Maths Basics Quick Sheet

Author: Matthieu Stombellini | Published on kb.zoroark.guru
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Derivatives

$$[f(u(x))]' = f'(u(x)) \times u'(x)$$

$$\left[e^{u(x)}\right]' = e^{u(x)} \times u'(x)$$

$$[u(x)^{\alpha}]' = \alpha u^{\alpha-1}(x) \times u'(x)$$

$$[\ln(u(x))]' = \frac{u'(x)}{u(x)}$$

$$\cos(x)' = -\sin(x)$$

$$\sin(x)' = \cos(x)$$

Primitives

$$u'e^{u} \to e^{u}$$

$$u^{\alpha}u' \to \frac{u^{\alpha+1}}{\alpha+1}$$

$$\frac{u'}{u} \to \ln(u(x))$$

$$u' \sin(u) \to -\cos(u)$$

$$u' \cos(u) \to \sin(u)$$

Taylor Expansions in 0

Exponential:

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + o(x^4)$$

$$(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + o(x^3)$$

$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 + o(x^4)$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + o(x^4)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + o(x^4)$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \left|\begin{matrix} o(x^4) \\ o(x^5) \end{matrix}\right.$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \left|\begin{matrix} o(x^5) \\ o(x^6) \end{matrix}\right.$$

Manipulation of $o(x^{\alpha})$

- $o(x^{\alpha}) - o(x^{\alpha}) = o(x^{\alpha})$
- $o(x^{\alpha}) - o(x^{\alpha+1}) = o(x^{\alpha})$
- $o(\lambda x^{\alpha}) = o(x^{\alpha})$
- $x^n \times o(x^{\alpha}) = o(x^n \times x^{\alpha}) = o(x^{\alpha+n})$
- $\frac{1}{x^n} o(x^2) = o\left(\frac{1}{x^n} \times x^2\right) = o(x^{2-n})$

Polynomials

Let r be a root of P

- r is a root of order of multiplicity at least m iff
$$(X-r)^m \mid P$$
- r is a root of order of multiplicity exactly m iff
$$\begin{cases}(X-r)^m & \mid P \\ (X-r)^{m+1} & \nmid P\end{cases}$$
- r is a root of order of multiplicity at least m iff
$$\left|\begin{array}{r} P(r)=0 \\ P'(r)=0 \\ \vdots \\ P^{(m-1)}(r)=0 \end{array}\right. \quad m \text{ conditions}$$
- r is a root of order of multiplicity exactly m iff
$$\begin{cases}(X-r)^m & \mid P \\ (X-r)^{m+1} & \nmid P\end{cases}$$

Differential equations

Don’t question these formulas. They Just Work™.

First order

With $ay' + by = c$

$$y_0 = ke^{-\int \frac{b}{a}}$$

$$y_p = y_0 \int \frac{c}{ay_0}$$

$$y = y_0 + y_p$$

Second order (constant terms for a b c)

With $ay'' + by' + cy = d(t)$

1. Compute root(s) of $ar^2 + br + c$
   a. $\Delta > 0$: Two real roots r_1 and r_2
   $$y_0 = k_1 e^{r_1 t} + k_2 e^{r_2 t}$$
   b. $\Delta = 0$: One real root r_1
   $$y_0 = (k_1 + k_2 t)e^{r_1 t}$$
   c. $\Delta < 0$: Two complex roots $r_1 = \alpha i + \beta$ and $r_2 = \alpha i - \beta$
   $$y_0 = e^{\alpha t}(k_1 \cos(\beta t) + k_2 \sin(\beta t))$$
2. Getting y_p
   a. $d(t) = P(t)$ (polynomial)
   Then $y_p = Q(t)$
   $$\begin{aligned} &c \neq 0 &&\rightarrow \deg(Q) = \deg(P) \\ &c = 0,\ b \neq 0 &&\rightarrow \deg(Q) = \deg(P) + 1 \\ &c = 0,\ b = 0 &&\rightarrow \deg(Q) = \deg(P) + 2 \end{aligned}$$
   We then know the expression of $Q(t)$.
   Compute the expressions of $Q'(t)$ and $Q''(t)$.
   $$aQ''(t) + bQ'(t) + cQ(t) = d(t)$$
   Use the coefficients of $d(t)$ to deduce the coefficients of the left side of the equation
   b. $d(t) = P(t)e^{mt}$ (polynomial times exponential)
   Then $y_p = Q(t)e^{mt}$
   Derive y_p twice to get y_p' and y_p''
   $$ay_p'' + by_p' + cy = P(t)e^{mt}$$
   Factorize the left side by e^{mt} and divide both sides by e^{mt}.
   You should find an equation
   $$\alpha Q''(t) + \beta Q'(t) + \gamma Q(t) = P(t)$$
   Once you get this, find $Q(t)$ using the previous method ($d(t) = P(t)$).
   c. For any other kind of $d(t)$
   I’ll quote Mehdi for this one:
   “You either Taylor the shit out of it and try to solve for a polynomial, or send it back to the hell it comes from because it won't be on MCQ anyway”
3. $y = y_0 + y_p$

Vector spaces

Direct sum/Supplementary subspaces

$E = F \oplus G$ if both conditions are true:

- $F \cap G = \{0_E\}$
- $F + G = E$
    - $\forall w \in E,\ \exists u \in F,\ \exists v \in G,\ w = u + v$

Linear (in)dependence

A set $X = (x_1, \cdots, x_n) \in E^n$ is linearly independent if

$$\forall (\lambda_i)_{i \in [\![1,n]\!]} \in \mathbb{K}^n, \left(\sum_{i=1}^{n} \lambda_i x_i = 0 \Rightarrow \forall i, \lambda_i = 0 \right)$$

If it is not linearly independent, it is linearly dependent.

- Adding vectors to a linearly dependent set makes it remain dependent.
- Removing vectors from (i.e. taking a subset of) a linearly independent set makes it remain independent.

Span(X)

Let E be a $\mathbb{R}$ vector space.

$$X = \{u_1, u_2, \dots, u_n\} \subset E$$

$$\begin{aligned} Span(X) &= \{\lambda_1 u_1 + \cdots + \lambda_n u_n \mid (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^{\mathbb{N}}\} \\ &= \{\text{linear combinations of } u_1, \cdots, u_n\} \end{aligned}$$

$Span(X)$ is a $\mathbb{R}$-vs with $X \subset Span(X)$

Spanning set

Let $X \subset E$. We say that X is a spanning set of E if $E = Span(X)$

Base

A linearly independent spanning set of E is called a basis of E.

$(e_1, \dots, e_n)$ is a basis of E $\Leftrightarrow$ $\forall u \in E$, there exists a unique decomposition of u as a linear combination of the basis $\left(u = \sum_{(i=1)}^{n} \lambda_i e_i\right)$

Linear maps

E and F two $\mathbb{R}$-vs.

$$f: E \to F$$

Then f is a linear map if $\forall (u, v) \in E^2, \forall \lambda \in \mathbb{R}$,

- $f(\lambda u + v) = \lambda f(u) + f(v)$

Or

- $f(u + v) = f(u) + f(v)$
  And $f(\lambda u) = \lambda f(u)$

Then:

- $f(0_E) = 0_F$
  Proof: $f(-u + u) = -f(u) + f(u) \Rightarrow f(0_E) = 0_F$
  All the linear maps $\mathbb{R}^n \to \mathbb{R}^p$ have the form

$$f\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right) = \begin{pmatrix} a_{1,1} x_1 + \cdots + a_{1,n} x_n \\ \vdots \\ a_{p,1} x_1 + \cdots + a_{p,n} x_n \end{pmatrix}$$

Kernel and image

Let $f: E \to F$ be a linear map ($f \in \mathcal{L}(E, F)$)

$$\begin{aligned} Ker(f) &= \{\text{preimages of } 0_F \text{ by } F\} \\ &= \{u \in E \text{ such that } f(u) = 0_F\} \\ &= f^{-1}(\{0_F\}) \end{aligned}$$

$$\begin{aligned} Im(f) &= f(E) \\ &= \{f(u), u \in E\} \\ &= \{v \in F \ \text{ such that } \exists v \in F, v = f(u)\} \end{aligned}$$