Methods of Theoretical Physics - BiteSized

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1 Introduction

You thought you were done with Maths after Christmas? Jokes on you, we have months of work on calculus and techniques for complex numbers! Honestly this is quite a nice course, rather than learning to solve an equation in an arbitrary form, here we go back to basics and learn to do basic calculus on complex functions.

2 What is an Complex Number?

To this point we have been dealing with numbers on the number line, however in Theoretical this is thinking a bit one-dimension-ally. To think about this line in the context of complex numbers is to imagine a this line turning into a 2D plane were a number can exist at any point.

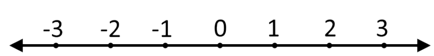

Figure 1: A one dimensional number line, on which all real numbers are located.

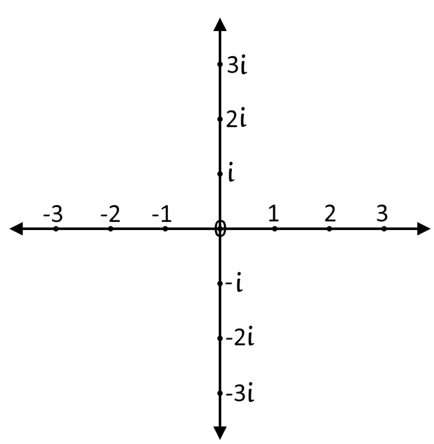

Figure 2: The one dimensional number line expanded into two dimensions to form a plane in which all numbers exist on. Note that the standard number line is still a part of this plane, it is just an infinitesimally small region.

As may be obvious from this picture the numbers on this plane carry magnitude and direction, because of this, these numbers are a vector quantity and must be referred to by two in-dependant scalars.

Figure 3: complex numbers are a vector because they have both magnitude and direction.

To differentiate between the standard number line and the other axis that defines the complex plane the new axis is in terms of a new imaginary unit i that is defined as the square root of -1.

$$i = \sqrt{-1}$$

Because of this, in Cartesian co-ordinates imaginary numbers (notation z) are written as:

$$z = x + iy$$

Or the imaginary plane can be thought of as a real vector space:

$$1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

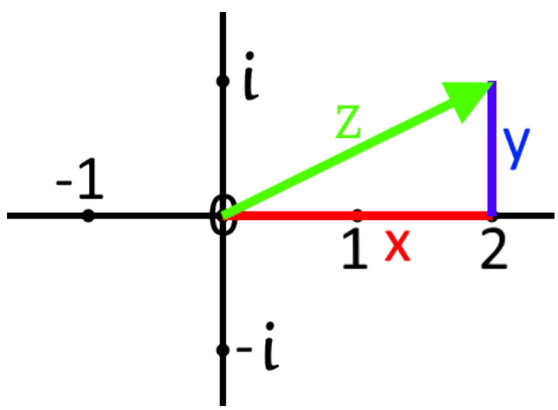

Figure 4: A diagram of the imaginary vector space with the point 2+i shown with its Cartesian components of x and y.

As stated, two scalars are needed to specify a place on the imaginary plane, this can be in the form of Cartesian co-ordinates, or in polar co-ordinates.

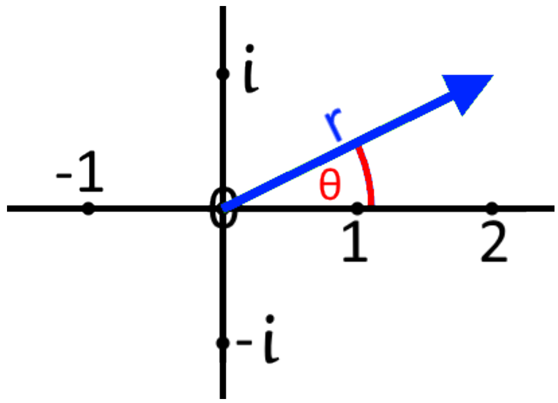

Figure 5: A diagram of the imaginary vector space with the point 2+i shown with its polar components of r and θ.

By defining the polar co-ordinates in this way an imaginary number can be written as:

$$z = r(\cos(\theta) + i\sin(\theta))$$

Where r is the distance from the origin to the complex number, |z|. the angle from the origin between the positive x- axis and the complex number.

$$r =\mid z \mid= x^2 + y^2$$

$$\theta = \arctan\left(\frac{y}{x}\right)$$

This is useful because it can be simplified by writing it in terms of Eula's equation:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$z = re^{i\theta}$$

For reference, in the complex plane all rotations are defines as being positive when anticlockwise and negative when anti-clockwise.

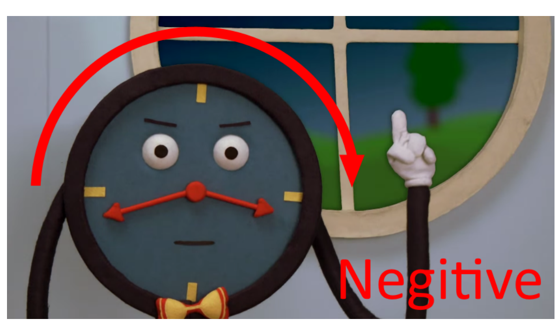

Figure 6: Clockwise is the negative direction in the imaginary plane.

Choosing the correct co-ordinate system is important as it can save a lot of number work! For example addition is easier to do in Cartesian co-ordinates and multiplication is easier to do in Polar co-ordinates.

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) = \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} e^{i \arctan(\frac{y_1+y_2}{x_1+x_2})}$$

$$z_1 * z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2) = r_1 r_2 e^{\theta_1 + \theta_2}$$

The conjugate of a complex number (z*) is the original number with its complex components magnitude reversed.

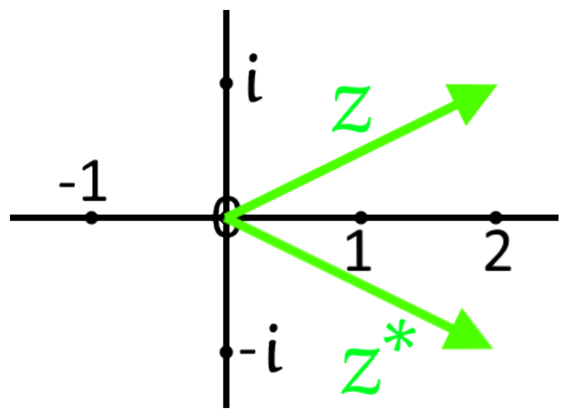

Figure 7: A complex number shown on an argand diagram with its conjugate.

$$z^* = x_z - iy_z = r_z e^{-i\theta_z}$$

The conjugates create a way to divide complex numbers in Cartesian co-ordinates, as we don't know how to divide by a number with complex and real components, but a number multiplied by its conjugate is a real number.

$$\frac{z_1}{z_2}\left(\frac{\mathbb{C}}{\mathbb{C}}\right) = \frac{z_1}{z_2} \times \frac{z_2^*}{z_2^*} = \frac{z_1 z_2^*}{z_2 z_2^*}\left(\frac{\mathbb{C}}{\mathbb{R}}\right)$$

Although it is still <u>much</u> easier to divide in polar co-ordinates.

3 Triangle Inequality

When adding two complex numbers together, the magnitude of the two numbers added together will be less than or equal to the sum of the two individual magnitudes. This can be seen algebraically but there is an intuitive way it can be demonstrated on an argand diagram.
An imaginary number can be represented by a line from the origin to the point the number resides on the argand diagram. The magnitude of this line is its length. Adding numbers can be shown as drawing these numbers out tip to tail, and the magnitudes of the added numbers is the distance from the origin to the end point.

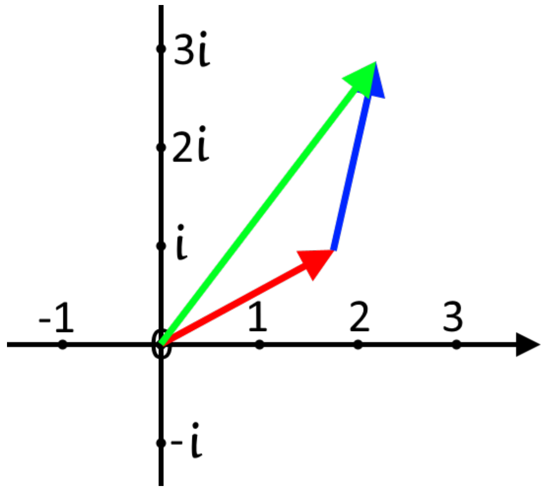

Figure 8: An argand diagram showing two complex numbers being added together to form another number.

From this plot, it is clear that the magnitude of the final number is related to the magnitudes of the original two numbers, by the difference of the argument (θ) between the two numbers.

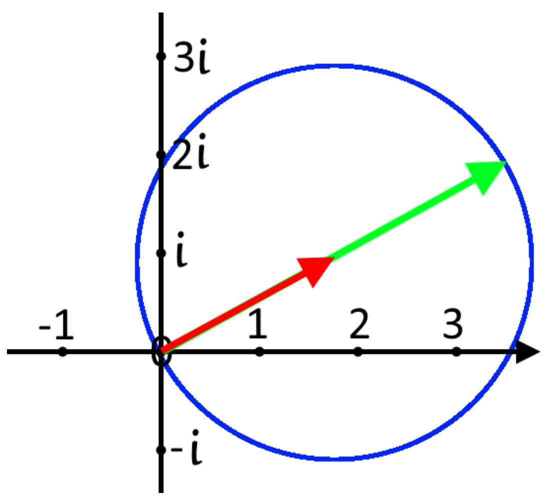

Figure 9: An argand diagram showing how the two complex numbers shown above are added together when the argument of the second number changes, the maximum value is shown in green.

As can be seen above the magnitude of the final number varies dramatically with the difference in argument of the two initial numbers. The magnitude of the final number is at maximum when the argument of the two numbers are the same.

$$| z_1 + z_2 | = || z_1 | + | z_2 | \cos(\Delta\theta) |$$

$$| z_1 + z_2 | \leq | z_1 | + | z_2 |$$

This can then be applied to summing N complex numbers.

$$| \sum_i z_i | \ \leq \ \sum_i | z_i |$$

And taking the limit this goes to an integral.

$$| \int z \, dz | \ \leq \ \int | z | \ dz$$

4 Multivaluedness

Because complex numbers can be expressed in terms of sinusoidal functions the multi valued-ness of these functions also carries over. Many to one functions are allowed because for the function at a point, there is a single solution; the fact it is not unique does not matter.

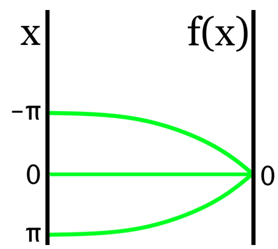

Figure 10: A sketch showing many inputs of a function having the same output. Many to one function.

However when the inverse is carried out this is not alright, because the function will be one to many, so each input will have more than one output.

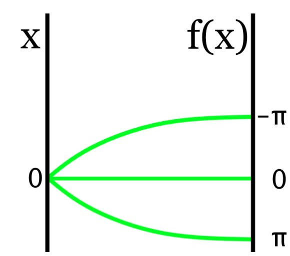

Figure 11: A sketch showing a single input of a function having more than one output. One to many function.

These kinds of functions are scary to me, but even more so to a computer, it wants to give you back a number, but it does not know which one! To help out the computer we define *principle* values to help us think about these problems. If there is more than one (or possible infinite) values we give the value that is part of the principle values. In <u>this course</u> we define our principle values to be:

$$0 \leq \theta < 2\pi$$

As an example, $\arccos(1) = 2\pi n$, $n \epsilon \mathbb{Z}$, so the single answer given would be 0, as this is the only value that fits in out principle values.

Typically the way a function is written determines if all possible solutions are needed or if only the principle value is the needed. If the function is the multi valued favour then the function starts with a capitel, if the principle value is needed then the function starts with a lower case letter e.g.:

$$arccos(1) = 0$$

$$Arccos(1) = 2\pi n$$

As such, when we write complex numbers in polar form we reduce them to be inside these principle values.

$$5e^{\frac{5}{2}i\pi} = 5e^{(\frac{1}{2}+2n)i\pi} = 5e^{\frac{1}{2}i\pi}$$

Taking the natural log of a complex number is actually pretty cool, you get to use lots of log rules and it all works out in the end! Start by writing the number out in its polar form and go from there.

$$ln\left|Ae^{i\theta}\right|$$

$$ln\left|e^{ln|A|} \times e^{i\theta}\right|$$

$$ln\left|e^{ln|A|}\right| + ln\left|e^{i\theta}\right|$$

$$ln\left|A\right| + i\theta$$

Leading to the general expression that the natural log of a complex number is the natural logarithm of the numbers magnitude added to the (imagenary) argument of the number. As the argument of the number is multi-valued the logarithm is also, and so combines two functions, the single-valued logarithm and the multi-valued logarithm.

$$Ln\left|1+i\right| = ln\left|\sqrt{2}\right| + i\pi\left(\frac{1}{4} + 2n\right)$$

$$ln\left|1+i\right| = ln\left|\sqrt{2})\right| + i\frac{\pi}{4}$$

As has been shown above, complex logarithms follow the same "log rules" and exponential rules real numbers do. This can help to solve problems.

$$i^i$$

$$(e^{i\frac{\pi}{2}})^i$$

$$e^{-\frac{\pi}{2}}$$

5 Continuance

In order for a complex function to be continuance and analytic it must be well defined everywhere. Being well defined at all points does not necessarily mean it it has a value everywhere, as values can be found with limits. Much like in 2D functions limits can have different valued depending on which direction they are approached from, but it can only be said to have a value at the point if the different limits agree.

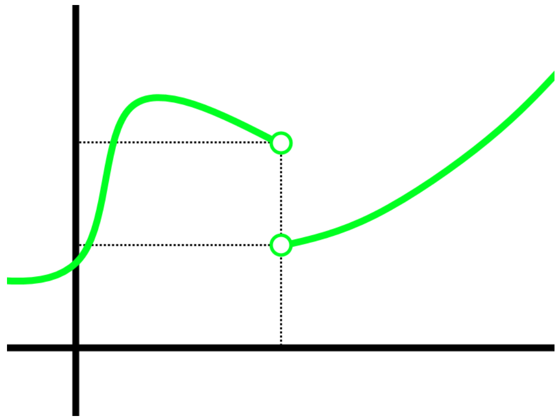

Figure 12: a real function that is not defined at a point because the limits approaching from each sides disagree.

A good example of this is derivatives, in real functions the derivative of a function is a single value, because it does not matter which direction the point is being approached from.
Note: some well defined functions are not differentiable.
This is true in the complex plain as well, however the point can be approached by any angle $0 < \theta 2\pi$.
In order for the function to be differentiable (also called analytic) we must check that the value from any approach are the same.
The space a point can be approached from is called the neighbourhood. The equation for this is $|Z_0 - Z| = \delta$

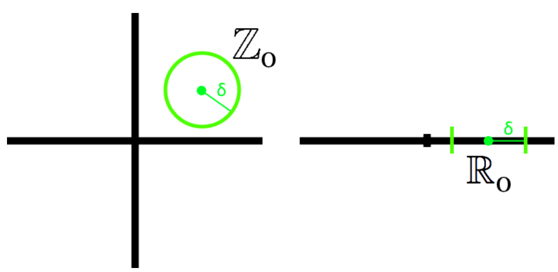

Figure 13: Neighborhood of a real and complex domain.

Derivatives can be taken of complex functions by taking a limit in the same way real functions can.

$$\frac{df}{dx} = \lim_{x \to 0} \frac{f(x+) - f(x)}{\delta x}$$

$$\frac{df}{dz} = \lim_{z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}$$

$$\frac{df}{dz} = \lim_{\delta x \to 0, \delta y \to 0} \frac{u(x + \delta x, y + \delta y) - u(x, y)}{\delta x + i\delta y} + i\frac{v(x + \delta x, y + \delta y) - v(x, y)}{\delta x + i\delta y}$$

This still does not allow us to take the derivative because there are two things going towards 0 and we don't know how quickly each of them are traveling. We can make this easier by looking at the simplest cases, holding either either δy or δx to be 0 so you are approaching Z_0 from parallel to 1 or i respectively.

$$\frac{df}{dz} = \lim_{\delta x \to 0} \frac{u(x + \delta x, y) - u(x, y)}{\delta x} + i\frac{v(x + \delta x, y) - v(x, y)}{\delta x} = \frac{du}{dx} + i\frac{dv}{dx}$$

$$\frac{df}{dz} = \lim_{\delta x \to 0} \frac{u(x + \delta x, y) - u(x, y)}{i\delta y} + i\frac{v(x + \delta x, y) - v(x, y)}{i\delta y} = \frac{du}{idy} + i\frac{dv}{idy} = -i\frac{du}{dy} + \frac{dv}{dy}$$

By setting these to be equal, which they have to be in order for the function to be analytic, and therefore differentiable, gives us the Cauchy-Riemann equations.

$$-i\frac{du}{dy} + \frac{dv}{dy} = \frac{du}{dx} + i\frac{dv}{dx}$$

$$-\frac{du}{dy} = \frac{dv}{dx}$$

$$\frac{dv}{dy} = \frac{du}{dx}$$

If these equations are satisfied, the equation is by definition analytic.

6 Complex Functions

Complex functions (analogously to real functions) take in a complex number, and give out a complex number, the two parts of the complex number can be treated as two independent variables.

$$f : \mathbb{C} \longrightarrow \mathbb{C}$$

$$x + iy :\longrightarrow u + iv$$

$$re^{i\theta} :\longrightarrow Re^{i\Theta}$$

In some cases it may be useful to think of the result as a function of two real variables.

$$f(z) = u(x, y) + iv(x, y)$$

However, this boils down to a $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ function, which is very hard to think about as 4D space is hard to visualise. Really the best way is to have one point in an Argand diagram going to another point on another Argand diagram.

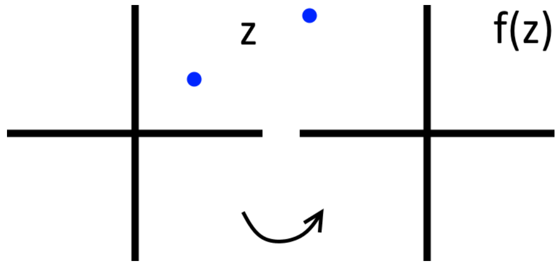

Figure 14: The input and output of a function (z^3) plotted on an argand diagram.

Because, as stated above, complex functions need to satisfy the Cauchy-Riemann equations in order to make sense, there are a lot of side effects complex functions fulfill to make them useful, as these functions are by definition, conservative.

These functions satisfy Laplace's equation:

$$\Delta^2 f = \frac{\delta^2 f}{\delta x^2} + \frac{\delta^2 f}{\delta y^2} = 0$$

Both $u(x, y)$ and $v(x, y)$ fulfill this separately and so are harmonic functions in there own right.

Partial derivatives commute:

$$\frac{\delta f}{\delta x \delta y} = \frac{\delta f}{\delta y \delta x}$$
$$\frac{\delta u}{\delta x \delta y} = \frac{\delta u}{\delta y \delta x}$$
$$\frac{\delta v}{\delta x \delta y} = \frac{\delta v}{\delta y \delta x}$$

The grad of each component of the complex function dotted together is zero (if C-R):

$$\Delta v \cdot \Delta u = \begin{bmatrix} \frac{\delta v}{\delta x} \\ \frac{\delta v}{\delta y} \end{bmatrix} \cdot \begin{bmatrix} \frac{\delta u}{\delta x} \\ \frac{\delta u}{\delta y} \end{bmatrix} = \frac{\delta v}{\delta x}\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y}\frac{\delta u}{\delta y} = \frac{\delta v}{\delta x}\frac{\delta v}{\delta y} - -\frac{\delta v}{\delta y}\frac{\delta u}{\delta x} = 0$$

7 Complex Integration

Similarly to real functions, an integral of a complex function can be taken. These are fairly intuitive and behave in a similarly way to a line integral of a real 2D function. Unlike differentials, integrals do not require the function to obey CR.

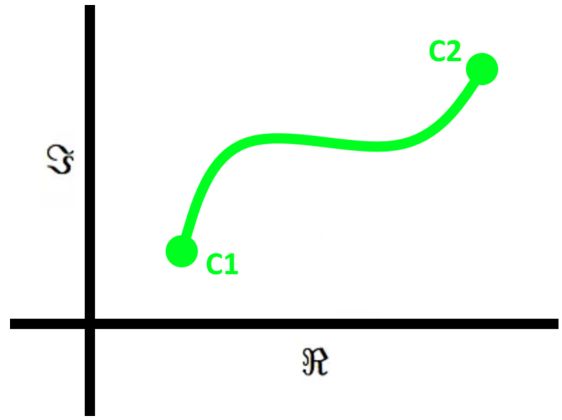

Figure 15: Firstly a contour must be defined, and each variable defined in terms of another variable t x(t) y(t) much like a parametric function.

From here an integral can be manipulated to give two real integrals, one with a factor of i.

$$\int z(x+iy)dz$$

$$\int u(x,y)+iv(x,y)dz$$

$$\int u(t)+iv(t)dz$$

$$z = x+iy$$

$$\frac{dz}{dt} = \frac{dx}{dt} + i\frac{dy}{dt}$$

$$dz = \left(\frac{dx}{dt} + i\frac{dy}{dt}\right) dt$$

$$\int \left(\frac{dx}{dt} + i\frac{dy}{dt}\right) (u(t)+iv(t))\, dt$$

$$\int \left(\frac{dx}{dt}u(t) - \frac{dy}{dt}v(t)\right) dt + i \int \left(\frac{dx}{dt}v(t) + \frac{dy}{dt}u(t)\right) dt$$

8 Curtain Theorem

This theorem is rather self explanatory, if the magnitude of a function is less than a value f_0 in the range C_1 and C_2 then the function integrated over the range will always be equal or less than f_0 times the range.

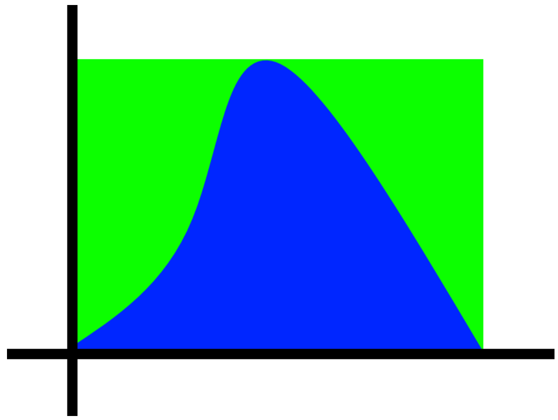

Figure 16: This easier to understand visually, you select an area that is guaranteed to be bigger that the function. green square > blue shape.

The The magnitude of a function is used here, so the blue function will never be negative. Think of this as the trapezium rule with a single strip, and never undershoots at any point.

9 Cauchy's Theorem

As you may have noticed, the conditions for a function to satisfy C-R means it is also a conservative field; as such most closed integrals are 0. The conditions for a closed integral being 0 are that the closed loop does not contain any singularities. Actually the shape of a closed loop has no effect on its value, only the singularities enclosed determine the value and so the shape can be heavily manipulated

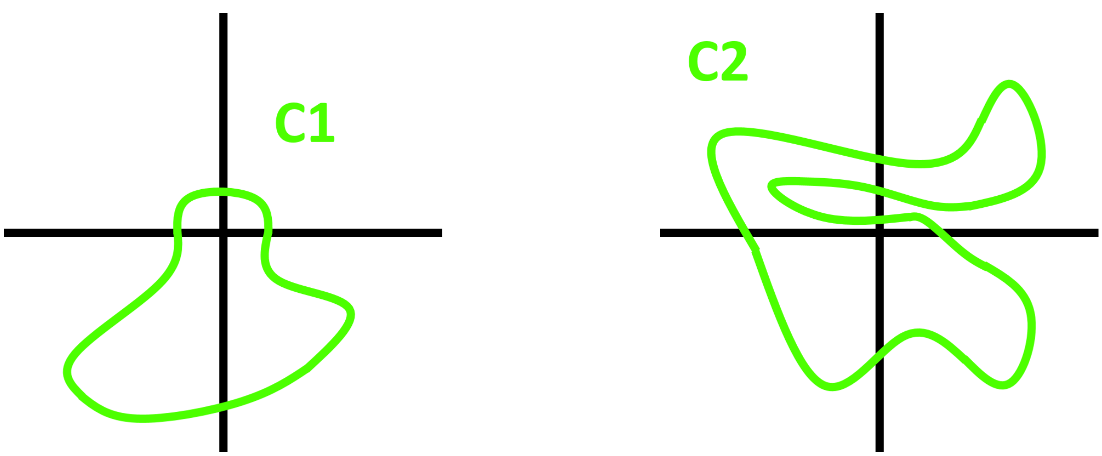

Figure 17: The contours above are of the function $\frac{1}{z}$, the only factor that defines the value is the singularities held in within the contour. As both contours shown include the single contour at (0, 0) so they are of the same value.

10 Cauchy's integral formula

$$\oint_c \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$$

This is a consequence of a complex function being a conservative field and Cauchy's theorem. This can be shown fairly easily by distorting the shape of a contour, but you just need to know the formula.

11 Complex Taylor Series

This isn't too hard to do, once given the formula the rest if fairly self explanatory assuming background knowledge of the $\mathbb{R}$ case.

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

The Taylor series is being expanded around a chosen z_0.The Taylor series converges to f(z) for $\mod z - z_0 \mod < \mod z - z_{singularity} \mod$ AKA z is within a circle originating at z_0 that extends to the closest singularity.

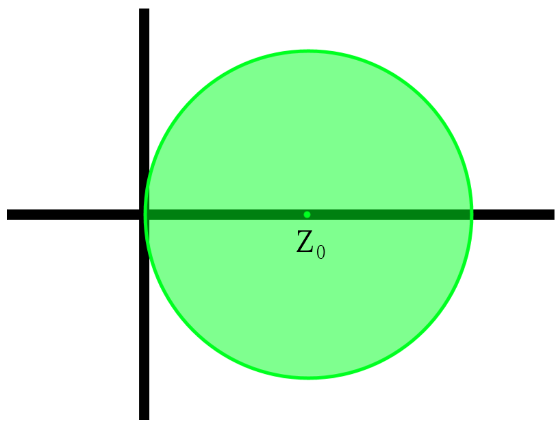

Figure 18: The radius of convergence of the function $\frac{1}{z}$. The Taylor series converges if the point is within the circle originated at z_0 with radius $\mod z - z_0 \mod$ as the only singularity is at 0 + 0i.

The Taylor series may or may not converge if z is on the radius of convergence.

12 Complex Power Series

Sometimes it is necessary to see if a power series converges at a point. There is little difference between this and the real case, so many of the same methods can be used. The direction of an imaginary number is irrelevant when it comes to a series converging, so only the magnitude is taken into account in methods analogous to real numbers. Ratio test:

$$\sum_{n=0}^{\infty} p_n, \text{ converges if } \lim_{x \to \infty} \left| \frac{p_{n+1}}{p_n} \right| < 1$$

When dealing with a power series the point the series is taken at matters.

$$\sum_{n=0}^{\infty} p_n z^n \longrightarrow \lim_{x\to\infty}\left|\frac{p_{n+1}z^{n+1}}{p_n z^n}\right| < 1 \longrightarrow \lim_{x\to\infty}\left|\frac{p_{n+1}}{p_n}\right||z| < 1 \longrightarrow |z| < \lim_{x\to\infty}\left|\frac{p_n}{p_{n+1}}\right|$$

This leads to a radius of convergence of the series, as the series will converge if the magnitude of where it converges as is under a certain Value.
Comparison test ect also work.
Handy reminder:

$$\sum_{n=0}^{\infty} z^n \longrightarrow 1 + z + z^2 + z^3.... \longrightarrow \frac{1}{1-z} \text{ for } z < 1$$

13 Laurent Series

Whereas the Taylor series converges between the point being expanded around and the closest singularity in a radius of convergence, the Laurent series can converge between two defined circles, with no singularities between them. The Laurent series does this by manipulating the function to converge in this region and having a power series that can go into negative powers.
A formal definition can be given as:

$$f(z) = \sum_{n=-\infty}^{\infty} p_n (z - z_0)^n$$

$$p_n = \frac{1}{2\pi i}\oint_{c_1} \frac{f(w)}{(w - z_0)^{n+1}} dw$$

However most of the time a function can be manipulated into a Laurent series by looking at the Taylor series of the different components.

$$\frac{e^z}{z^2} \longrightarrow \frac{\sum_{n=0}^{\infty}\frac{z^n}{n!}}{z^2} \longrightarrow \sum_{n=0}^{\infty}\frac{z^{n-2}}{n!} \longrightarrow \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2} + \frac{z}{6} + ...$$

Unfortunately sometimes we need to do more work to get the series to converge in different regions e.g. to avoid a singularity.

E.g. The function $\frac{1}{z}\frac{1}{1+z^2}$ has a singularity at i and -i so in order to get the full range of the function in a power series two functions bust be constructed, one function from 0$\leq |z| <$1 and one function from 1$< |z| \leq \infty$.

First case : 0$\leq |z| <$1

$$\frac{1}{z}\frac{1}{1+z^2}$$

$$\frac{1}{z}\frac{1}{1-(-z^2)}$$

$$0 \leq |z| < 1 \longrightarrow 0 \leq \left|z^2\right| < 1$$

$$\frac{1}{z}\sum_{n=0}^{\infty}(-1)^n z^{2n-1}$$

$$\frac{1}{z} - z + z^3 - z^5 + z^7 + ...$$

Second case : 1$<$ $|z| \leq \infty$

$$\frac{1}{z}\frac{1}{1+z^2}$$

$$\frac{1}{z}\frac{1}{z^2(z^{-2}+1)}$$

$$0 \leq |z| < 1 \longrightarrow 0 \leq \left|z^{-2}\right| < 1$$

$$\frac{1}{z^3}\frac{1}{1-(-z^{-2}))}$$

$$\frac{1}{z^3}\sum_{n=0}^{\infty}(-1)^n z^{-2n}$$

$$\frac{1}{z^3} - \frac{1}{z^5} + \frac{1}{z^7} - \frac{1}{z^9} + ...$$

Note that in order for a Laurent series to be defined at a point the ”neighbourhood” of the function at that point. This simply means that a singularity cannot be infinitely close to the point being expanded around.

14 Poles

This is a really interesting topic, and further reading is advised and can be found here.

Every singularity can be taken to be a pole, to find out more about this pole a Laurent series must be constructed around it and looked at. Each pole has an associated order and is defined as the largest non zero negative power in the Laurent series. If the pole has an order one it is defined to be a simple pole, while if the order is ∞ then the pole is called an essential singularity.

I like to think about this as how fast the function approaches the singularity and as such can be worked out intuitively a lot of the time.

$\frac{1}{z}$ Has a singularity at 0 of order 1 (simple pole).

$\frac{1}{z^2}$ Has a singularity at 0 of order 2.

$\frac{1}{z}\frac{1}{(1+z)^2}$ Has a singularity at 0 of order 1 (simple pole) and a singularity at -1 of order 2.

A pole of order m can be removed by multiplying the original function by $(z - z_0)^m$

15 Residue

The z^-1 term has special properties, the constant in front of this term is called the residue and is different depending on which point (z_0) is expanded around.

As stated before, a closed loop integral of an analytic function is always 0 if the loop does not contain any singularities. If a loop does contain singularity then the value of this integral is decided by the values of the residues calculated at the singularities contained.

$$\oint_c f(z)dz = 2\pi i \sum_k Res[f]_{z_k}$$

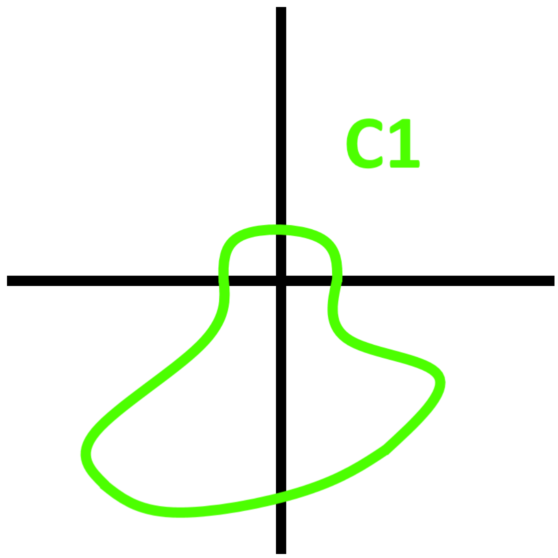

Figure 19: This is a contour over the function $\frac{1}{z}$, it contains a single singularity at 0.

The Laurent series of this function (at 0) is itself, so z^{-1}, making its residue one, and so the integral around this point is $2\pi i$.

Now that we have a link between a contour integral and a residue we can work backwards from here to find a another way to calculate the residue. By defining a function g(w) that is equal to f(w)(w-z_0)^m and using the standardised form of Cauchy's integral formula it can be shown that:

$$res\,[f]_{z=z_0} = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]_{z=z_0}$$

Depending on the equation, sometimes this is easier to calculate and sometimes it is harder.

16 Analytic Continuation

If two functions that are valid over different domains, and they agree on the area that the domains intersect, then they are on the same function.

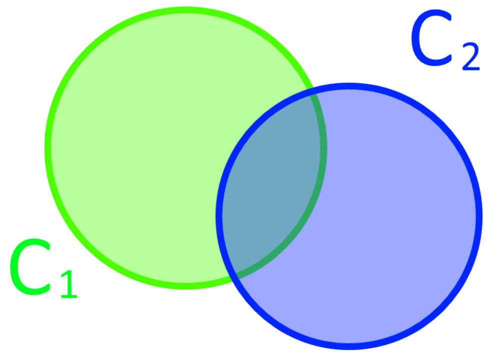

Figure 20: If C_1 and C_2 (different functions on different domains) produce the same value in the area C_1 = C_2 then the functions are analytic continuations of each other.

This is especially true with real functions, they are defined on the real line but can be continued into the complex plane.

17 Real Integrals

As with real integrals, complex integrals can be continued from the end of each other; as complex functions are conservative.

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

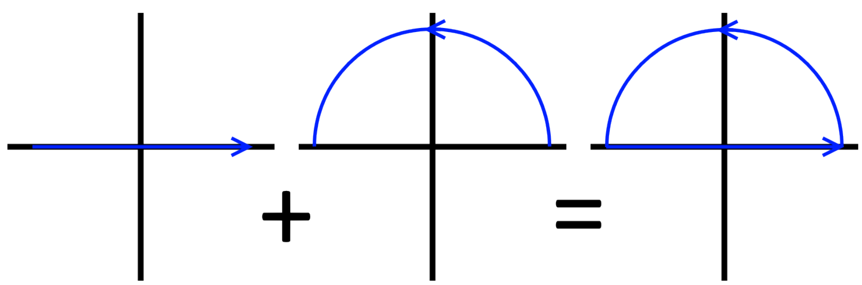

Figure 21: Adding up integrals to result a closed contour integral.

Combine this with the fact that closed contours are usually pretty easy to calculate and a real integral can be calculated but subtracting a complex line integral from a complex contour integral; lots of time this is much easier.

This is a bit easier with an example:

$$f(z) = \frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$$

This function has two simple poles, one at i and one at -i. Finding the integral of the function between -infinity and infinity:

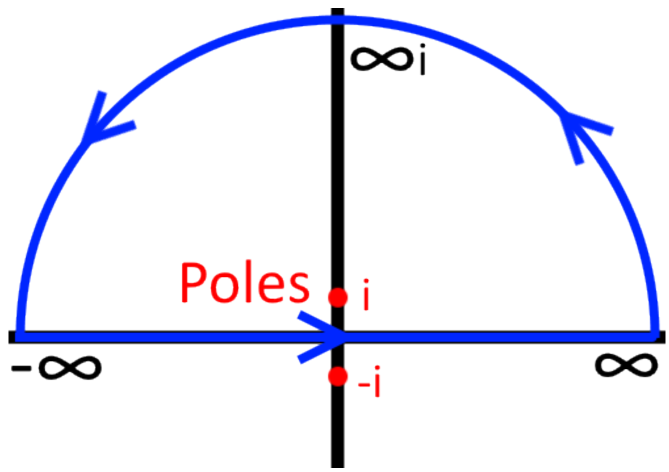

Figure 22: Diagram of the path of integration taken to turn this real integral into an imaginary one.

$$\int_{RealLine} f(z)dz + \int_{ImaginaryCurve} f(z)dz = \int_{ContourIntegral} f(z)dz$$

$$\int_{RealLine} f(x)dx = \int_{ContourIntegral} f(z)dz - \int_{ImaginaryCurve} f(z)dz$$

$$\int_{ImaginaryCurve} f(z)dz = \int_{-\infty}^{\infty} \frac{1}{z^2+1} dz \text{ on path: } z = \infty \times e^{i\theta}, 0 \leq \theta \leq 2\pi$$

As $\mid z \mid = \infty$ for the whole of this integral f(z) = 0 for the whole of this integral, because of this the integral of this function is adding up zeros at every point, and so when the integral is evaluated it is 0.

$$\int_{ImaginaryCurve} f(z)dz = 0$$

Looking at the functions it is clear that the singularities are simple poles, and by using the residue formula it can be seen that the poles (i and $-i$) have residues $\frac{1}{2i}$ and -$\frac{1}{2i}$ respectively. This leads to contour integrals of π and $-\pi$ respectively, the semicircle can be taken up or down and it will work fine as the downward semicircle will be negative due to the direction taken.

$$\int_{ContourIntegral} f(z)dz = \pi$$

$$\int_{RealLine} f(x)dx = \pi - 0$$

$$f(z) = \int_{-\infty}^{\infty} \frac{1}{z^2+1} dz = \pi$$

A complex integral can also be used in place of a trigonometric integral by replacing the trig functions by there exponential form.

$$cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}, sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

As This will transform a trig integral to a closed contour in the complex plane.

There is a general expression for a specific case.

$$\int_{-\infty}^{\infty} f(x)e^{iax}dz = 2\pi i \sum_k Res[f(z)e^{iaz}]_{z_k}$$

$$\int_{-\infty}^{\infty} f(x)cos(iax)dz = \Re[2\pi i \sum_k Res[f(z)e^{iaz}]_{z_k}]$$

$$\int_{-\infty}^{\infty} f(x)sin(iax)dz = \Im[-2\pi i \sum_k Res[f(z)e^{iaz}]_{z_k}]$$

18 Principle Value integrals

In order to deal with a contour on the path of the integral we want to take, we go around it in an infinitely small semi circle around the pole. Note that here we only deal with simple poles.]

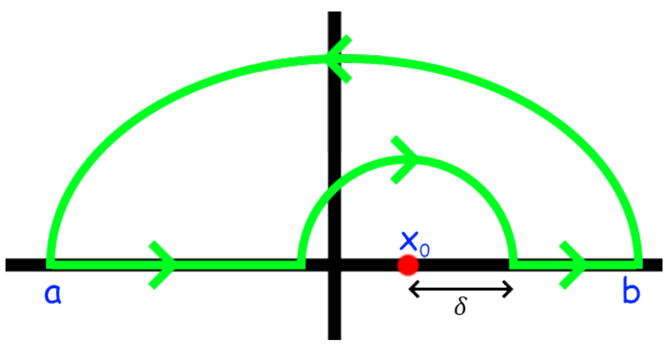

Figure 23: Breaking up a contour integral that would go over a pole with a integral that goes around the pole in a clockwise semicircle in the imaginary plane.

$$\int_{Contour} f(z)dz = \int_{a}^{x_0-\delta} f(z)dz - \int_{x_0-\delta}^{x_0+\delta} f(z)dz + \int_{x_0+\delta}^{b} f(z)dz + \int_{b}^{a} f(z)dz$$

Note that the semi circle avoiding the pole is negative because it is in the clockwise direction. The contour taken to avoid the poles is $z = z_0 + \delta e^{i\theta}$ This leads to $dz = i(z - z_0)d\theta$

$$\int_{Contour} f(z)dz = \int_{Contour} \frac{g(z)}{z - z_0}dz = \int_0^{\pi} i\frac{g(z)}{z - z_0}(z - z_0)d\theta$$

$$\lim_{\delta \to 0} \text{Contour around pole} = Z_0$$

$$\int_0^{\pi} ig(z_0)d\theta = ig(z_0)\int_0^{\pi} d\theta = i\pi g(z_0)$$

This can help with real integrals as when $\delta \longrightarrow 0$ the integral covers the whole of the real line

$$\lim_{\delta \to 0} \int_{a}^{x_0-\delta} f(z)dz + \int_{x_0+\delta}^{b} f(z)dz = \int_{a}^{b} f(z)dz$$

$$\int_{Contour} f(z)dz = \int_{a-Line}^{b} f(z)dz - \int_{x_0-\delta-Curve}^{x_0+\delta} f(z)dz + \int_{b-Curve}^{a} f(z)dz$$

$$\int_{a}^{b} f(x)dx = \int_{b-Curve}^{a} f(z)dz + \int_{x_0-\delta-Curve}^{x_0+\delta} f(z)dz$$

$$\int_{a}^{b} f(x)dx = 2\pi i \sum_{k} Res[f]_{z_k} + \pi i Res[f]_{z_0}$$