SPE Maths Quick Sheet

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Series

- The series of general term u_n, denoted $\sum u_n$, is the sequence of partial sum

$$(S_n)_{n\in\mathbb{N}}, S_n = \sum_{k=0}^{n} u_k$$

- (S_n) converges $\Leftrightarrow \sum u_n$ converges
- Let $\sum u_n$ and $\sum v_n$ be two sequences. Let $\lambda \in \mathbb{R}$
  - $\sum u_n$ CVG and $\sum v_n$ CVG $\Rightarrow \sum(u_n + v_n)$ CVG
  - $(\sum u_n \text{ CVG}) \Rightarrow (\sum \lambda u_n \text{ CVG})$
  - $(\sum \mathrm{u_n} \text{ CVG and } \sum v_n \text{ DVG}) \Rightarrow (\sum u_n + v_n \text{ CVG})$
- **Necessary condition of convergence**

$$\sum u_n \text{ is convergent} \Rightarrow \lim_{n\to+\infty} u_n = 0$$

Series of nonnegative terms

- $\sum u_n$ is a series of nonnegative terms if, $\forall n \in \mathbb{N}, u_n \geq 0$.
- $\sum u_n$ is a series of nonnegative terms, (S_n) the sequence of partial sums: $\sum u_n$ converges $\Leftrightarrow u_n$ is upper-bounded
- Let (u_n) and (v_n) be two sequences such that $\forall n \in \mathbb{N}, 0 \leq u_n \leq v_n$

$$\begin{cases} \sum v_n \text{ CVG} \Rightarrow \sum u_n \text{ CVG} \\ \sum u_n \text{ DVG} \Rightarrow \sum v_n \text{ DVG} \end{cases}$$

- **Geometric series:** $\sum q^n$ CVG $\Leftrightarrow |q| < 1$
- **Riemann series**: $\sum \frac{1}{n^\alpha}, \alpha \in \mathbb{R}$
  - $\sum \frac{1}{n^\alpha}$ converges $\Leftrightarrow \alpha > 1$
- **Criteria of comparison**
  Let (u_n) and (v_n) be two nonnegative real sequences.
  - $u_n = o(v_n) \Rightarrow \begin{cases} \sum v_n \text{ CVG} \Rightarrow \sum u_n \text{ CVG} \\ \sum u_n \text{ DVG} \Rightarrow \sum v_n \text{ DVG} \end{cases}$
  - $u_n \underset{n\to+\infty}{\sim} v_n \Rightarrow \sum u_n$ and $\sum v_n$ are of the same nature
  - $u_n = O(v_n) \Rightarrow$ same as $u_n = o(v_n)$
- *Reminder* **Landau notation**
  - $u_n = o(v_n) \Leftrightarrow \frac{u_n}{v_n} \xrightarrow[n\to+\infty]{} 0$
  - $u_n \underset{n\to+\infty}{\sim} v_n \Leftrightarrow \frac{u_n}{v_n} \xrightarrow[n\to+\infty]{} 1$
  - $u_n = O(v_n) \Leftrightarrow \frac{u_n}{v_n}$ is bounded towards $n \to +\infty$
  - $u_n + o(u_n) \sim u_n$
- Let (u_n) be a real sequence of nonnegative terms. Then

$$\sum(u_n - u_{n-1}) \text{ CVG} \Leftrightarrow (u_n) \text{ CVG}$$

- **Riemann's rule**
  Let (u_n) be a real nonnegative sequence. If $\exists \alpha > 1$ such that $n^\alpha u_n \xrightarrow[n\to+\infty]{} 0$ then $\sum u_n$ CVG
- **D'Alembert/Cauchy test**

$$\text{If either } \begin{cases} \frac{u_{n+1}}{u_n} \to l \\ \sqrt[n]{u_n} \to l \end{cases} \text{ then } \begin{cases} l < 1 \Rightarrow \sum u_n \text{ CVG} \\ l > 1 \Rightarrow \sum u_n \text{ DVG} \\ l = 1 \Rightarrow ? \end{cases}$$

  Using the ratio, it is a D'Alembert test. Using the root, it is a Cauchy test.

Series of arbitrary terms

- **Alternating sequence**
  Let (u_n) be a real sequence. (u_n) is an alternating sequence if there exists a nonnegative sequence (v_n) such that for all $n \in \mathbb{N}$: $u_n = (-1)^n v_n$ or $u_n = (-1)^{n+1} v_n$
- **Alternating series**
  If u_n is an alternating sequence, then the series $\sum u_n$ is called an alternating series.

- Let (u_n) be a real alternating sequence
  (u_n) is decreasing and $\lim_{n\to+\infty} u_n = 0$, then $\sum u_n$ CVG
  $$\left.\begin{array}{c}(|u_n|) \text{ is decreasing} \\ \lim_{n\to+\infty} u_n = 0\end{array}\right\} \Rightarrow \sum u_n \text{ CVG}$$
- We say $\sum u_n$ converges absolutely if the series $\sum |u_n|$ converges.
  - $\sum u_n$ converges absolutely $\Rightarrow \sum u_n$ converges
  - We say $\sum u_n$ is semi-convergent (or conditionally convergent) if it converges but does not converge absolutely.
- Let $\alpha \in \mathbb{R}$. Then $\sum \frac{(-1)^n}{n^\alpha}$ CVG $\Leftrightarrow \alpha > 0$

Generating functions

Ah yes, enslaved probabilities

- **Definition**
  If the possible values for X are $[\![0,n]\!]$, then
  $$G_X(t) = t^0P(X=0) + t^1P(X=1) + \cdots + t^nP(X=N)$$
  $G_X(t)$ is a polynomial
- **General properties**
  - $G_X(1) = 1$
  - $E(X) = G'_X(1)$
  - $Var(X) = G''_X(1) + G'_X(1) - \left(G'_X(1)\right)^2$
- If X and Y are **independent** random variables, then:
  $$G_{X+Y}(t) = G_X(t) \times G_Y(t)$$
- *Reminder* **Bernoulli distribution**
  $$X \rightsquigarrow \text{Bernoulli}(p) \Rightarrow \begin{cases} X \in \{0,1\} \\ P(X=1) = p \\ P(X=0) = 1-p \end{cases}$$
  X has a Bernoulli distribution with parameter p
- *Reminder* **Binomial distribution**
  A binomial distribution is the repetition of a Bernoulli distribution n times. With $i \in [\![1,n]\!]$,
  $$X_i \rightsquigarrow \text{Bernoulli}(p)$$
  $$Y = X_1 + \cdots + X_n \rightsquigarrow \text{B}(n,p)$$
  The probability that there are k X variables equal to 1 is:
  $$P(Y=k) = \binom{n}{k}(1-p)^{n-k}p^k$$
- If X can take an infinite number of values in $\mathbb{N}$,
  $$G_X(t) = \sum_{k=0}^{+\infty} P(X=k)t^k$$

Power series

- **Definition**
  $$\sum a_n x^n = a_0 + a_1x + a_2x^2 + \cdots$$
  - a_n does not depend on x
  - If $x = 0$, all terms of $\sum a_n x^n$ are equal to 0, except when $n = 0$, since $a_0 \times x^0 = a_0 \times 1$
  - If $\sum a_n x^n$ CVG for some values of x, then we have $f(x) = \sum_{n=0}^{+\infty} a_n x^n$
- **Sum and product of power series**
  Let $f(x) = \sum_{n=0}^{+\infty} a_n x^n$ with its radius of convergence R_f and $g(x) = \sum_{n=0}^{+\infty} b_n x^n$ with its radius of convergence R_g, then
  - $f(x) + g(x) = \sum_{n=0}^{+\infty} (a_n + b_n)x^n$
  - $f(x) \times g(x) = \sum_{n=0}^{+\infty} c_n x^n$ where $c_n = \sum_{k=0}^{+\infty} a_k \times b_{n-k}$
- **Radius of convergence**
  Let (a_n) be a real sequence, $\sum a_n x^n$ the P.S. defined by this sequence and the function $f: x \mapsto \sum_{n=0}^{+\infty} a_n x^n$.
  Then $\exists R \in \mathbb{R}_+ \cup \{+\infty\}$ such that
  - $\forall x \in \mathbb{R}, |x| < R$, then $\sum a_n x^n$ CVG ABS
  - $\forall x \in \mathbb{R}, |x| > R$, then $\sum a_n x^n$ DVG

  R is called the radius of convergence of this P.S.
  The set $\{x \in \mathbb{R}, |x| < R\} =]-R, R[$ is called the open disk of convergence of the P.S.
- **Determining the radius of convergence** of a power series
  Ratio test (D'Alembert's rule)
  Let $\sum a_n x^n$ be a power series. If $\left|\frac{a_{n+1}}{a_n}\right| \to l$, then $R = \frac{1}{l}$
  (considering that $\frac{1}{0} = +\infty$ and $\frac{1}{+\infty} = 0$)
- **Power series of basic functions**
  - e^x
  $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
  $$= \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

$$R = +\infty$$

- $\ln(1+x)$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$
$$= \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{x^n}{n}$$
$$R = 1$$
$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$$
$$= -\sum_{n=1}^{+\infty} \frac{x^n}{n}$$

- $(1+x)^\alpha$

$$(1+x)^\alpha = \sum_{n=0}^{+\infty} a_n x^n$$
$$a_n = \frac{\alpha(\alpha-1)\dots(\alpha-n)}{n!}$$
$$R = \begin{cases} 1 \text{ if } \alpha \notin \mathbb{N} \\ +\infty \text{ if } \alpha \in \mathbb{N} \end{cases}$$

- $\sin(x)$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$
$$= \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$
$$R = +\infty$$

- $\cos(x)$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$
$$= \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$
$$R = +\infty$$

Infinite discrete random variable

- **Definition**

  Let $(\Omega, P, \mathcal{P}(\Omega))$ be a probability space, and $x: \Omega \to \mathbb{R}$ our random variable. X is a discrete infinite random variable if the values of $X(\Omega)$ are indexable by $\mathbb{N}$:
  $$X(\Omega) = \{x_0, x_1, x_2, \dots\} = \{x_k, k \in \mathbb{N}\}$$
  (We will study IDRVs where $X(\Omega) \subset \mathbb{N}$, so X is an Infinite Integer Random variable)
  - $P(\Omega) = 1 \Leftrightarrow \sum_{n=0}^{+\infty} P(X=n) = 1$
    (The series of general term $P(X=n)$ is equal to 1)
  - $P(X \in A) = \sum_{n \in A} P(X=n)$
- **Geometric distribution**

  Let $p \in]0,1[$ and $X \rightsquigarrow \text{Bernoulli}(p)$. Let Y be the number of tries needed to get the first $X = 1$, with each try being independent.
  $$\forall n \in \mathbb{N}^*, P(Y=n) = (1-p)^{n-1} \times p$$
  Y is a geometric distribution R.V. $\Leftrightarrow Y \rightsquigarrow \mathcal{G}(p)$
- **Expected value and variance**
  $$E(X) = \sum_{n \in X(\Omega)} n \times P(X=n)$$
  $$V(X) = \sum_{n \in X(\Omega)} \big(n - E(X)\big)^2 P(X=n)$$
  $$\sigma(X) = \sqrt{V(X)}$$
  - If the sum of power series in $E(X)$ diverges, X has no expected value or variance
  - If the sum of power series in $E(X)$ converges but the one in $V(X)$ diverges, X has an expected value but no variance
- **Generating function**
  $$G_X(t) = P(X=0)t^0 + P(X=1)t^1 + \cdots + P(X=n)t^n + \cdots$$
  $$= \sum_{n=0}^{+\infty} P(X=n)t^n$$

- The convergence radius of the resulting series is ≥ 1
- G_X exists and is continuous over at least $[-1, 1]$, and $G_X(1) = 1$
- G_X is C^∞ over $]-1, 1[$
  - *Reminder*
    f is C^0 over I means that it is continuous over I
    f is C^1 over I means that it is differentiable and that f' is continuous over I
    f is C^n over I means that it is differentiable and that $f^{(n+1)}$ is continuous
- If X and Y are independent IIRV, $G_{X+Y} = G_X + G_Y$
- If X has an expected value ($\Leftrightarrow \sum_{n \in X(\Omega)} n \times P(X = n)$ converges), then G_X is differentiable for $t = 1$ and $E(X) = G_X'(1)$
- If X has a variance, G_X' is differentiable for $t = 1$ and $V(X) = G_X''(1) + G_X'(1) - \left(G_X'(1)\right)^2$