CST-305: Project 2 – Runge-Kutta-Fehlberg (RKF) For ODE

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CST-305: Principles of Modeling and Simulation

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Responsibilities and completed tasks by each team member

All tasks are done by Jack Utzerath

Specific Problem Solved

I am using Greens funcntion to solve $y''+4y=x; y(0)=y'(0)=0$ and $y''+y=4$; $y(0)=y'(0)=0$. Then I am comparing them to other solving methods including Variation of Parameters and Simple Form of the Method

The mathematical approach to solving it

Green Function Equation 1:

Green's function

$y'' + 4y = x \,;\; y(0) = y'(0) = 0$

$y'' + 4y = 0$

$\lambda^2 + 4 = 0$

$\lambda^2 = -4$

$\lambda = 0 \pm 2i$

Inhomogenous equation

$G''(t,s) + 4G(t,s) = \delta(t-s)$

$y_H(x) = C_1 \cos 2x + C_2 \sin 2x$

$C_1 \cos 2x + C_2 \sin 2x = 0$

$$G(t,s) = \begin{cases} 0 & t<s \\ C_1 \cos(2x) + C_2 \sin(2x) & t>s \end{cases}$$

$G(s,s) = 0 \rightarrow C_1 \sin(2t) + C_2 \cos(2t) = 0$

$C_1 = -C_2 \frac{\sin(2s)}{\cos(2s)}$

$\frac{d}{dt} g(t,s)\big|_{t=s} = 1$

$C_1 \cos 2t + C_2 \sin 2t = 0$

$C_1 = -C_2 \frac{\sin 2t}{\cos 2t}$

$\frac{d}{ds}(C_1 \cos 2s + C_2 \sin 2s) = 1$

$-2C_1 \sin 2s + 2C_2 \cos 2s = 1$

$-2\left(C_2 \frac{\sin 2s}{\cos 2s}\right) \sin 2s + 2C_2 \cos 2s = 1$

$2C_2 \sin^2 2s + 2C_2 \cos^2 2s = \cos 2s$

$2C_2 = \cos 2s$

$C_2 = \frac{\cos(2s)}{2}$

$G(t,s) = C_1 \cos 2(x) + C_2 \sin(2x)$

$G(t,s) = -\frac{\cos(2s)}{2} \frac{\sin(2t)}{\cancel{\cos(2x)}} \cdot \cos 2x + \frac{\cos(2s)}{2} \cdot \sin(2x)$

$G(t,s) = -\frac{\sin 2s \cdot \cos(2s)}{} + \frac{\cos(2s) \cdot \sin(2s)}{}$

$G(t,s) = -\frac{\sin(2s)\cos(2s) + \cos(2s) \cdot \sin(2s)}{2}$

$\int_0^t G(t,s)\,t\,ds)$

$\int_0^t \frac{\sin(2t)\cos(2s) - \cos(2t)\sin(2s)}{2}\,ds$

$= \frac{t\sin(2t)}{2} \int \cos(2s)\,ds - \frac{t\cos(2t)}{2} \int \sin(2s)\,ds$

$= \int \cos(2s)\,ds \rightarrow \sin(u)$

$= \int \sin(2s)\,ds \rightarrow \frac{1}{2}\int \sin(u)\,du \rightarrow -\frac{\cos(u)}{2}$

$\frac{t\sin(2t)\sin(2s)}{4} + \frac{t\cos(2t)\cos(2t)}{4}$

$\frac{t}{4} - \frac{\sin(2t)}{8}$

Greens Function Equation 2:

Green function

$y'' + y = 4 \quad y(0) = y'(0) = 0$

$y'' + y = 0$

$\lambda^2 + 1 = 0$

$\lambda^2 = 1$

$\lambda = 0 \pm i$

$C_1 \cos x + C_2 \sin x$

$G''(t,s) + G(t,s) = \delta(t-s)$

$G(t,s) \begin{cases} 0 & t<s \\ c_1\cos x + c_2 \sin x & t>s \end{cases}$

$C_1 \cos x + C_2 \sin x = 0$

$C_1 \cos x = -C_2 \sin x$

$C_1 = -C_2 \frac{\sin x}{\cos x}$

$-C_1 \sin(x) + C_2 \cos x = 1$

$C_2 \frac{\sin x}{\sin x} \sin x + C_2 \cos x = 1$

$C_2 \sin^2 x + C_2 \cos^2(x) = \cos x$

$C_2 = \cos x$

$G(t,s) = C_1 \cos(x) + C_2 \sin(x)$

$-\sin x \cos t + \cos(s) \sin(t)$

$y(x) = \int_0^t (-\sin(s)\cos(t) + \cos(s)\sin(t))\, 4\, ds$

$4\int_0^t -\sin(s)\cos(t) + \int \cos(s)\sin(t))\, ds$

$4 \cos(t) \int_0^t -\sin(s) ds + \sin(t) \int_0^t \cos(s)\, ds$

$4 \cos(t) \cdot \cos(s)|_0^t \quad \sin(t) \cdot \sin(s)|_0^t$

$4(\cos(t)(\cos t - \cos(0)) + \sin t(\sin t - \sin(0))$

$4(\cos^2 t - \cos(t) + \sin^2 t)$

$4(1 - \cos(t))$

$y(t) = 4 - 4\cos(t)$

Simple Method Equation 1:

Undertumned Coeffients

Simple Methods

$y'' + 4y = x; \quad y(0) = y'(0) = 0$

$y'' + 4y = 0$

$\lambda^2 + 4 = 0$

$\lambda^2 = -4$

$\lambda = \sqrt{-4}$

$\lambda = 0 \pm 2i$

$y_H(x) = C_1 \cos(2x) + C_2 \sin(2x)$

$y'' + 0y' + 4y = 0x^2 + x + 0$

$y_p(x) = Ax^2 + Bx + C$

$y'_p(x) = 2Ax + B$

$y''_p(x) = 2A$

$(2A) + 0(\cancel{2Ax+B}) + 4(Ax^2 + Bx + C) = 0x^2 + x + 0$

$2A + 4Ax^2 + 4Bx + 4C = 0x^2 + x + 0$

$(4A)x^2 + (4B)x + (2A + 4C) = 0x^2 + x + 0$

$4A = 0 \quad 4B = 1 \quad \cancel{2A} + 4C = 0$

$A = 0 \quad B = \frac{1}{4} \quad C = 0$

$y_p(x) = 0x^2 + \frac{1}{4}x + 0 = \frac{1}{4}x$

$y_p(x) = \frac{x}{4}$

$$\boxed{y(x) = C_1 \cos 2x + C_2 \sin 2x + \frac{x}{4}}$$

$y(0) = C_1 \cos(0) + C_2 \sin(0) + \frac{0}{4} = 0$

$= C_1 + 0 + 0 = 0$

$C_1 = 0$

$y'(0) = -C_1 \sin(0) + C_2 \cos(0) + \frac{1}{4} = 0$

$2C_2 + \frac{1}{4} = 0$

$2C_2 = -\frac{1}{4}$

$C_2 = -\frac{1}{8}$

$y(x) = 0 \cos(2x) + -\frac{1}{8} \sin 2x + \frac{x}{4}$

$$y(x) = -\frac{\sin 2x}{8} + \frac{x}{4}$$

Simple Method Equation 2:

Underdmined Coefficents
simple method

Problem 2

$y''+y=4 \qquad y(0)=y'(0)=0$

$\lambda^2+1=0$

$\lambda=\pm i$

$y_H(x)=C_1\cos(x)+C_2\sin(x)$

$y''+0y'+y=0x^2+0x+4$

$y_p(x)=Ax^2+Bx+C$

$y'_p(x)=2Ax+B$

$y''_p(x)=2A$

$(2A)+0(2Ax+B)+(Ax^2+Bx+C)=0x^2+0x+4$

$A=0 \qquad B=0 \qquad \cancel{2A}+C=4$

$C=4$

$y_p(x)=0x^2+0x+4$

$y_p(x)=4$

$\boxed{y(x)=C_1\cos(x)+C_2\sin x+4}$

$y(0)=C_1\cos(0)+\cancel{C_2\sin(0)}+4$

$C_1+4=0$

$C_1=-4$

$y'(0)=-\cancel{C_1\sin(0)}+C_2\cos(0)=0$

$C_2=0$

$y(x)=-4\cos(x)+4$

Variation of Parameters Equation 1:

Undetermined Coefficents
Variation of Parameters

$y''+4y=x\,;\quad y(0)=y'(0)=0$

$y''+4y=0$

$\lambda^2+4=0$

$\lambda=\pm 2i$

$y_H(x)=C_1\cos(2x)+C_2\sin(2x)$

$$\begin{cases} v_1'\cos(2x)+v_2'\sin(2x)=0 & \cdot\, 2\sin 2x \\ -2v_1'\sin 2x+2v_2'\cos(2x)=x & \cdot\, \cos 2x \end{cases}$$

$2v_1'\cos 2x\,\cancel{\sin 2x}+2v_2'\sin^2 2x=0$

$-2v_1'\cancel{\cos 2x}\sin 2x+2v_2'\cos^2 2x=x\cos 2x$

$2v_2'\cos(2x)+2v_2'\sin 2x=x\cos(2x)$

$2v_2'(\cos^2(2x)+\sin^2(2x))=x\cos(2x)$

$2v_2'\cos(2x)+2v_2'\sin 2x=x\cos 2x$

$2v_2'(1)=x\cos 2x$

$v_2'=\frac{1}{2}x\cos 2x$

$v_2=\frac{1}{2}\int x\cos 2x$

$u=x \quad dv=\cos 2x$

$du=1 \quad v=\int\cos(2x)$

$\int\cos 2x$
$u=2x$
$\frac{1}{2}du=dx$
$\frac{1}{2}\int\cos u\,du$
$\frac{1}{2}\sin 2x$

$V_2=\frac{1}{2}\left[\frac{x\sin 2x}{2}-\int\frac{1}{2}\sin 2x\,dx\right]$

$V_2=\frac{1}{2}\left[\frac{x\sin 2x}{2}-\frac{1}{2}\int\sin 2x\,dx\right]$

$V_2=\frac{1}{2}\left[\frac{x\sin 2x}{2}-\frac{1}{4}(-\cos 2x)\right]$

$V_2=\frac{1}{2}\left[\frac{x\sin 2x}{2x}+\frac{\cos 2x}{4}\right]$

$V_2=\frac{x\sin 2x}{4}+\frac{\cos 2x}{8}$

V_1

$v_1'\cos(2x)+(\frac{1}{2}x\cos 2x)\sin 2x=0$

$v_1'\cos(2x)+\frac{1}{2}x\cos 2x\sin 2x=0$

$v_1'\cos(2x)=-\frac{1}{2}x\cos 2x\sin 2x$

$v_1'=-\frac{1}{2}x\sin 2x$

$V_1=\int-\frac{1}{2}x\sin 2x\,dx$

$V_1=-\frac{1}{2}\int x\sin 2x\,dx$

$u=x \quad dv=\sin(2x)$

$du=1 \quad v=\int\sin(2x)$

$V_1=-\frac{1}{2}\left[-\frac{1}{2}x\cos(2x)-\int-\frac{1}{2}\cos 2x\,dx\right]$

$V_1=-\frac{1}{2}\left[-\frac{1}{2}x\cos 2x+\frac{1}{2}\int\cos 2x\,dx\right]$

$u=2x$

$\frac{1}{2}du=dx$

$V_1=-\frac{1}{2}\left[-\frac{1}{2}x\cos 2x+\frac{1}{4}\sin 2x\right]$

$V_1=\frac{1}{4}x\cos 2x-\frac{1}{8}\sin 2x$

$V_1=\frac{x\cos 2x}{4}-\frac{\sin 2x}{8}$

$y_p(x)=V_1y_1+V_2y_2$

$y_p(x)=\left(\frac{x\cos(2x)}{4}-\frac{\sin 2x}{8}\right)\cos 2x+\left(\frac{x\sin 2x}{4}+\frac{\cos 2x}{8}\right)\sin 2x$

$y_p(x)=\frac{x\cos^2 2x}{4}-\frac{\cos 2x\sin 2x}{8}+\frac{x\sin^2 2x}{4}+\frac{\cos 2x\sin 2x}{8}$

$y_p(x)=\frac{x(\cos^2 2x+\sin^2 2x)}{4}=\frac{x}{4}$

$\boxed{y(x)=C_1\cos 2x+C_2\sin 2x+\frac{x}{4}}$

$y(0)=C_1\cos(0)+C_2\sin(0)+0=0$

$C_1=0$

$y'(0)=-2C\sin(2\cdot 0)+2C_2\cos(2\cdot 0)+\frac{1}{4}=0$

$2C_2(1)+\frac{1}{4}=0$

$2C_2=-\frac{1}{4}$

$C_2=-\frac{1}{8}$

$y(x)=(0)\cos 2x+-\frac{1}{8}\sin 2x+\frac{x}{4}$

$y(x)=-\frac{\sin 2x}{8}+\frac{x}{4}$

Variation of Parameters Equation 2:

Undetermined Coefficients

Variation of Parameters

$y'' + y = 4 \qquad y(0) = y'(0) = 0$

$\lambda^2 + 1 = 0$

$\lambda = \pm i$

$y_H(x) = C_1\cos(x) + C_2\sin(x)$

$$\begin{cases} v_1'\cos x + v_2'\sin x = 0 \cdot \sin x \\ -v_1'\sin x + v_2'\cos(x) = 4 \cdot \cos x \end{cases}$$

$$\begin{array}{l} v_1'\cos x\sin(x) + v_2'\sin^2 x = 0 \\ -\ v_1'\cos x\sin(x) + v_2'\cos^2 x = 4\cos x \\ \hline v_2'\cos^2 x + v_2'\sin^2 x = 4\cos x \end{array}$$

$v_2'(\cos^2 x + \sin^2 x) = 4\cos x$

$v_2'(1) = 4\cos x$

$v_2' = 4\cos(x)$

$v_2 = 4\int\cos(x)$

$v_2 = 4\sin x$

$v_1'\cos(x) + (4\cancel{\cos x})\sin(x) = 0$

$v_1'\cos(x) + 4\cos x\sin(x) = 0$

$v_1'\cos x = -4\cos x\sin x$

$v_1' = -4\sin x$

$v_1 = \int -4\sin x$

$v_1 = -4\int\sin x$

$v_1 = +4\cos x$

$y_p(x) = v_1 y_1 + v_2 y_2$

$y_p(x) = 4\cos(x)\cos(x) + 4\sin(x)(\sin(x))$

$y_p(x) = 4\cos^2 x + 4\sin^2 x$

$y_p(x) = 4[\cos^2 x + \sin^2 x]$

$y_p(x) = 4$

$y(x) = C_1\cos(x) + C_2\sin(x) + 4$

$y(0) = C_1\cos(0) + C_2\sin(0) + 4$

$C_1 + 4 = 0$

$C_1 = -4$

$y'(0) = -C_1\sin(x) + C_2\cos(x) = 0$

$C_2 = 0$

$y'(x) = -4\cos(x) + 4$

The approach for implementation in code

For the implementation in Code, I used four packages, Numpy, sympy matplotlib, and scipy. This code solves and visualizes second-order ordinary differential equations (ODEs). It involves symbolic manipulation with SymPy for solving ODEs symbolically and numerical integration with NumPy, SciPy, and Matplotlib for plotting. The first part solves the ODE 'y'' + 4y = x' symbolically and plots the solution with initial conditions. The second part solves the ODE 'y'' + y = 4' symbolically and plots the solution with initial conditions. The third part solves 'y'' + 4y = x' using the Green's function method and plots the solution. The fourth part solves 'y'' + y = 4' using the Green's function method and plots the solution.

these functions and compare them both on the same graph.

References for theory and code sources

Green's function. from Wolfram MathWorld. (n.d.).

https://mathworld.wolfram.com/GreensFunction.html

Readme Document written in Markdown detailing how to install and run the program

https://github.com/utzerath/Courses/tree/main/Projects

````
# README for Second Order ODE Solver

This repository contains Python code for solving and visualizing second-order ordinary differential
equations (ODEs) using various libraries. Before running the code, you'll need to ensure that the required
packages are installed. This README provides instructions on how to set up the environment.

## Prerequisites

Make sure you have Python and a package manager, such as pip, installed on your system.

## Installation

You can install the required packages by running the following commands in your terminal or command prompt:

1. **SymPy**: Used for symbolic mathematics and ODE solving.

   ```bash
pip install sympy
pip install numpy
pip install scipy
pip install matplotlib
````

Full code submitted to GitHub

https://github.com/utzerath/Courses/tree/main/Projects
````

Code Basics:

How ODE is solved and outputed symbolically through package sympy (Equation 1 in pic):

```
#Solving the ODEs
import sympy as sp

# Define the symbolic variables
x = sp.symbols('x')
y = sp.Function('y')(x)

# Define the ODE "y'' + 4y = x"
ode = y.diff(x, x) + 4 * y - x

# Specify the initial conditions
y0 = 0  # y(0) = 0
y1 = 0  # y'(0) = 0

# Solve the ODE symbolically with initial conditions
solution = sp.dsolve(ode, y, ics={y.subs(x, 0): y0, y.diff(x).subs(x, 0): y1})

# Print the symbolic solution with initial conditions
print("Symbolic Solution to y'' + 4y = x with Initial Conditions:")
print(solution)
```

How I implemented Greens Function and Plotted it (Equation 1 in Pic):

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp

# Define the second-order ODE: y'' + 4y = x
def ode(t, y):
    return [y[1], -4 * y[0] + t]

# Define the Green's function method for y'' + 4y = x
def green_function(t, t_prime):
    if t >= t_prime:
        return (1/8) * (np.sin(2 * (t - t_prime)) - np.sin(2 * (t + t_prime)))
    else:
        return (1/8) * (np.sin(2 * (t_prime - t)) - np.sin(2 * (t_prime + t)))

# Define a function to compute the solution using the Green's function
def solve_with_green_function(t, t_prime, f):
    G = np.zeros_like(t)
    for i, t_i in enumerate(t):
        G[i] = np.trapz([green_function(t_i, tp) * f[j] for j, tp in enumerate(t_prime)], t_prime)
    return G

# Define the time span for the solution
t_span = (0, 5)

# Initial conditions
y0 = [0, 0]

# Solve the ODE using the Green's function method
t_values = np.linspace(t_span[0], t_span[1], 500)
y_solution = solve_with_green_function(t_values, t_values, t_values)

# Plot the solution
plt.figure(figsize=(10, 6))
plt.plot(t_values, y_solution, label='Solution')
plt.xlabel('t')
plt.ylabel('y(t)')
plt.title("Second Order ODE Solution using Green's Function (y'' + 4y = x)")
plt.grid(True)
plt.legend()
```

How I used a variation of parameters to Plot function (Equation 2 in pic):

```python
#y'' + y = 4
# Define the second-order ODE
def ode(t, y):
    y1, y2 = y
    return [y2, 4 - y1]

# Define the time span for the solution
t_span = (0, 5)

# Initial conditions
y0 = [0, 0]

# Define the particular solution
def particular_solution(t):
    return t

# Define a function to solve the homogeneous equation
def homogeneous_solution(t):
    return np.sin(t)

# Solve the homogeneous ODE to obtain the homogeneous solution
t_values = np.linspace(t_span[0], t_span[1], 500)
homogeneous_solution_values = homogeneous_solution(t_values)

# Solve the particular ODE with a particular solution
sol = solve_ivp(ode, t_span, y0, t_eval=t_values, vectorized=True)

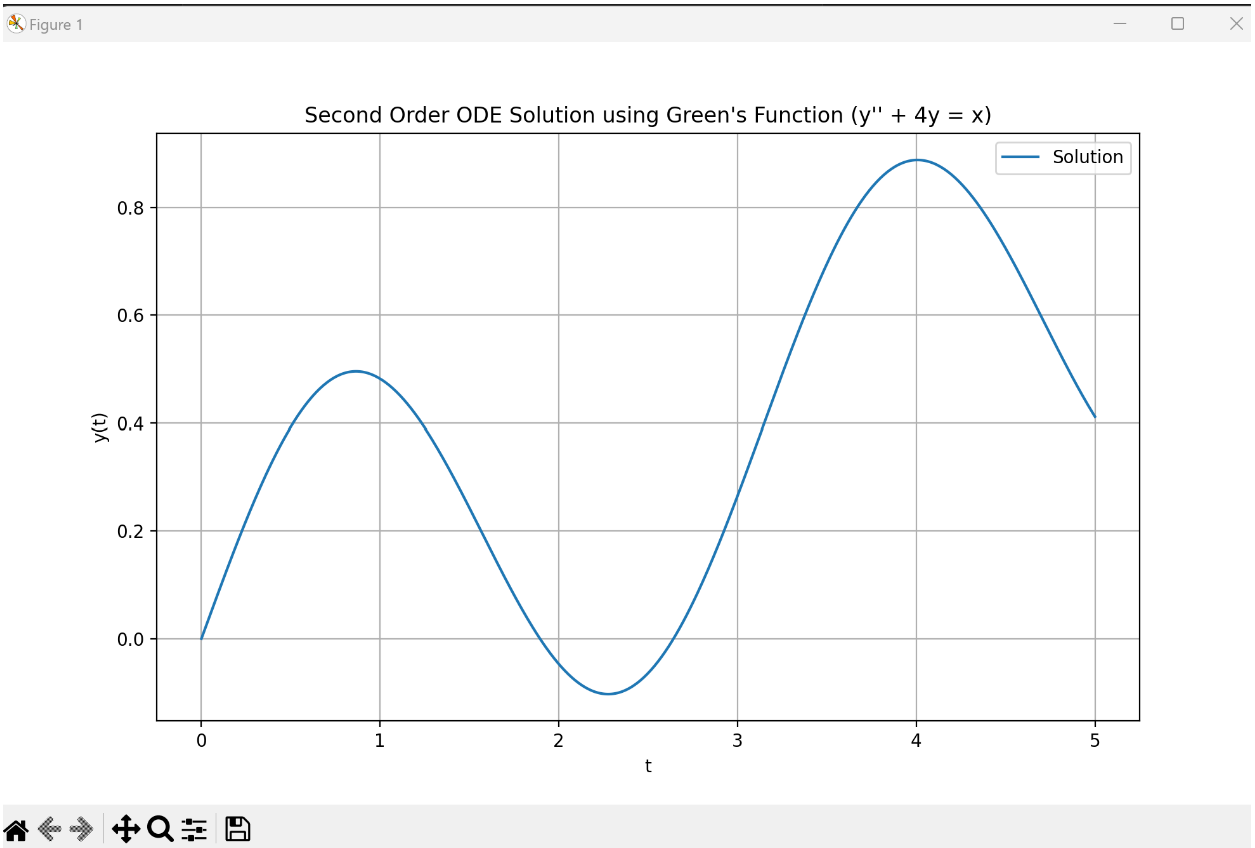
# Combine the homogeneous and particular solutions to get the general solution
general_solution = homogeneous_solution_values + sol.y[0]

# Plot the general solution
plt.figure(figsize=(10, 6))
plt.plot(t_values, general_solution, label='General Solution')
plt.xlabel('t')
plt.ylabel('y(t)')
plt.title('Second Order ODE Solution: y\'\' + y = 4')
plt.grid(True)
plt.legend()
plt.show()
```

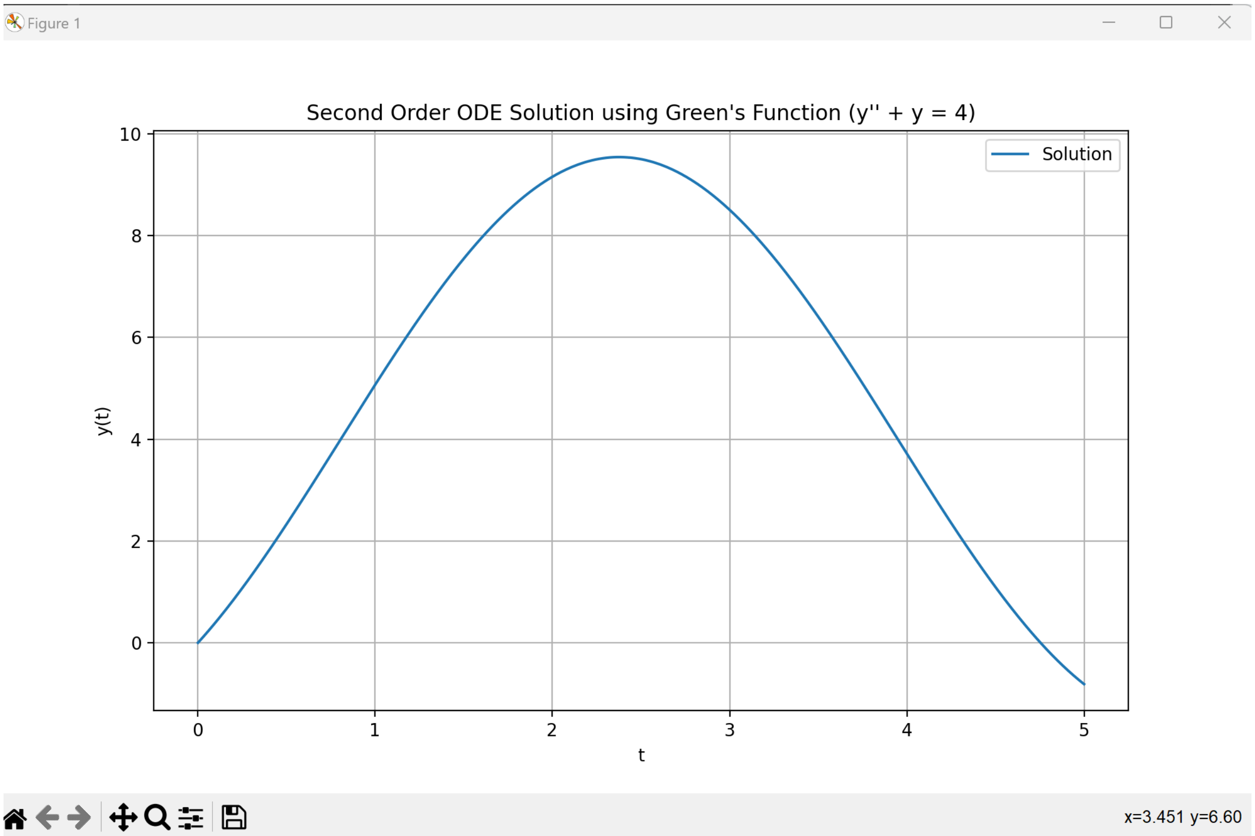
Code Output:

```
Symbolic Solution to y'' + 4y = x with Initial Conditions:
Eq(y(x), x/4 - sin(2*x)/8)
Symbolic Solution to y'' + y = 4 with Initial Conditions:
Eq(y(x), 4 - 4*cos(x))
```

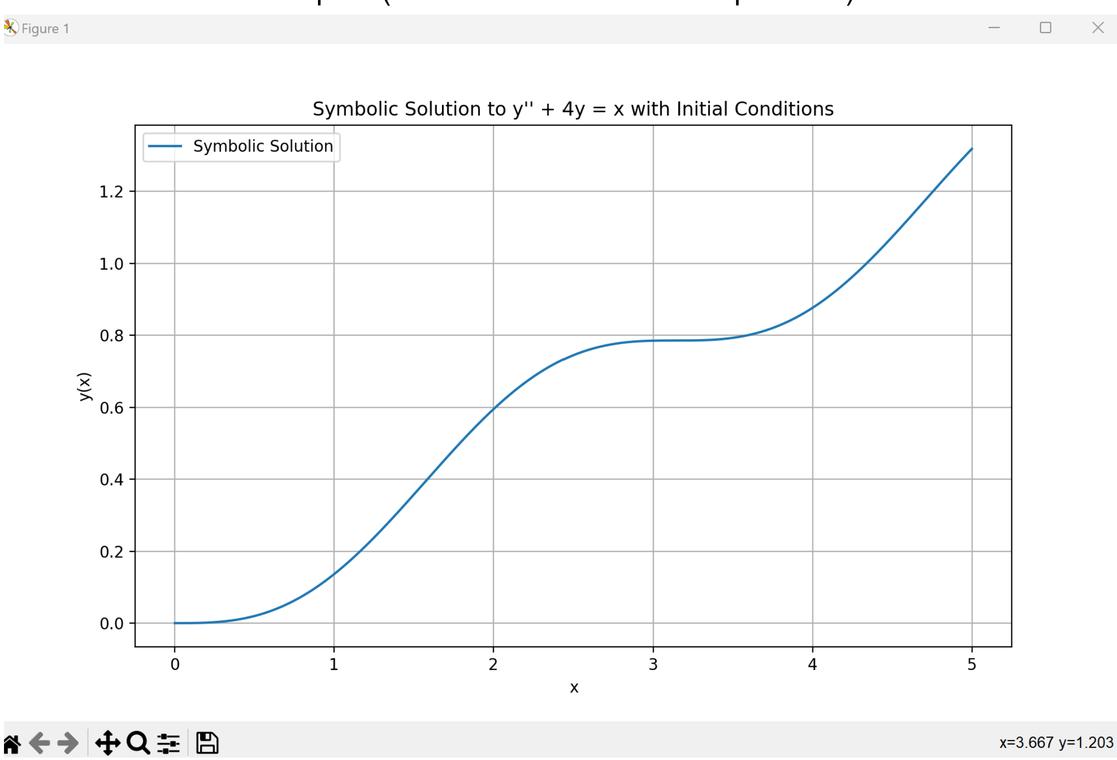
Graph 1 (Green's function Equation 1)



Graph 2 (Greens Function Equation 2)



Graph 3 (Variation of Parameters Equation 1)



Graph 4 (Variation of Parameters Equation 2)