SPE Maths Quick Sheet

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Series

- The series of general term u_n, denoted $\sum u_n$, is the sequence of partial sum

$$(S_n)_{n\in\mathbb{N}}, S_n = \sum_{k=0}^{n} u_k$$

- (S_n) converges $\Leftrightarrow \sum u_n$ converges
- Let $\sum u_n$ and $\sum v_n$ be two sequences. Let $\lambda \in \mathbb{R}$
  - $\sum u_n$ CVG and $\sum v_n$ CVG $\Rightarrow \sum(u_n + v_n)$ CVG
  - $(\sum u_n \text{ CVG}) \Rightarrow (\sum \lambda u_n \text{ CVG})$
  - $(\sum \mathrm{u_n} \text{ CVG and } \sum v_n \text{ DVG}) \Rightarrow (\sum u_n + v_n \text{ CVG})$
- **Necessary condition of convergence**

$$\sum u_n \text{ is convergent} \Rightarrow \lim_{n\to+\infty} u_n = 0$$

Series of nonnegative terms

- $\sum u_n$ is a series of nonnegative terms if, $\forall n \in \mathbb{N}, u_n \geq 0$.
- $\sum u_n$ is a series of nonnegative terms, (S_n) the sequence of partial sums: $\sum u_n$ converges $\Leftrightarrow u_n$ is upper-bounded
- Let (u_n) and (v_n) be two sequences such that $\forall n \in \mathbb{N}, 0 \leq u_n \leq v_n$

$$\begin{cases} \sum v_n \text{ CVG} \Rightarrow \sum u_n \text{ CVG} \\ \sum u_n \text{ DVG} \Rightarrow \sum v_n \text{ DVG} \end{cases}$$

- **Geometric series:** $\sum q^n$ CVG $\Leftrightarrow |q| < 1$
- **Riemann series**: $\sum \frac{1}{n^\alpha}, \alpha \in \mathbb{R}$
  - $\sum \frac{1}{n^\alpha}$ converges $\Leftrightarrow \alpha > 1$
- **Criteria of comparison**
  Let (u_n) and (v_n) be two nonnegative real sequences.
  - $u_n = o(v_n) \Rightarrow \begin{cases} \sum v_n \text{ CVG} \Rightarrow \sum u_n \text{ CVG} \\ \sum u_n \text{ DVG} \Rightarrow \sum v_n \text{ DVG} \end{cases}$
  - $u_n \underset{n\to+\infty}{\sim} v_n \Rightarrow \sum u_n$ and $\sum v_n$ are of the same nature
  - $u_n = O(v_n) \Rightarrow$ same as $u_n = o(v_n)$
- *Reminder* **Landau notation**
  - $u_n = o(v_n) \Leftrightarrow \frac{u_n}{v_n} \xrightarrow[n\to+\infty]{} 0$
  - $u_n \underset{n\to+\infty}{\sim} v_n \Leftrightarrow \frac{u_n}{v_n} \xrightarrow[n\to+\infty]{} 1$
  - $u_n = O(v_n) \Leftrightarrow \frac{u_n}{v_n}$ is bounded towards $n \to +\infty$
  - $u_n + o(u_n) \sim u_n$
- Let (u_n) be a real sequence of nonnegative terms. Then

$$\sum(u_n - u_{n-1}) \text{ CVG} \Leftrightarrow (u_n) \text{ CVG}$$

- **Riemann's rule**
  Let (u_n) be a real nonnegative sequence. If $\exists \alpha > 1$ such that $n^\alpha u_n \xrightarrow[n\to+\infty]{} 0$ then $\sum u_n$ CVG
- **D'Alembert/Cauchy test**

$$\text{If either } \begin{cases} \frac{u_{n+1}}{u_n} \to l \\ \sqrt[n]{u_n} \to l \end{cases} \text{ then } \begin{cases} l < 1 \Rightarrow \sum u_n \text{ CVG} \\ l > 1 \Rightarrow \sum u_n \text{ DVG} \\ l = 1 \Rightarrow ? \end{cases}$$

  Using the ratio, it is a D'Alembert test. Using the root, it is a Cauchy test.

Series of arbitrary terms

- **Alternating sequence**
  Let (u_n) be a real sequence. (u_n) is an alternating sequence if there exists a nonnegative sequence (v_n) such that for all $n \in \mathbb{N}$: $u_n = (-1)^n v_n$ or $u_n = (-1)^{n+1} v_n$
- **Alternating series**
  If u_n is an alternating sequence, then the series $\sum u_n$ is called an alternating series.

- Let (u_n) be a real alternating sequence
  (u_n) is decreasing and $\lim_{n\to+\infty} u_n = 0$, then $\sum u_n$ CVG
  $$\left.\begin{array}{c}(|u_n|) \text{ is decreasing} \\ \lim_{n\to+\infty} u_n = 0\end{array}\right\} \Rightarrow \sum u_n \text{ CVG}$$
- We say $\sum u_n$ converges absolutely if the series $\sum |u_n|$ converges.
  - $\sum u_n$ converges absolutely $\Rightarrow \sum u_n$ converges
  - We say $\sum u_n$ is semi-convergent (or conditionally convergent) if it converges but does not converge absolutely.
- Let $\alpha \in \mathbb{R}$. Then $\sum \frac{(-1)^n}{n^\alpha}$ CVG $\Leftrightarrow \alpha > 0$

Generating functions

Ah yes, enslaved probabilities

- **Definition**
  If the possible values for X are $[\![0, n]\!]$, then
  $$G_X(t) = t^0 P(X = 0) + t^1 P(X = 1) + \cdots + t^n P(X = N)$$
  $G_X(t)$ is a polynomial
- **General properties**
  - $G_X(1) = 1$
  - $E(X) = G'_X(1)$
  - $Var(X) = G''_X(1) + G'_X(1) - \left(G'_X(1)\right)^2$
- If X and Y are **independent** random variables, then:
  $$G_{X+Y}(t) = G_X(t) \times G_Y(t)$$
- *Reminder* **Bernoulli distribution**
  $$X \rightsquigarrow \text{Bernoulli}(p) \Rightarrow \begin{cases} X \in \{0,1\} \\ P(X = 1) = p \\ P(X = 0) = 1 - p \end{cases}$$
  X has a Bernoulli distribution with parameter p
- *Reminder* **Binomial distribution**
  A binomial distribution is the repetition of a Bernoulli distribution n times. With $i \in [\![1, n]\!]$,
  $$X_i \rightsquigarrow \text{Bernoulli}(p)$$
  $$Y = X_1 + \cdots + X_n \rightsquigarrow \text{B}(n, p)$$
  The probability that there are k X variables equal to 1 is:
  $$P(Y = k) = \binom{n}{k} (1 - p)^{n-k} p^k$$
- If X can take an infinite number of values in $\mathbb{N}$,
  $$G_X(t) = \sum_{k=0}^{+\infty} P(X = k) t^k$$
- **Power series**
  $$\sum u_n(x) \text{ with } u_n(x) = a_n x^n$$
  $$\sum a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$
  - a_n does not depend on x
  - If $x = 0$, all terms of $\sum a_n x^n$ are equal to 0, except when $n = 0$, since $a_0 \times x^0 = a_0 \times 1$
  - If $\sum a_n x^n$ CVG for some values of x, then we have $f(x) = \sum_{n=0}^{+\infty} a_n x^n$
- **Radius of convergence**
  Let (a_n) be a real sequence, $\sum a_n x^n$ the P.S. defined by this sequence and the function $f: x \mapsto \sum_{n=0}^{+\infty} a_n x^n$.
  Then $\exists R \in \mathbb{R}_+ \cup \{+\infty\}$ such that
  - $\forall x \in \mathbb{R}, |x| < R$, then $\sum a_n x^n$ CVG ABS
  - $\forall x \in \mathbb{R}, |x| > R$, then $\sum a_n x^n$ DVG

  R is called the radius of convergence of this P.S.
  The set $\{x \in \mathbb{R}, |x| < R\} =]-R, R[$ is called the open disk of convergence of the P.S.
- **Determining the radius of convergence** of a power series
  Ratio test (D'Alembert's rule)
  Let $\sum a_n x^n$ be a power series. If $\left|\frac{a_{n+1}}{a_n}\right| \to l$, then $R = \frac{1}{l}$
  (considering that $\frac{1}{0} = +\infty$ and $\frac{1}{+\infty} = 0$)
- **Power series of basic functions**
  - e^x
  $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
  $$= \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$
  $$R = +\infty$$
  - $\ln(1 + x)$
  $$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$
  $$= \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{x^n}{n}$$

$$R = 1$$

$$\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$$

$$= -\sum_{n=1}^{+\infty} \frac{x^n}{n}$$

- $(1 + x)^\alpha$

$$(1 + x)^\alpha = \sum_{n=0}^{+\infty} a_n x^n$$

$$a_n = \frac{\alpha(\alpha - 1) \ldots (\alpha - n)}{n!}$$

$$R = \begin{cases} 1 \text{ if } \alpha \notin \mathbb{N} \\ +\infty \text{ if } \alpha \in \mathbb{N} \end{cases}$$

- $\sin(x)$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$= \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n + 1)!}$$

$$R = +\infty$$

- $\cos(x)$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$= \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$R = +\infty$$