# SPE Maths Quick Sheet

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## Series

- The series of general term $u_n$, denoted $\sum u_n$, is the sequence of partial sum
$$(S_n)_{n\in\mathbb{N}}, S_n = \sum_{k=0}^{n} u_k$$
- $(S_n)$ converges $\Leftrightarrow \sum u_n$ converges
- Let $\sum u_n$ and $\sum v_n$ be two sequences. Let $\lambda \in \mathbb{R}$
    - $\sum u_n$ CVG and $\sum v_n$ CVG $\Rightarrow \sum(u_n + v_n)$ CVG
    - $(\sum u_n \text{ CVG}) \Rightarrow (\sum \lambda u_n \text{ CVG})$
    - $(\sum \mathrm{u_n} \text{ CVG and } \sum v_n \text{ DVG}) \Rightarrow (\sum u_n + v_n \text{ CVG})$
- **Necessary condition of convergence**
$$\sum u_n \text{ is convergent} \Rightarrow \lim_{n\to+\infty} u_n = 0$$

### Series of nonnegative terms

- $\sum u_n$ is a series of nonnegative terms if, $\forall n \in \mathbb{N}, u_n \geq 0$.
- $\sum u_n$ is a series of nonnegative terms, $(S_n)$ the sequence of partial sums: $\sum u_n$ converges $\Leftrightarrow u_n$ is upper-bounded
- Let $(u_n)$ and $(v_n)$ be two sequences such that $\forall n \in \mathbb{N}, 0 \leq u_n \leq v_n$
$$\begin{cases} \sum v_n \text{ CVG} \Rightarrow \sum u_n \text{ CVG} \\ \sum u_n \text{ DVG} \Rightarrow \sum v_n \text{ DVG} \end{cases}$$
- **Geometric series:** $\sum q^n$ CVG $\Leftrightarrow |q| < 1$
- **Riemann series**: $\sum \frac{1}{n^\alpha}, \alpha \in \mathbb{R}$
    - $\sum \frac{1}{n^\alpha}$ converges $\Leftrightarrow \alpha > 1$
- **Criteria of comparison**
  Let $(u_n)$ and $(v_n)$ be two nonnegative real sequences.
    - $u_n = o(v_n) \Rightarrow \begin{cases} \sum v_n \text{ CVG} \Rightarrow \sum u_n \text{ CVG} \\ \sum u_n \text{ DVG} \Rightarrow \sum v_n \text{ DVG} \end{cases}$
    - $u_n \underset{n\to+\infty}{\sim} v_n \Rightarrow \sum u_n$ and $\sum v_n$ are of the same nature
    - $u_n = O(v_n) \Rightarrow$ same as $u_n = o(v_n)$
- Reminder:
    - $u_n = o(v_n) \Leftrightarrow \frac{u_n}{v_n} \xrightarrow[n\to+\infty]{} 0$
    - $u_n \underset{n\to+\infty}{\sim} v_n \Leftrightarrow \frac{u_n}{v_n} \xrightarrow[n\to+\infty]{} 1$
    - $u_n = O(v_n) \Leftrightarrow \frac{u_n}{v_n}$ is bounded towards $n \to +\infty$
    - $u_n + o(u_n) \sim u_n$
- Let $(u_n)$ be a real sequence of nonnegative terms. Then
$$\sum(u_n - u_{n-1}) \text{ CVG} \Leftrightarrow (u_n) \text{ CVG}$$
- **Riemann's rule**
  Let $(u_n)$ be a real nonnegative sequence. If $\exists \alpha > 1$ such that $n^\alpha u_n \xrightarrow[n\to+\infty]{} 0$ then $\sum u_n$ CVG
- **D'Alembert/Cauchy test**
$$\text{If either } \begin{cases} \frac{u_{n+1}}{u_n} \to l \\ \sqrt[n]{u_n} \to l \end{cases} \text{ then } \begin{cases} l < 1 \Rightarrow \sum u_n \text{ CVG} \\ l > 1 \Rightarrow \sum u_n \text{ DVG} \\ l = 1 \Rightarrow ? \end{cases}$$
  Using the ratio, it is a D'Alembert test. Using the root, it is a Cauchy test.

### Series of arbitrary terms

- **Alternating sequence**
  Let $(u_n)$ be a real sequence. $(u_n)$ is an alternating sequence if there exists a nonnegative sequence $(v_n)$ such that for all $n \in \mathbb{N}$: $u_n = (-1)^n v_n$ or $u_n = (-1)^{n+1} v_n$
- **Alternating series**
  If $u_n$ is an alternating sequence, then the series $\sum u_n$ is called an alternating series.

- Let $(u_n)$ be a real alternating sequence
  $(u_n)$ is decreasing and $\lim_{n\to+\infty} u_n = 0$, then $\sum u_n$ CVG

$$\left.\begin{array}{c}(|u_n|) \text{ is decreasing} \\ \lim\limits_{n\to+\infty} u_n = 0\end{array}\right\} \Rightarrow \sum u_n \text{ CVG}$$

- We say $\sum u_n$ converges absolutely if the series $\sum |u_n|$ converges.
  - $\sum u_n$ converges absolutely $\Rightarrow \sum u_n$ converges
  - We say $\sum u_n$ is semi-convergent (or conditionally convergent) if it converges but does not converge absolutely.
- Let $\alpha \in \mathbb{R}$. Then $\sum \frac{(-1)^n}{n^\alpha}$ CVG $\Leftrightarrow \alpha > 0$