Oscillations and Waves - BiteSized

lr17771

January 2019

1 Introduction

Oscillations and waves is the shortest section of the Classical Physics I course, consisting of only one system, solved in a variety of ways. Because of this it was glossed over very fast in the first few lectures, meaning although it is simple, its probably the part that people are least familiar with.

2 Single Mass on a Spring

This is the simpler system that should be familiar from both A-level and first year mechanics. You may call it a mass on a spring, but here in second year, us big boys call it a harmonic oscillator.

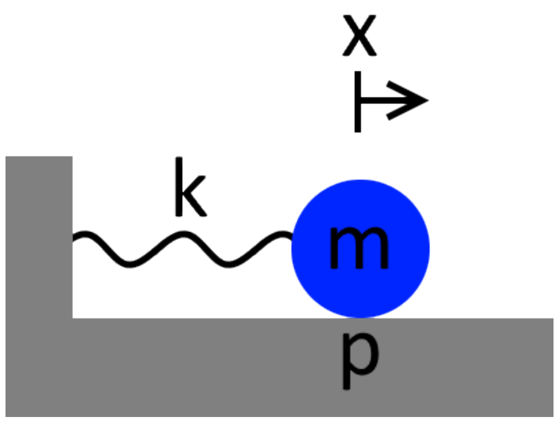

Figure 1: Double mass on a system of springs.

This is fairly easy to explain, to begin with Newton's second law is applied.

$$F = ma$$

$$-kx = m\ddot{x}$$

$$\ddot{x} = -\frac{k}{m}x$$

The motion here is linear, as can be seen from the equations.

It is standard to define this equation in terms of its angular velocity, ω.

$$\ddot{x} = -\omega^2 x$$

Where ω = -$\sqrt{\frac{k}{m}}$
As we know the angular velocity of the motion we can define it as the real part of a phasor.

$$x = \Re(Ae^{i(\theta+\delta)})$$

The force equation can be integrated to find energy stored.

$$E = \int kx \;\; dx$$

$$E = \frac{kx^2}{2}$$

3 Double Mass on a Spring

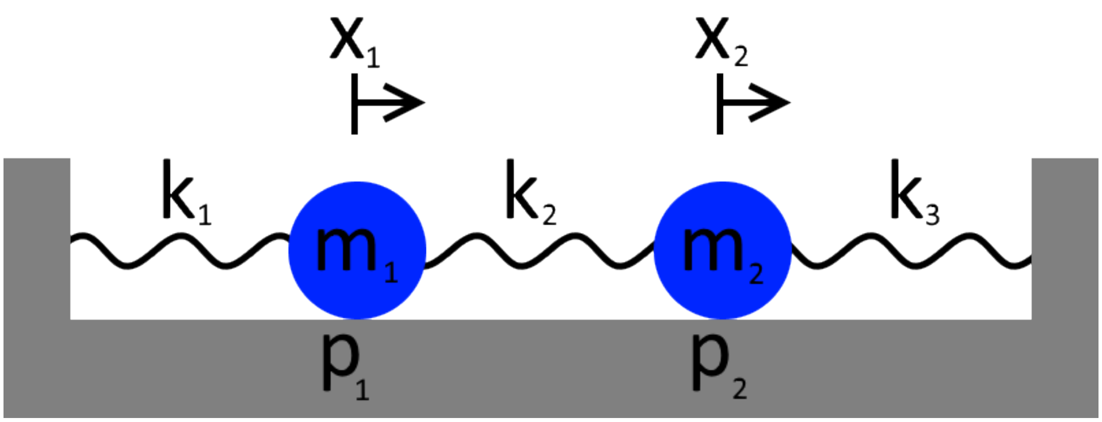

Figure 2: Double mass on a system of springs.

This is the system that is studied in this part of the course. This is an expansion of the system from the first year, going from a single mass on the end of a spring, to two masses connected to each other, as well as a wall either side, with a spring. For convenience the position of each mass is described as x_1 and x_2.

4 Simple Equations of Motion

By applying Newton's second law to each particle individually, basic equations of motion can be derived.

$$(x_2 - x_1)k_2 - x_1 k_1 = m_1 \ddot{x}_1$$

$$-x_2 k_3 - (x_2 - x_1)k_2 = m_2 \ddot{x}_2$$

As can be seen, the motion of each mass is affected by the other, they are not independent.

5 Matrix Algebra

These equations can be we-written in terms of matrices in order to make it simpler to manipulate.

$$\begin{bmatrix}(x_1k_1 + x_1k_2) & x_2k_2 \\ x_1k_2 & -(x_2k_2 + x_2k_3)\end{bmatrix} = \begin{bmatrix}m_1\ddot{x}_1 \\ m_2\ddot{x}_2\end{bmatrix}$$

$$-\begin{bmatrix}k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} = \begin{bmatrix}m_1 & 0 \\ 0 & m_2\end{bmatrix}\begin{bmatrix}\ddot{x}_1 \\ \ddot{x}_2\end{bmatrix}$$

To make these easier to understand we define some of these matricies to them resemble the single mass on a spring.

$$M = \begin{bmatrix}m_1 & 0 \\ 0 & m_2\end{bmatrix}$$

$$K = \begin{bmatrix}k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3\end{bmatrix}$$

$$\underline{\ddot{x}} = \begin{bmatrix}\ddot{x}_1 \\ \ddot{x}_2\end{bmatrix}$$

$$\underline{x} = \begin{bmatrix}x_1 \\ x_2\end{bmatrix}$$

Leading to:

$$-K\underline{x} = M\underline{\ddot{x}}$$

6 Deriving Normal Co-Ordinates

Because both particles are oscillating it is assumed they can be represented as a complex exponential.

$$\begin{bmatrix}x_1 \\ x_2\end{bmatrix} = \begin{bmatrix}c_1 \\ c_2\end{bmatrix} e^{i\omega\theta} \implies \begin{bmatrix}\ddot{x}_1 \\ \ddot{x}_2\end{bmatrix} = -\omega^2 \begin{bmatrix}c_1 \\ c_2\end{bmatrix} e^{i\omega\theta} \implies \begin{bmatrix}\ddot{x}_1 \\ \ddot{x}_2\end{bmatrix} = -\omega^2 \begin{bmatrix}x_1 \\ x_2\end{bmatrix}$$

By putting these back into the original equation it can be solved. To simplify the maths $k_1 = k_2 = k_3$ and m_1=m_2, meaning M reduced to mI.

$$-K\begin{bmatrix}c_1 \\ c_2\end{bmatrix} e^{i\omega\theta} = -mI\omega^2 \begin{bmatrix}c_1 \\ c_2\end{bmatrix} e^{i\omega\theta}$$

$$\begin{bmatrix}2k & -k \\ -k & 2k\end{bmatrix}\begin{bmatrix}c_1 \\ c_2\end{bmatrix} = m\omega^2 \begin{bmatrix}c_1 \\ c_2\end{bmatrix}$$

$$\frac{k}{m}\begin{bmatrix}2 & -1 \\ -1 & 2\end{bmatrix}\begin{bmatrix}c_1 \\ c_2\end{bmatrix} = \omega^2 \begin{bmatrix}c_1 \\ c_2\end{bmatrix}$$

$$\left(\frac{k}{m}\begin{bmatrix}2 & -1 \\ -1 & 2\end{bmatrix} - \omega^2\right)\begin{bmatrix}c_1 \\ c_2\end{bmatrix} = 0$$

This is a form of the eigenvalue equation, so we can solve for ω.

$$det\left(\begin{bmatrix}\frac{2k}{m} & -\frac{k}{m}\\ -\frac{k}{m} & \frac{2k}{m}\end{bmatrix} - \omega^2 I\right) = 0$$

$$det\left(\begin{bmatrix}\frac{2k}{m} - \omega^2 & -\frac{k}{m}\\ -\frac{k}{m} & \frac{2k}{m} - \omega^2\end{bmatrix}\right) = 0$$

This leads to the value for ω^2 being ω_1^2 = $\frac{k}{m}$ and ω_2^2 = $\frac{3k}{m}$. we can now put these values for ω back into the eigenvalue equation.

6.1 ω_1

$$\begin{bmatrix}\frac{2k}{m} - \frac{k}{m} & -\frac{k}{m}\\ -\frac{k}{m} & \frac{2k}{m} - \frac{k}{m}\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix} = 0$$

$$\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix} = 0$$

$$\begin{bmatrix}c_1 - c_2\\ -c_1 + c_2\end{bmatrix} = 0$$

$$c_1 = c_2$$

6.2 ω_2

$$\begin{bmatrix}\frac{2k}{m} - \frac{3k}{m} & -\frac{k}{m}\\ -\frac{k}{m} & \frac{2k}{m} - \frac{3k}{m}\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix} = 0$$

$$\begin{bmatrix}-1 & -1\\ -1 & -1\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix} = 0$$

$$\begin{bmatrix}-c_1 - c_2\\ -c_1 - c_2\end{bmatrix} = 0$$

$$c_1 = -c_2$$

This leads to the normalised co-ordinates for this system being $\begin{bmatrix}1\\ 1\end{bmatrix}$ and $\begin{bmatrix}-1\\ 1\end{bmatrix}$, let us call these q_1 and q_2 respectively. Each of these represent a "normal mode" of oscillation and there are as many of these as there are degrees of freedom (in this case masses) in the system.

7 Meaning of Normal Co-Ordinates

While these numbers appear meaningless, they actually have physical meaning, much like the θ and τ in polar co-ordinates. Note that the co-ordinates are defined in terms of the relation between c_1 and c_2 the negatives of these 2x1 matrices are still valid co-ordinate systems, the negative would only change the positive direction for each co-ordinate, think of it like a scale factor of -1. The negatives of these systems would be the shown velocity arrows pointing in the opposite directions. Any state of the system can be written out as a superposition of these normal modes.

7.1 $\mathbf{q}_1$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ represents the two particles travelling in the same direction, with the same speed.

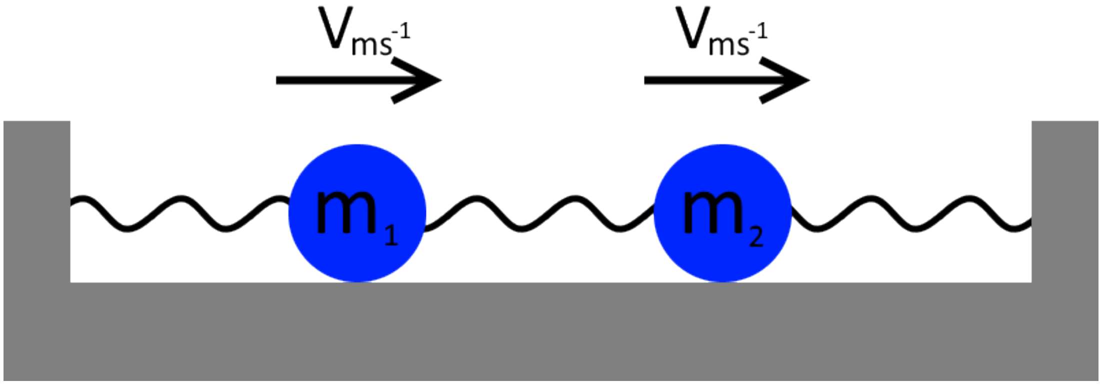

Figure 3: The two particles in the double mass on a spring system traveling in the same direction with the same speed.

7.2 $\mathbf{q}_2$

$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ represents the two particles travelling in the opposite directions, with the same speed, but opposite velocity.

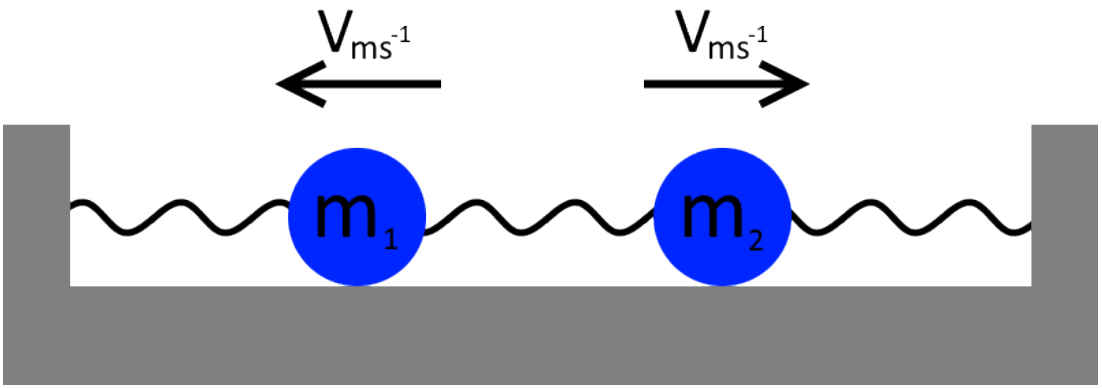

Figure 4: The two particles in the double mass on a spring system traveling in opposite directions with the same speed.

As these have been derived from the equations in this way, any state of the system can be represented by a combination of these two states.

8 Converting to Normal Co-Ordinates

Converting to these normal co-ordinates is fairly easy once they have been calculated, and it uses our favorite mathematical method, $\begin{bmatrix} m & a & t \\ r & i & c \\ i & e & s \end{bmatrix}$

To begin with, the two matrices of q_1 and q_2 are placed together to form a 2x2 matrix. Note that these could be combined in a different order, this would not change the maths and is perfectly valid, the final solutions obtained for $\ddot{q}_1$ and $\ddot{q}_2$ would just be be the other way around.

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} and \begin{bmatrix} -1 \\ 1 \end{bmatrix} \implies \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

This matrix will be called P. This matrix is important because it defines how to convert between normal X co-ordinates and our new normalised co-ordinates Q.

$$P^{-1}X = Q$$

$$PQ = X$$

By subbing in the values of this example we can obtain the conversions between X and Q co-ordintes.

$$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} X = Q$$

$$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2}x_1 & \frac{1}{2}x_2 \\ -\frac{1}{2}x_1 & \frac{1}{2}x_2 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$\frac{1}{2}x_1 + \frac{1}{2}x_2 = q_1$$

$$-\frac{1}{2}x_1 + \frac{1}{2}x_2 = q_2$$

Because we are defining our own co-ordinate system we can scale it to our will. As we can see if we double the scale of the new system the equations become much easier to handle.

An example here is defining your own arbitrary co-ordinate , such that  = 27x, addition and subtraction would still work fine, however it would be easier to do these with 27.

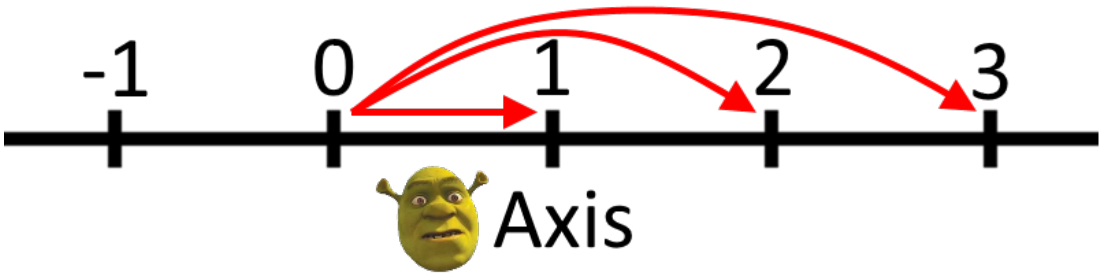

Figure 5: Adding together factors of 27 is really easy in my new co-ordinate system simply because of the scale factor, other than that it doesn't really change anything.

$$x_1 + x_2 = q_1$$

$$-x_1 + x_2 = q_2$$

These can then be combined to give equations in terms of x_1 and x_2

$$x_1 = \frac{1}{2}(q_1 - q_2) \implies \ddot{x}_1 = \frac{1}{2}(\ddot{q}_1 - \ddot{q}_2)$$

$$x_2 = \frac{1}{2}(q_1 + q_2) \implies \ddot{x}_2 = \frac{1}{2}(\ddot{q}_1 + \ddot{q}_2)$$

These can then be put back into the equations derives in section 4; remembering at this point that all ks and ms are equal, but some are more equal than others.

$$-2x_1k + x_2k = m\ddot{x}_1$$

$$-2\left[\frac{1}{2}(q_1 - q_2)k\right] + \frac{1}{2}(q_1 + q_2)k = m\frac{1}{2}(\ddot{q}_1 - \ddot{q}_2)$$

$$x_1k - 2x_2k = m\ddot{x}_2$$

$$\frac{1}{2}(q_1 - q_2)k - 2\left[\frac{1}{2}(q_1 + q_2)\right]k = m\frac{1}{2}(\ddot{q}_1 + \ddot{q}_2)$$

These can then be used to to get equations in terms of $\ddot{p}_1$ and $\ddot{q}_2$. We define ω as $\frac{k}{m}$.

$$\ddot{q}_1 = -\omega^2 q_1$$

$$\ddot{q}_2 = -3\omega^2 q_2$$

Because, in this system, the acceleration of each co-ordinate is independent of each other, they can be considered completely separate, and no energy will be transferred between each mode of oscillation. This means all of the energy is mutually exclusive of the other, so the energy of the total system is the energy of each mode added together.

$$E_{Total} = E_{Mode1} + E_{Mode2} + E_{Mode3} + \ldots$$