calculate the curvature information of a given function. (Sharma, 2022)

A Hessian matrix is given as below:

$$H_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 x_2} & \cdots & \frac{\partial^2 f}{\partial {x_n}^2} \\ \frac{\partial^2 f}{\partial x_2 x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 x_n} \\ . & . & . & . \\ . & . & . & . \\ \frac{\partial^2 f}{\partial x_n x_1} & \frac{\partial^2 f}{\partial x_n x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

Hessian matrix is always a square matric with its dimension equal to the number of variables of a function. Therefore, for the function $f(x, y)$ with two variables, the dimension will be 2×2 and can be written as:

$$H_{f(x,y)} = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial xy} \\ \frac{\partial^2 f}{\partial yx} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

Referring to equation (11), we have already obtained the First order partial derivatives of $f(x, y)$:

$$\frac{\partial f}{\partial x} = 2xy^3 + 2(y+1)$$
$$\frac{\partial f}{\partial y} = 3x^2y^2 + 2y + 2x$$

Second order partial derivatives of $f(x, y)$ are as below:

$$\frac{\partial^2 f}{\partial x^2} = 2y^3$$
$$\frac{\partial^2 f}{\partial y^2} = 6x^2y + 2$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = \frac{\partial}{\partial x}(3x^2y^2 + 2y + 2x) = 6xy^2 + 2$$
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = \frac{\partial}{\partial y}(2xy^3 + 2(y+1)) = 6xy^2 + 2$$

Therefore, the final Hessian matrix is as given below:

$$\boxed{H_{f(x,y)} = \begin{bmatrix} 2y^3 & 6xy^2 + 2 \\ 6xy^2 + 2 & 6x^2y + 2 \end{bmatrix}}$$

Question 3

3. Take this system of linear equations

$$x + y - z = -3$$
$$2x + 3y - 8z = -18$$
$$5x + 6y - 10z = -25)$$

a) Write this system as a Matrix vector equation

b) Calculate the determinant and thereby determine if there is a unique solution to this system of equations

c) Write this as an augmented matrix and solve this system of equations

Provide appropriate justification and explanation to all your answers, detailing the methods used.

Solution:

a) Matrix vector equation

A matrix equation has the form (Fred,2015):

$$Ax = b \tag{A}$$

where,

A is any $m \times m$ or $m \times n$ matrix called as coefficient matrix

x is a vector of unknown variables, $n \times 1$ column vector

b is $m \times 1$ column vector

Lipschutz says that a system of linear equations is a list of linear equations with same unknowns (2009,pp.58). In this problem statement we have 3 linear equations with 3 unknowns in a 3×3 system.

The vector addition and scaler multiplication properties of a vector are given by,

$$If\ a = \begin{bmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{bmatrix}, then\ a + b = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ . \\ . \\ a_n + b_n \end{bmatrix} \tag{13}$$

$$If\ 'k'\ is\ a\ scaler,\ a = \begin{bmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{bmatrix}, then\ ka = \begin{bmatrix} ka_1 \\ ka_2 \\ . \\ . \\ ka_n \end{bmatrix} \tag{14}$$

Referring to these properties, we can separate and extract 3 vectors from the given system of linear equations:

$$u : x \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, v : y \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}, w : x \begin{bmatrix} -1 \\ -8 \\ -10 \end{bmatrix}$$

We can further rewrite the system of equations in below form:

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -8 \\ 5 & 6 & -10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ -18 \\ -25 \end{bmatrix} \tag{15}$$

Comparing (15) to formula (A), we have the representation of the system of linear equations in the Matrix vector equation.

b) Calculate determinant

Determinant is an essential tool for gathering and inspecting properties of square matrices (Lipschutz,2009). In this problem statement, the given system of lienar equations has a unique solution if and only if its determianat i.e. $|A| \neq 0$.

$$\det A = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & -8 \\ 5 & 6 & -10 \end{vmatrix}$$

$$\begin{aligned} |A| &= (1)[(3) \times (-10) - (-8) \times 6] - (1)[(2) \times (-10) - (-8) \times (5)] + (-1)[2 \times 6 - 3 \times 5] \\ &= 18 - 20 + 3 \end{aligned}$$

$$\boxed{|A| = 1} \tag{16}$$

Since $|A| \neq 0$, this system has a unique solution.

c) Solving this system of linear equations

The linear equations can be presented as an Augmented matrix as below:

$$A = \left[\begin{array}{ccc|c} 1 & 1 & -1 & -3 \\ 2 & 3 & -8 & -18 \\ 5 & 6 & -10 & -25 \end{array}\right]$$

Using Gaussian elimination, we find the Row Echelon Form of the matrix as below:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -3 \\ 2 & 3 & -8 & -18 \\ 5 & 6 & -10 & -25 \end{array}\right] \to R_2 = R_2 - 2R_1 \to \left[\begin{array}{ccc|c} 1 & 1 & -1 & -3 \\ 1 & 1 & -6 & -12 \\ 5 & 6 & -10 & -25 \end{array}\right] \to R_3 = R_3 - 5R_1 \to \left[\begin{array}{ccc|c} 1 & 1 & -1 & -3 \\ 1 & 1 & -6 & -12 \\ 0 & 1 & -5 & -10 \end{array}\right] \to$$

$$R_3 = R_3 - R_2 \to \left[\begin{array}{ccc|c} 1 & 1 & -1 & -3 \\ 1 & 1 & -6 & -12 \\ 0 & 0 & 1 & 2 \end{array}\right] \to R_1 = R_1 - R_2 \to \left[\begin{array}{ccc|c} 1 & 0 & 5 & 9 \\ 0 & 1 & -6 & -12 \\ 0 & 0 & 1 & 2 \end{array}\right] \to R_2 = R_2 + 6R_3 \to$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 9 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right] \to R_1 = R_1 - 5R_3 \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

From the reduced Echelon form matrix, we see that

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}}$$

In other words, $x = -1, y = 0, z = 2$

Question 4

For a linear transformation T: $\mathbf{R}^3 -> \mathbf{R}^4$.

$$T = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 + x_3 \\ x_1 - 3x_2 \\ x_1 + x_2 + x_3 \\ 3x_1 + 2x_2 + x_3 \end{bmatrix}$$

a) Determine the transformation matrix A

b) Determine rank(A)

c) Find the kernel and image of transformation T, and their dimension

Provide appropriate justification and explanation to all your answers, detailing the methods used.

Solution:

a) Transformation matrix A

We have the property of linear tranformation:

$$T(\alpha_1 v_1, \alpha_2 v_2, .., \alpha_n v_n) = \alpha_1 T(v_1) + \alpha_2 T(v_2) + ... + \alpha_n T(v_n)$$

$$for\ all\ v_1, v_2 ... v_n \in V\ and\ \alpha_1, \alpha_2 ... \alpha_n\ are\ scalers$$

Applying the same to given $T(x)$, we get

$$T(x) = \begin{bmatrix} 2x_1 + 3x_2 + x_3 \\ x_1 - 3x_2 \\ x_1 + x_2 + x_3 \\ 3x_1 + 2x_2 + x_3 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ -3 \\ 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$T(x) = \begin{bmatrix} 2 & 3 & 1 \\ 1 & -3 & 0 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{17}$$

We have a Linear transformation formula:

$$T(x) = Ax \tag{A}$$

Comparing equation (17) with formula (A), we get matrix A:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & -3 & 0 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix} \tag{18}$$

b) Determining the $rank(A)$

In order to find solutions of a linear system, there is a method to eliminate variables step by step using a technique known as Gaussian Elimination (Lipschutz,2009). With this reduction, we get a system of equations in its matrix form callen Row Echelon Form (REF). The below transitions of matrices demonstrate the Gaussian elimination to get the matrrix A in its REF.

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & -3 & 0 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix} \rightarrow R_1 = \frac{R_1}{2} \rightarrow \begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ 1 & -3 & 0 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix} \rightarrow R_2 = R_3 - R_2 \rightarrow \begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 4 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix} \rightarrow R_3 = 3R_3 - R_4 \rightarrow$$

$$\begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 4 & 1 \\ 0 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix} \rightarrow R_4 = R_4 - 3R_1 \rightarrow \begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 4 & 1 \\ 0 & 1 & 2 \\ 0 & \frac{-5}{2} & \frac{-1}{2} \end{bmatrix} \rightarrow R_2 = \frac{R_2}{4} \rightarrow \begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{4} \\ 0 & 1 & 2 \\ 0 & \frac{-5}{2} & \frac{-1}{2} \end{bmatrix} \rightarrow R_4 = \frac{5}{2}R_3 + R_4 \rightarrow$$

$$\begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{4} \\ 0 & 1 & 2 \\ 0 & 0 & \frac{9}{2} \end{bmatrix} \rightarrow R_3 = R_3 - R_2 \rightarrow \begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & \frac{7}{4} \\ 0 & 0 & \frac{9}{2} \end{bmatrix} \rightarrow R_3 = \frac{4}{7}R_3 \rightarrow \begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 1 \\ 0 & 0 & \frac{9}{2} \end{bmatrix} \rightarrow R_4 = \frac{9}{2}R_3 - R_4 \rightarrow$$

$$\begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow R_2 = R_2 - \frac{1}{4}R_3 \rightarrow \begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow R_1 = R_1 - \frac{1}{2}R_3 \rightarrow \begin{bmatrix} 1 & \frac{3}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow R_1 = R_1 - \frac{3}{2}R_2 \rightarrow$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

In the Row Echelon form, the number of pivot columns or linearly independent columns is 3.

$$\boxed{rank(A) = 3}$$

c) Kernel and Image of Transformation and theor dimension

c.1) Determining Kernel $ker(T)$

As per definition, if $F : V -> U$, then kernel of F is the set of elements in V that map into the zero vector 0 in U (Lipschutz,2009),

$$Ker\ F = \{v \in V : F(v) = 0\} \tag{B}$$

So essentially, kernel is a set of all inputs taken to zero. Hence, for the given linear transformation, we have,

$$ker(T) = \{x \in R^3 | Ax = 0\}$$

Using the matrix in Row Echelon Form (REF), and rewriting it in in form $Ax = 0$ as below:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus, the kernel contains only zero vector with a dimension 3:

$$\boxed{ker(T) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

c.2) Determining Image

The *image* is a span of column vectors of matrix A (UoL,2025). It is given by,

$$\begin{aligned} Im(T) &= Ax | x \in R^n \\ &= (\alpha_1, \alpha_2 ... \alpha_n)x | x \in R^n \\ &= span[(\alpha_1, \alpha_2 ... \alpha_n)] \in R^n \end{aligned} \tag{C}$$

Using this formula, we compute $Im(T)$ as the column space of REF of matrix A as below:

$$\boxed{Im(T) = span[\left(\begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right)]}$$

Question 5

$$Matrix, A = \begin{bmatrix} 2 & -3 \\ 1 & 6 \end{bmatrix}$$

a) Determine the inverse of matrix A

b) Find the singular value decomposition (SVD) of matrix A

c) Given the SVD of matrix, find the SVD of

Provide appropriate justification and explanation to all your answers, detailing the methods used.

Solution:

a) Determine Inverse of A

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 6 \end{bmatrix}$$

A square matrix $A \in R^{n\times n}$ is said to be invertible if its deteminant is not equal to 0 (Lipschutz,2009).

$$|A| = 2 \times 6 - (-3) \times 1 = 15 \tag{19}$$

Since $|A| \neq 0$, the given matrix A is invertible.

We have the below mathematical formula to determine inverse of a matrix:

$$A^{-1} = \frac{1}{|A|} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{|A|} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{B}$$

Applying formula (B) and using deteminant from equation (19), we get,

$$A^{-1} = \frac{1}{15} \times \begin{bmatrix} 6 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 \cdot \frac{1}{15} & (3) \cdot \frac{1}{15} \\ -1 \cdot \frac{1}{15} & 2 \cdot \frac{1}{15} \end{bmatrix}$$

$$\boxed{A^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{-1}{15} & \frac{2}{15} \end{bmatrix}}$$

b) Singular Value Decomposition (SVD) Calculation of matrix A

Singular Value Decomposition states that any matrix A could be decomposed into a product of three matrices - two orthogonal matrices U and V, and a diagonal matrix Σ.

The SVD of a matrix A is given by a decomposition of the below form:

$$A = U \cdot \Sigma \cdot V^T \tag{C}$$

Denoted as:

$$\underbrace{A}_{original\ matrix} = \underbrace{U}_{Orthogonal\ matrix,\ L(A)} \cdot \underbrace{\Sigma}_{Diagonal\ matrix} \cdot \underbrace{V^T}_{orthogonal\ matrix,\ R(A)} \tag{20}$$

Since the given matrix A is 2×2 , the dimensions of SVD matrices will be,

$$A_{n\times n} = U_{n\times n}\Sigma_{n\times n}V^T_{n\times n}$$

Step 1: Find the right singular vectors of A to compute matrix V^T

We will perform below sub-tasks in this step:

i) Compute A^TA.

ii) Find Eigenvalues of A^TA.

iii) Using eigenvalues from step ii., compute eigenvectors of A^TA

iv) Find V^T

i) Determine A^TA

$$A^TA = \begin{bmatrix} 2 & 1 \\ -3 & 6 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 \\ 1 & 6 \end{bmatrix}$$
$$= \begin{bmatrix} 4+1 & -6+6 \\ -6+6 & 9+36 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 5 & 0 \\ 0 & 45 \end{bmatrix} \tag{21}$$

ii)Find Eigen values of A^TA

As per the definition, for a square matrix A, a scaler λ is called *eigenvalue* of A if there exists a non-zero column vector v such that (Lipschutz,2009),

$$Av = \lambda v$$
$$Av - \lambda v = 0$$
$$(A - \lambda I)v = 0 \tag{D}$$

Any vector satisfying formula (D) is called *eigenvector* of A corresponding to *eigenvalue* λ.

This equation has a non-zero solution if and only if,

$$|A - \lambda I| = 0 \tag{E}$$

Using formula (E) to determine eigenvalues as below:

$$|A^T A - \lambda I| = 0$$

$$\left| \begin{bmatrix} 5 & 0 \\ 0 & 45 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 5-\lambda & 0 \\ 0 & 45-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(45-\lambda) = 0$$

Computing eigenvalues as,

$$5 - \lambda = 0, \lambda = -5$$

$$45 - \lambda = 0, \lambda = -45$$

The eigen values are:

$$\lambda = 5 \ and \ \lambda = 45 \tag{22}$$

iii)Find Eigenventors

Using eigen values from (22), we now find Eigen vectors by rewriting formula (D) in terms of x:

$$(A^T A - \lambda I)x = 0$$

$$\left(\begin{bmatrix} 5 & 0 \\ 0 & 45 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 5 & 0 \\ 0 & 45 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Calculating for $\lambda = 45$

$$\left(\begin{bmatrix} 5 & 0 \\ 0 & 45 \end{bmatrix} - \begin{bmatrix} 45 & 0 \\ 0 & 45 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -40 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Applying Gaussian elimination to the augmented matrix to reduce it to Row Echelon Form:

$$\left[\begin{array}{cc|c} -40 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] \rightarrow R_1 \times -\tfrac{1}{40} \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

Solving for null space:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system in terms of free variables, we get below values of x_1 and x_2 :

$x_1 = 0$, $x_2 = x_2$

The solution vector is as given below:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$$

Representing it as a linear combination of vectors:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

For $\lambda = 45$, the eigen vector is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Calculating for $\lambda = 5$:

Reiterating the eigenvector formula:

$$(A - \lambda I)x = 0$$

$$\left(\begin{bmatrix} 5 & 0 \\ 0 & 45 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 40 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Applying Gaussian elimination to the augmented matrix to reduce it to Row Echelon Form. The transformation begins by swapping the rows because as per Row-Echelon matrix property, all rows with only zeros must be at the bottom of the matrix. (Vincent, 2021)

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 40 & 0 \end{array} \right] \rightarrow Swapping R_2 and R_1 \rightarrow \left[\begin{array}{cc|c} 0 & 40 & 0 \\ 0 & 0 & 0 \end{array} \right] \rightarrow R_2 \frac{1}{40} \rightarrow \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

Solving for null space:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system in terms of free variables, we get below values of x_1 and x_2 :

$x_2 = 0$ $x_1 = x_1$

Hence, the solution vector is as given below:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

Representing it in a linear combination of vectors:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

For $\lambda = 5$, the eigen vector is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

iv)Find V^T

The right singular vector or orthogonal matrix is a cmbination of eigen vectors of $A^T A$.

Hence we can write it as below:

$$V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\boxed{V^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \tag{23}$$

Step 2: Find Σ

Σ is an $n \times n$ matrix with singular values of A on the main diagonal and the rest of the entries as 0 (Zuniga,2021). The singular values σ_i of A are square roots of eigen values of $A^T A$. Thus, referring to (22), we get

$$\sigma_1 = \sqrt{5}\ and\ \sigma_2 = \sqrt{45} \tag{24}$$

$$\boxed{\Sigma = \begin{bmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{45} \end{bmatrix}} \tag{25}$$

Step 3: Find left signular vectors of A for matrix U

Let us find the left singular vectors u_1 and u_2 for matrix U

From formula (C), we have,

$$A = U \cdot \Sigma \cdot V^T$$

Multiply each side by V,

$$A \cdot V = U \cdot \Sigma \cdot V^T \cdot V$$

Since V and V^T are orthogonal, it follows that $V^T V = I$. Therefore the equation we get is,

$$A \cdot V = U \cdot \Sigma$$

Thus, for finding invidual left singular vectors u_i , we can use the below formula,

$$A \cdot v_i = u_i \cdot \sigma_i \qquad \text{(F)}$$
$$u_i = \frac{A \cdot v_i}{\sigma_i}$$

Using formula (F), to find vectors u_1 and u_2:

$$u_1 = \frac{1}{\sigma_1} \cdot A \cdot v_1$$
$$= \frac{1}{\sqrt{5}} \cdot \begin{bmatrix} 2 & -3 \\ 1 & 6 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-3}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{6}{\sqrt{5}} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$u_1 = \begin{bmatrix} \frac{-3}{\sqrt{5}} \\ \frac{6}{\sqrt{5}} \end{bmatrix}$$

$$u_2 = \frac{1}{\sigma_2} \cdot A \cdot v_2$$
$$= \frac{1}{\sqrt{45}} \cdot \begin{bmatrix} 2 & -3 \\ 1 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{45}} & \frac{-3}{\sqrt{45}} \\ \frac{1}{\sqrt{45}} & \frac{6}{\sqrt{45}} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$u_2 = \begin{bmatrix} \frac{2}{3\sqrt{5}} \\ \frac{1}{3\sqrt{5}} \end{bmatrix}$$

Computing U as a combination of u_1 and u_2 as,

$$U = \begin{bmatrix} u_1 & u_2 \end{bmatrix} = \begin{bmatrix} \frac{-3}{\sqrt{5}} & \frac{2}{3\sqrt{5}} \\ \frac{6}{\sqrt{5}} & \frac{1}{3\sqrt{5}} \end{bmatrix}$$

$$\boxed{U = \frac{1}{\sqrt{5}} \cdot \begin{bmatrix} -3 & \frac{2}{3} \\ 6 & \frac{1}{3} \end{bmatrix}} \qquad (26)$$

Step 4: Calculate SVD of matrix A

From results (23),(25),(26), we calculate the SVD of matrix A, which should be exactly equal to the original given matrix A.

$$A = U \cdot \Sigma \cdot V^T$$

$$\begin{aligned}
&= \frac{1}{\sqrt{5}} \cdot \begin{bmatrix} -3 & \frac{2}{3} \\ 6 & \frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{45} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{-3}{\sqrt{5}} & \frac{2}{3\sqrt{5}} \\ \frac{6}{\sqrt{5}} & \frac{1}{3\sqrt{5}} \end{bmatrix} \cdot \begin{bmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{45} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\end{aligned}$$

Simplifying the denominator for matrix U by multiplying all elements by $\frac{\sqrt{5}}{\sqrt{5}}$ and rewriting $\sqrt{45}$ as $3\sqrt{5}$

$$\begin{aligned}
A &= \begin{bmatrix} \frac{-3\sqrt{5}}{5} & \frac{2\sqrt{5}}{15} \\ \frac{6\sqrt{5}}{5} & \frac{\sqrt{5}}{15} \end{bmatrix} \cdot \begin{bmatrix} \sqrt{5} & 0 \\ 0 & 3\sqrt{5} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{-3\times5}{5} + 0 & 0 + \frac{2\times3\times5}{15} \\ \frac{6\times5}{5} + 0 & 0 + \frac{5\times3}{15} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -3 & 2 \\ 6 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\end{aligned}$$

$$\boxed{A = \begin{bmatrix} 2 & -3 \\ 1 & 6 \end{bmatrix}} \tag{27}$$

And this is our original given matrix A.

c) Determine SVD of A^T

For the given matrix A, its tranpose is computed as,

$$A^T = \begin{bmatrix} 2 & 1 \\ -3 & 6 \end{bmatrix}$$

Referring to SVD formula (C), its transpose version as per the property of transpose is given by,

$$\begin{aligned}
A^T &= (U \cdot \Sigma \cdot V^T)^T \\
&= U^T \cdot \Sigma^T \cdot V
\end{aligned}$$

Since the singular values in Σ remain the same, we reorder the terms (Reilly,2025) in the above equation to derive the SVD of A^T,

$$A^T = V \cdot \Sigma \cdot U^T \tag{G}$$

From equations (23), (25) and (26), we compute transpose of U, Σ. We get:

$$U^T = \frac{1}{\sqrt{5}} \cdot \begin{bmatrix} -3 & 6 \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}, \Sigma^T = \begin{bmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{45} \end{bmatrix}, V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Calculating SVD of A^T as below:

$$\begin{aligned} A^T &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{45} \end{bmatrix} \cdot \frac{1}{\sqrt{5}} \cdot \begin{bmatrix} -3 & 6 \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{45} \end{bmatrix} \cdot \begin{bmatrix} \frac{-3}{\sqrt{5}} & \frac{6}{\sqrt{5}} \\ \frac{2}{3\sqrt{5}} & \frac{1}{3\sqrt{5}} \end{bmatrix} \\ &= \begin{bmatrix} 0 \cdot \sqrt{5} + 1 \cdot 0 & 0 \cdot 0 + 1 \cdot \sqrt{45} \\ 1 \cdot \sqrt{5} + 0 & 0 + 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{-3}{\sqrt{5}} & \frac{6}{\sqrt{5}} \\ \frac{2}{3\sqrt{5}} & \frac{1}{3\sqrt{5}} \end{bmatrix} \end{aligned}$$

Simplifying the denominator for matrix U by multiplying all elements by $\frac{\sqrt{5}}{\sqrt{5}}$ and rewriting $\sqrt{45}$ as $3\sqrt{5}$

$$\begin{aligned} A &= \begin{bmatrix} 0 & \sqrt{45} \\ \sqrt{5} & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{-3}{\sqrt{5}} & \frac{6}{\sqrt{5}} \\ \frac{2}{3\sqrt{5}} & \frac{1}{3\sqrt{5}} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 3\sqrt{5} \\ \sqrt{5} & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{-3\sqrt{5}}{5} & \frac{6\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{15} & \frac{\sqrt{5}}{15} \end{bmatrix} \end{aligned}$$

$$\boxed{A^T = \begin{bmatrix} 2 & 1 \\ -3 & 6 \end{bmatrix}} \tag{28}$$

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