Lecture. 2. Eliminate with Matrices.

1. Overview.

- Elimination：计算机方程求解
- Back - Substitution
- Elimination Matrics.
- Matrix Multiplication.

2. Elimination to solve equations：elimination & Back-subst.

2.1 Elimination：

主要以下2种情形：
- Success：Can find all non-zero pivots. 可逆
- Failure：At least one "pivot" is zero. 不可逆.

eg. 求解.
$$\begin{cases} x + 2y + z = 2 \\ 3x + 8y + z = 12 \\ 4y + z = 2 \end{cases}$$

改写为 $Ax=b$ 的形式：

$$\underset{A}{\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}} \underset{x}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{b}{\begin{bmatrix} 2 \\ 12 \\ 2 \end{bmatrix}}$$

- Pivot can not be Zero.

$$\underset{A}{\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}} \xrightarrow[(2,1)]{} \overset{\text{First Pivot.}}{\begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}} \xrightarrow[(3,2)]{} \overset{\text{Second Pivot}}{\begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 0 & 5 \end{bmatrix}} \longrightarrow \overset{\text{Third Pivot}}{\underset{U}{\begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 0 & \boxed{5} \end{bmatrix}}}$$

(upper triangular)

如果 pivot 为0，则需换行，找到一个非0数。如果找不到，则矩阵不可逆，elimination 求出的结果不唯一。

2.2 Back-Substitution. 其实，该方法与 elimination 同时进行.

eg.

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 2 \\ 12 \\ 2 \end{bmatrix}}_{b}$$

Augment Matrix

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right] \quad A \quad b.$$

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right] \xrightarrow[(2,1)]{} \left[\begin{array}{ccc:c} \boxed{1} & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array}\right] \xrightarrow[(3,2)]{} \left[\begin{array}{ccc:c} \boxed{1} & 2 & 1 & 2 \\ 0 & \boxed{2} & -2 & 6 \\ 0 & 0 & \boxed{5} & -10 \end{array}\right]$$

$$\Rightarrow \begin{cases} 1x + 2y + z = 2 \\ 2y - 2z = 6 \\ 5z = -10 \end{cases} \Rightarrow \quad \begin{matrix} x = 2 \\ y = 1 \\ z = -2 \end{matrix}$$

3. 从矩阵角度看 elimination.

3.1 矩阵乘法：

- 矩阵与向量乘法：矩阵列向量的线性组合.

$$\begin{bmatrix} | & | & | \\ | & | & | \\ | & | & | \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a \times \underline{Col1} + b \cdot \underline{Col2} + c \cdot \underline{Col3}$$

- 行向量与矩阵的乘积：矩阵行向量的线性组合.

$$\underset{1\times 3}{\begin{bmatrix} a & b & c \end{bmatrix}} \underset{3\times 3}{\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}} = a \times \underline{row1} + b \cdot \underline{row2} + c \cdot \underline{row3}.$$

3.2. Elimination. (矩阵角度)：行变换：利用向量左乘矩阵.

- Step 1. Substract 3 times row 1 from row 2.

Row1 don't change

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \underset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

Row3 don't change.

- Step 2. Substract 2 times row 2 from 3.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \underset{E_{32}.}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

☆ $$\boxed{E_{32}(E_{21} \cdot A) = U}$$

$$\Downarrow$$

$$EA = U$$

4. Permutation Matrix：行变换和列变换

4.1 行变换 $\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{P}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

* $AB \neq BA$

$(AB)C = A(BC)$

4.2 列变换 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{P} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$

左乘 = 行变换，右乘 = 列变换.

5、Inverses：$U \longrightarrow A$.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E^{-1}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I}$$

$$A \longrightarrow EA \longrightarrow U$$

$$U \longrightarrow E^{-1}U \longrightarrow A$$