A1) $$\frac{d^2y}{dt^2} + \omega^2 y = 0$$

i) $$y = Ae^{i\omega t} + Be^{-i\omega t}$$

$$\frac{dy}{dt} = i\omega A e^{i\omega t} - i\omega B e^{-i\omega t}$$

$$\frac{d^2y}{dt^2} = -\omega^2 A e^{i\omega t} - \omega^2 B e^{-i\omega t}$$ ✓

ii) $$\boxed{y = y_0 \quad \frac{dy}{dt} = 0 \quad t = 0}$$

$$y = A + B \qquad \frac{dy}{dt} = i\omega A - i\omega B$$

$$y_0 = A + B \qquad 0 = i\omega A - i\omega B$$

$$0 = A - B$$

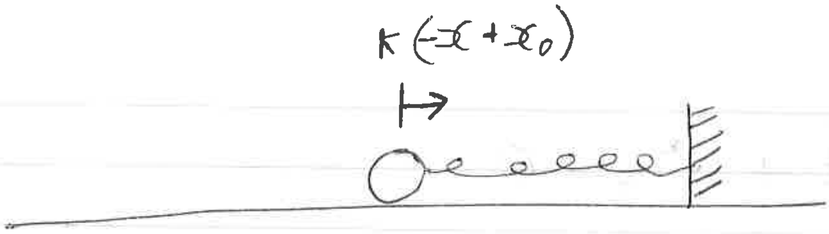
$$y_0 = A + A \qquad B = A$$

$$y_0 = 2A$$

$$\frac{y_0}{2} = A = B$$

$$\underline{y = \frac{y_0}{2} e^{i\omega t} + \frac{y_0}{2} e^{-i\omega t}}$$ ✓

iii) a)



$$F = ma$$

$$k(-x + x_0) = m\left(\frac{d^2x}{dt^2}\right)$$

$$\frac{k}{m}(-x + x_0) = \left(\frac{d^2x}{dt^2}\right)$$

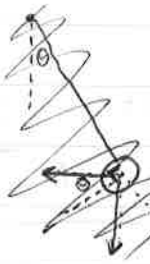
$$\sqrt{\frac{k}{m}}^{\,2}(-x + x_0) = \frac{d^2x}{dt^2}$$

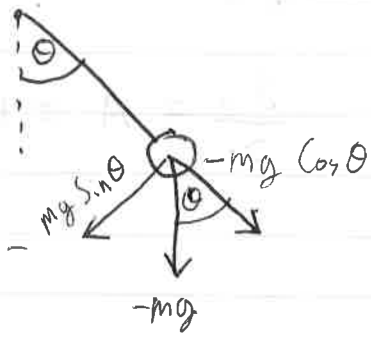
$$\sqrt{\frac{k}{m}} = \omega \qquad x_0 = 0$$

$$-\omega^2 x = \frac{d^2x}{dt^2}$$

$$0 = \frac{d^2x}{dt^2} + \omega^2 y \quad ✓$$

b)





$$\Gamma = I\alpha$$

$$F = ma \qquad -l\,mg\sin\theta = I\ddot{\theta}$$

$$-\cancel{l}\,\cancel{m}g\sin\theta = \cancel{m}l^{\cancel{2}}\,\ddot{\theta}$$

$$-\frac{g}{l}\sin\theta = \ddot{\theta}$$

$$-\sqrt{\frac{g}{L}}^{\,2}\sin\theta = \ddot{\theta} \qquad \sqrt{\frac{g}{L}} = \omega$$

$$\sin\theta = 0$$

$$-\omega^2\theta = \ddot{\theta}$$

$$0 = \ddot{\theta} + \omega^2\theta \quad ✓$$

iii)

c) $$\sum_i V_i = 0$$

$$V_{capacitor} + V_{inductor} = 0$$

$$\frac{Q}{C} + \frac{dI}{dt} L = 0$$

$$I = \frac{dQ}{dt} \qquad \frac{dI}{dt} = \frac{d^2Q}{dt^2}$$

$$\frac{1}{C} Q + \frac{d^2Q}{dt^2} L = 0$$

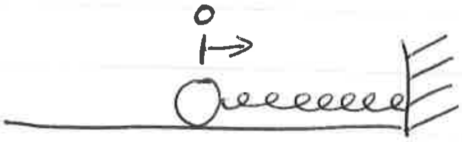
$$\left(\frac{1}{LC}\right) Q + \frac{d^2Q}{dt^2} = 0$$

$$\sqrt{\frac{1}{LC}}^2 Q + \frac{d^2Q}{dt^2} = 0 \qquad \sqrt{\frac{1}{LC}} = \omega$$

$$\omega^2 Q + \frac{d^2Q}{dt^2} = 0$$

$$\frac{d^2Q}{dt^2} + \omega^2 Q = 0 \quad \checkmark$$

iv)



drag term

$$\underbrace{}_{} F = ma$$

$$-kx + bv = ma$$

$$a = \frac{dv}{dt} \qquad v = \frac{dx}{dt} \qquad a = \frac{d^2x}{dt^2}$$

$$0 = ma - bv + kx$$

$$0 = m\frac{d^2x}{dt^2} - b\frac{dx}{dt} + kx$$

$$\text{term} \propto \frac{dx}{dt}$$

Constant b

A2

$$\text{energy} = \frac{1}{2}\beta\left(\frac{dy}{dt}\right)\left(\frac{dy}{dt}\right) + \frac{1}{2}\alpha(y)(y)$$

$$\frac{d(\text{energy})}{dt} = ~~\text{Consta}~~\ 0$$

$$2\frac{1}{2}\beta\left(\frac{dy}{dt}\right)\left(\frac{d^2y}{dt^2}\right) + 2\frac{1}{2}\alpha\left(\frac{dy}{dt}\right)(y) = 0$$

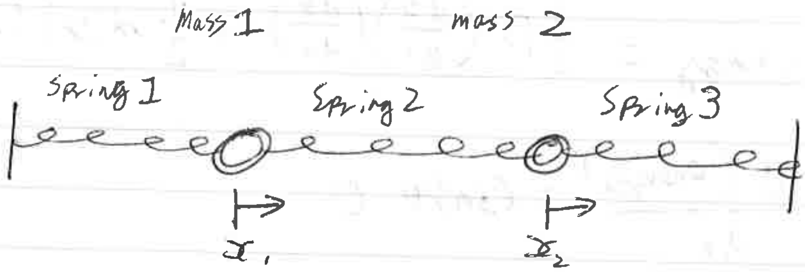
$$\underbrace{\left(\frac{dy}{dt}\right)}_{\text{trivial Solution}}\left(\beta\frac{d^2y}{dt^2} + \alpha y\right) = 0$$

$$\frac{d^2y}{dt^2} + \frac{\alpha}{\beta}y = 0$$

$$\omega^2 = \frac{\alpha}{\beta} \qquad \omega = \sqrt{\frac{\alpha}{\beta}}$$

$$\omega = \frac{2\pi}{T} \qquad T = \frac{2\pi}{\omega} \qquad T = 2\pi\sqrt{\frac{\beta}{\alpha}}$$

A3 i)



Spring Constant $= k$ all springs

mass $= m$ ~~all~~ both masses

for mass 1

$F = ma$

$F_{spring 2} - F_{spring 1} = m a_{mass 1}$

$k(x_2 - x_1) - k x_1 = m\ddot{x}_1$

$k(x_2 - 2x_1) = m\ddot{x}_1$

$-\frac{k}{m}(2x_1 - x_2) = \ddot{x}_1$

$-\omega_0^2(2x_1 - x_2) = \ddot{x}_1$

for mass 2

$F = ma$

$F_{spring 3} - F_{spring 2} = m a_{mass_2}$

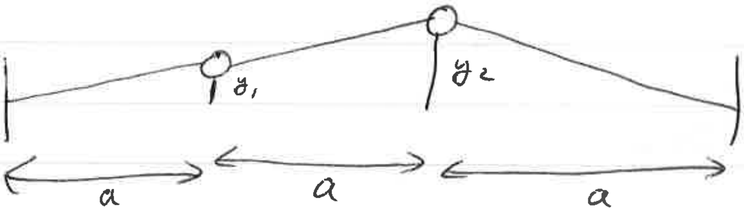
$k(x_2) - k(x_1 - x_2) = m\ddot{x}_2$

$k(2x_2 - x_1) = m\ddot{x}_2$

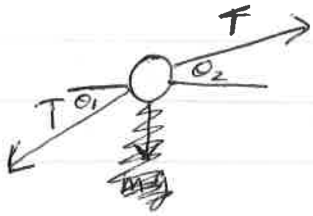
$-\frac{k}{m}(2x_2 - x_1) = \ddot{x}_2$

$-\omega_0^2(2x_2 - x_1) = \ddot{x}_2$ ✓

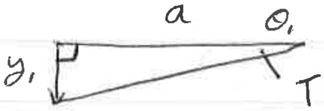
ii)



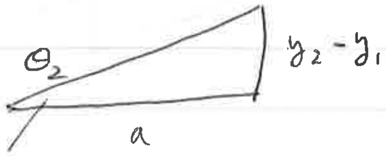
y_1



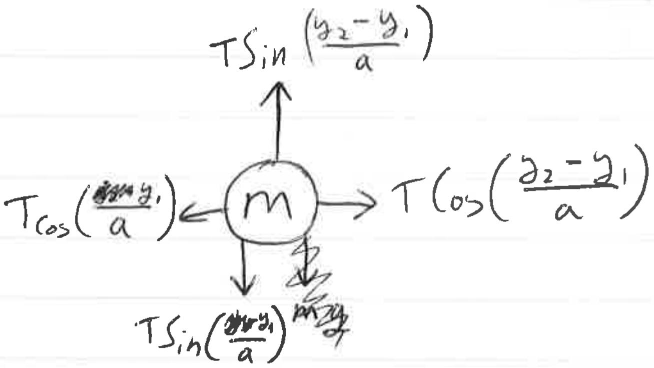
S^{o}_{h} C^{a}_{h} T^{o}_{a}



$\tan^{-1}\left(\frac{y_1}{a}\right) \approx \frac{y_1}{a}$



$\tan^{-1}\left(\frac{y_2 - y_1}{a}\right) \approx \frac{y_2 - y_1}{a}$



$F = ma$ vertically

$$m\ddot{y}_1 = T\sin\left(\frac{y_2 - y_1}{a}\right) - T\sin\left(\frac{y_1}{a}\right) \cancel{- mg}$$

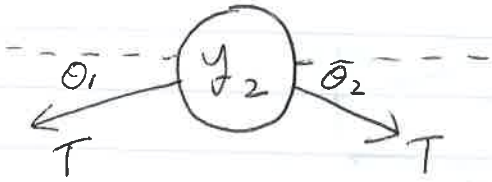
$$m\ddot{y}_1 = T\,\frac{y_2 - y_1}{a} - T\,\frac{y_1}{a} \cancel{- mg}$$

$$\ddot{y}_1 = \frac{T}{am}(y_2 - y_1 - y_1)$$

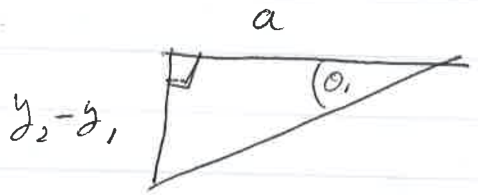
$$\ddot{y}_1 = \frac{T}{am}(y_2 - 2y_1)$$

$$\ddot{y}_1 = -\frac{T}{am}(2y_1 - y_2)$$

$$\ddot{y}_1 = -\omega_0^2(2y_1 - y_2)$$

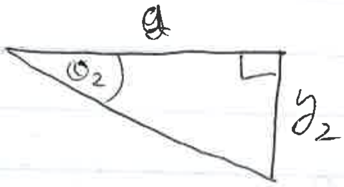


$S\frac{o}{h}\ C\frac{a}{h}\ T\frac{o}{a}$



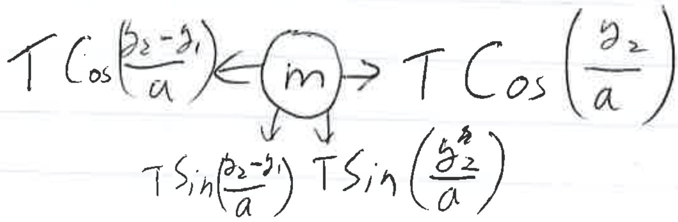
$$\theta_1 = Tan^{-1}\left(\frac{y_2 - y_1}{a}\right)$$

$$\theta_1 \approx \frac{y_2 - y_1}{a}$$



$$\theta_2 = Tan^{-1}\left(\frac{y_2}{a}\right)$$

$$\theta_2 \approx \frac{y_2}{a}$$



$$F = ma$$

$$-T\,Sin\left(\frac{y_2}{a}\right) + T\,Sin\left(\frac{y_2 - y_1}{a}\right) = m\ddot{y}_2$$

$$-T\frac{y_2}{a} + T\frac{y_2 - y_1}{a} = m\ddot{y}_2$$

$$\frac{T}{ma}\left(-y_2 + y_2 + y_1\right) = \ddot{y}_2$$

$$\frac{T}{ma}\left(-2y_2 + y_1\right) = \ddot{y}_2$$

$$-\omega_0^2\left(2y_2 - y_1\right) = \ddot{y}_2$$

iii)

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\omega_0^2 \begin{pmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 \end{pmatrix}$$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\omega_0^2 \begin{pmatrix} +2 & -1 \\ -1 & +2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\ddot{X} = \omega_0^2 \, K \, X$$

$$\det(K - I\lambda) = 0 \qquad \det\begin{pmatrix} +2-\lambda & -1 \\ -1 & +2-\lambda \end{pmatrix} = 0$$

$$(+2-\lambda)^2 - (-1)^2 = 0 \qquad 4 + 2\lambda + 2\lambda + \lambda^2 - 1 = 0$$

$$\lambda^2 + 4\lambda + 3 = 0$$

$$(\lambda+3)(\lambda+1) = 0$$

$$\lambda = 3, 1$$

$$\begin{pmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0$$

$\lambda = 3$

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0 \qquad \begin{pmatrix} -u - v \\ -u - v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-u - v = 0$$

$$-u = v$$

e.igenvector $\rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ or $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\lambda = 1$

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0 \qquad \begin{pmatrix} u - v \\ -u + v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$u - v = 0$$

$$u = v$$

eigenvector $\rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$

Putting $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ together into a matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = P$$

$$PQ = X$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$q_1 + q_2 = x_1$$

$$q_1 - q_2 = x_2$$

$$2q_1 = x_1 + x_2$$

$$q_1 = \tfrac{1}{2}x_1 + \tfrac{1}{2}x_2$$

$$2q_2 = x_1 - x_2$$

$$q_2 = \tfrac{1}{2}x_1 - \tfrac{1}{2}x_2$$

to make things simpler double the scale of the new system

$$\downarrow$$

$$q_1 = x_1 + x_2 \qquad \tfrac{1}{2}(q_1 + q_2) = x_1$$

$$q_2 = x_1 - x_2 \qquad \tfrac{1}{2}(q_1 - q_2) = x_2$$

$$\ddot{q}_1 = \ddot{x}_1 + \ddot{x}_2 \qquad\qquad \frac{1}{2}(\ddot{q}_1 + \ddot{q}_2) = \ddot{x}_1$$

$$\ddot{q}_2 = \ddot{x}_1 - \ddot{x}_2 \qquad\qquad \frac{1}{2}(\ddot{q}_1 - \ddot{q}_2) = \ddot{x}_2$$

$$-\omega_0^2(2x_1 - x_2) = \ddot{x}_1 \qquad\qquad -\omega_0^2(2x_2 - x_1) = \ddot{x}_2$$

$$-\omega_0^2\left(\frac{1}{2}(q_1 + q_2) + q_2\right) = \frac{1}{2}(\ddot{q}_1 + \ddot{q}_2)$$

$$-\omega_0^2\left(\frac{1}{2}q_1 + \frac{3}{2}q_2\right) = \frac{1}{2}(\ddot{q}_1 + \ddot{q}_2)$$

$$-\omega_0^2(q_1 + 3q_2) = \ddot{q}_1 + \ddot{q}_2 \qquad (1)$$

$$-\omega_0^2\left(\frac{1}{2}(q_1 - q_2) - q_2\right) = \frac{1}{2}(\ddot{q}_1 - \ddot{q}_2)$$

$$-\omega_0^2\left(\frac{1}{2}q_1 - \frac{3}{2}q_2\right) = \frac{1}{2}(\ddot{q}_1 - \ddot{q}_2)$$

$$-\omega_0^2(q_1 - 3q_2) = \ddot{q}_1 - \ddot{q}_2 \qquad (2)$$

---

(1) + (2) $\quad -2\omega_0^2 q_1 = 2\ddot{q}_1$

$$-\omega_0^2 q_1 = \ddot{q}_1 \quad ✓$$

(1) - (2) $\quad -6\omega^2 q_2 = 2\ddot{q}_2$

$$-3\omega_0^2 q_2 = \ddot{q}_2 \quad ✓$$

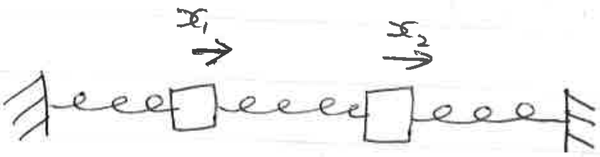
$$\omega^2 = 3\omega_0^2$$

$$\omega = \sqrt{3}\,\omega_0$$

q_1

$$q_1 = x_1 + x_2$$

$$\dot{q}_1 = \dot{x}_1 + \dot{x}_2$$



q_1

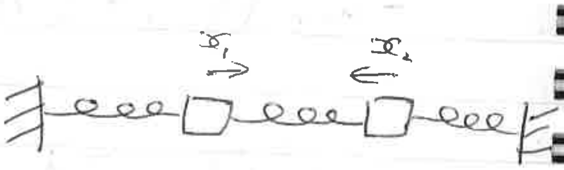
$$\omega^2 = \omega_0^2$$

$$\omega = \omega_0$$

q_2

$$q_2 = x_1 - x_2$$

$$\dot{q}_2 = \dot{x}_1 - \dot{x}_2$$



q_2

✓

A4)

$\cancel{\frac{T}{2\pi}}$ $\omega = \frac{2\pi}{T}$ for X

$T_x = 1$ $\omega_x = 2\pi$ $\omega_x^2 = 4\pi^2$

$-4\pi^2 X = \ddot{X}$ $X = e^{i 2\pi t}$

for $\cancel{y}$

$T_y = 0.5$ $\omega_y = \frac{2\pi}{0.5} = 4\pi$ $\omega_y^2 = 16\pi^2$

$-16\pi^2 Y = \ddot{Y}$ $Y = e^{i 4\pi t}$

$t = 0$, $y_1 = 0$ $y_2 = 1$

$X_0 = \frac{1}{2}(y_1 + y_2)$ $\cancel{X_0} = \frac{1}{2}(y_1 - y_2)$

$X_0 = \frac{1}{2}(0 + 1)$ $Y_0 = \frac{1}{2}(0 - 1)$

$X_0 = \frac{1}{2}$ $Y_0 = -\frac{1}{2}$

$X = X_0 \cos(\omega_x t)$ $Y = Y_0 \cos(\omega_y t)$

$X_{2.2} = \frac{1}{2} \cos\left(\frac{2.2}{\cancel{2\pi}} \cdot 2\pi\right)$ $Y_{2.2} = \frac{1}{2} \cos\left(\frac{2.2}{\cancel{4\pi}} \cdot 4\pi\right)$

$X_{2.2} + Y_{2.2} = y_1$ $X_{2.2} - Y_{2.2} = y_2$

$-\frac{1}{4}$ 0.559

-0.25 ✓ 0.56 ✓

$\vec{SDC}$ $\vec{CD-S}$

$\dot{X} = -\omega_x X_0 \sin(\omega_x t)$

$\dot{Y} = -\omega_y Y_0 \sin(\omega_y t)$

$\dot{X} + \dot{Y} = \dot{y}_1$

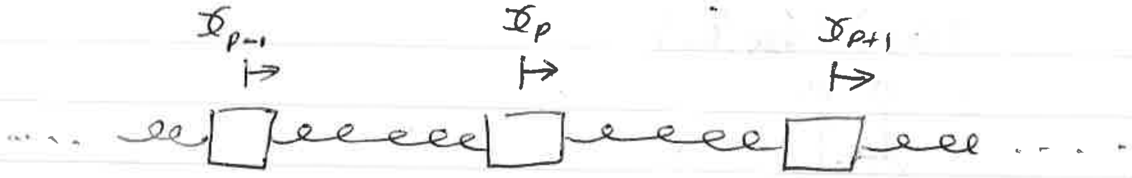
-6.68 ✓

$\dot{X} - \dot{Y} = \dot{y}_2$

0.705

0.71 ✓

A5) i)



$$F = ma$$

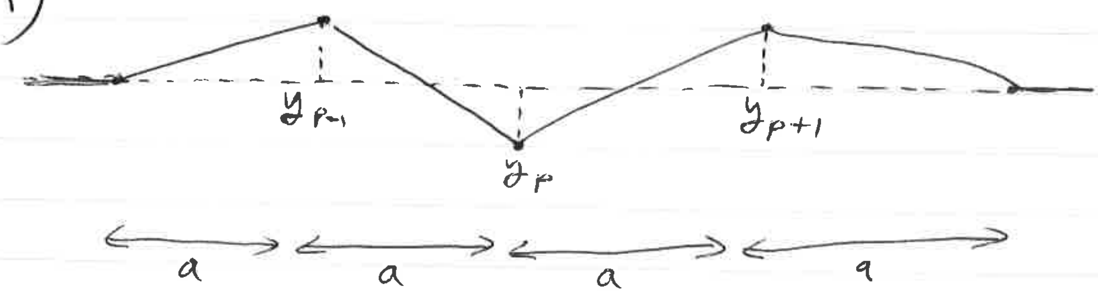
$$k(x_{p+1} - x_p) + k(x_{p-1} - x_p) = m\ddot{x}_p$$

$$k(x_{p+1} - x_p + x_{p-1} - x_p) = m\ddot{x}_p$$

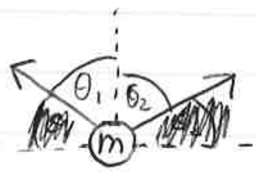
$$k(x_{p+1} + x_{p-1} - 2x_p) = m\ddot{x}_p$$

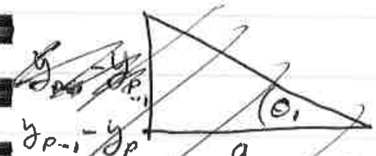
$$-k(2x_p - x_{p+1} - x_{p-1}) = m\ddot{x}_p \qquad \square$$

ii)



$s^{o}_{n}\ c^{q}_{n}\ T^{o}_{a}$

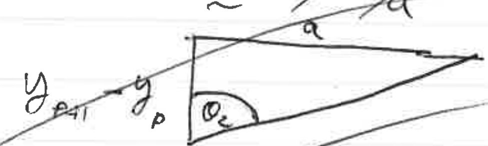




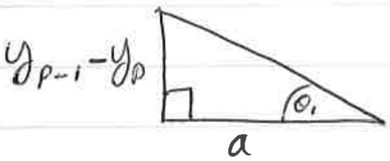
$$\theta_1 = \mathrm{Tan}\left(\frac{y_{p-1} - y_p}{a}\right) \approx \frac{y_{p-1} - y_p}{a}$$

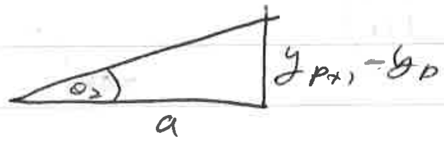
$$\theta_1 = \mathrm{Tan}\left(\frac{a}{y_{p-1} - y_p}\right) \approx \frac{a}{y_{p-1} - y_p}$$

$$\theta_1 = \mathrm{Tan}\left(\frac{y_{p+1} - y_p}{a}\right) \approx \frac{y_{p+1} - y_p}{a}$$



$$\theta_2 = \mathrm{Tan}\left(\frac{a}{y_{p+1} - y_p}\right) \approx \frac{a}{y_{p+1} - y_p}$$



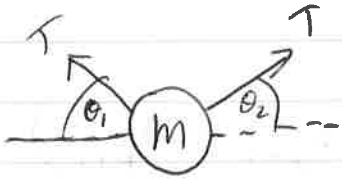


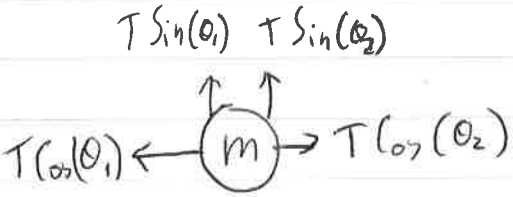
$$\tan(\theta_1) = \frac{y_{p-1} - y_p}{a}$$

$$\theta_1 \approx \frac{y_{p-1} - y_p}{a}$$

$$\tan(\theta_2) = \frac{y_{p+1} - y_p}{a}$$

$$\theta_2 \approx \frac{y_{p+1} - y_p}{a}$$





$F = ma$ Vertically

$$\sin(\theta) \approx \theta$$

$$T\left(\frac{y_{p-1} - y_p}{a} + \frac{y_{p+1} - y_p}{a}\right) = m\ddot{y}_p$$

$$\frac{T}{a}\left(y_{p-1} - y_p + y_{p+1} - y_p\right) = m\ddot{y}_p$$

$$\frac{T}{a}\left(y_{p-1} + y_{p+1} - 2y_p\right) = m\ddot{y}_p$$

$$-\frac{T}{a}\left(2y_p - y_{p-1} - y_{p+1}\right) = m\ddot{y}_p \qquad \square$$

iii)

from (i)

$$m\ddot{x}_p = -K(2x_p - x_{p-1} - x_{p+1})$$

$$\ddot{x}_p = -\frac{K}{m}(2x_p - x_{p-1} - x_{p+1})$$

$\omega_0 = \sqrt{\frac{K}{m}}$ $\qquad$ $\omega_0^2 = \frac{K}{m}$

$$\ddot{x}_p = -\omega_0^2(2x_p - x_{p-1} - x_{p+1})$$

$$\omega^2 = 2\omega_0^2(1 - \cos\theta)$$

$$\frac{\omega^2}{2(1-\cos\theta)} = \omega_0^2$$

$$\ddot{x}_p = \frac{\omega^2(2x_p - x_{p-1} - x_{p+1})}{2(1-\cos\theta)}$$

?