Introduction to Kalman-Filter, SDE, and LFM

Shikai Fang
2021/10 for group present

Material form PRML ch13, books/paper/poster from Simo Sarkka/Aron Solin

Skeleton

SDE
(stochastic differential equation)

Brown Motion

Ito integration

State Space
Gaussian Process

Latent Force Model
(LFM)

Markov Models

State Space Model &
Linear Dynamic Sys(LDS)

Kalman Filter &
Smoother

Non-linear & Cont time
Kalman

Skeleton

We start here!

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Non-linear & Cont time Kalman

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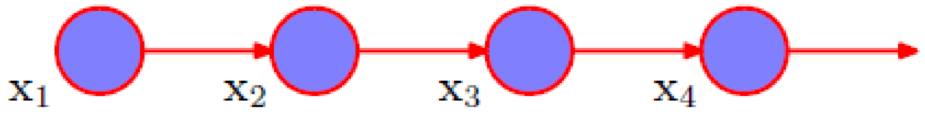
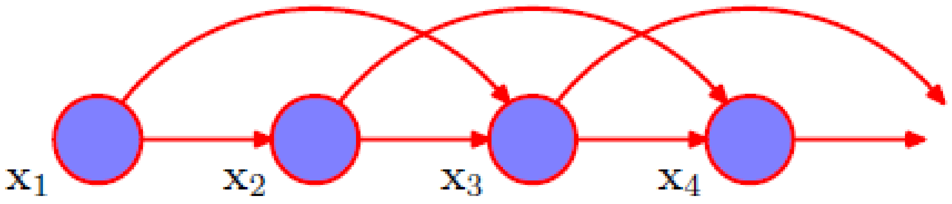
Markov Models :

model the relation of seq data

$$p(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \prod_{n=1}^{N} p(\mathbf{x}_n | \mathbf{x}_1, \ldots, \mathbf{x}_{n-1}).$$



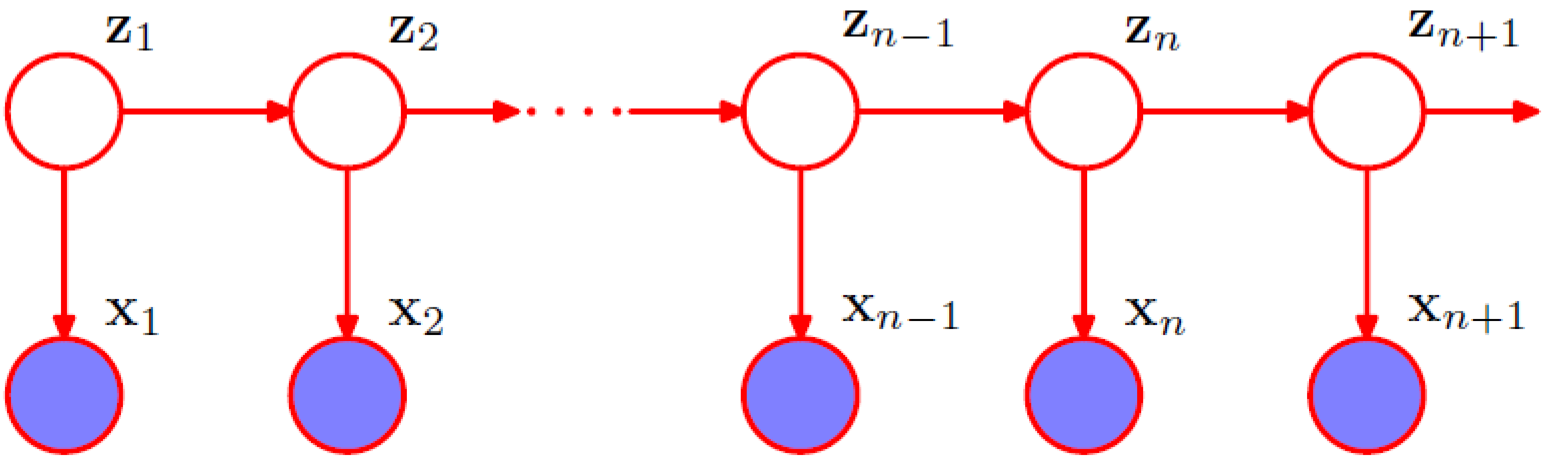
$$p(\mathbf{x}_1, \ldots, \mathbf{x}_N) = p(\mathbf{x}_1) \prod_{n=2}^{N} p(\mathbf{x}_n | \mathbf{x}_{n-1}).$$



$$p(\mathbf{x}_1, \ldots, \mathbf{x}_N) = p(\mathbf{x}_1) p(\mathbf{x}_2 | \mathbf{x}_1) \prod_{n=3}^{N} p(\mathbf{x}_n | \mathbf{x}_{n-1}, \mathbf{x}_{n-2}).$$

Markov chains

- First order
- Second order

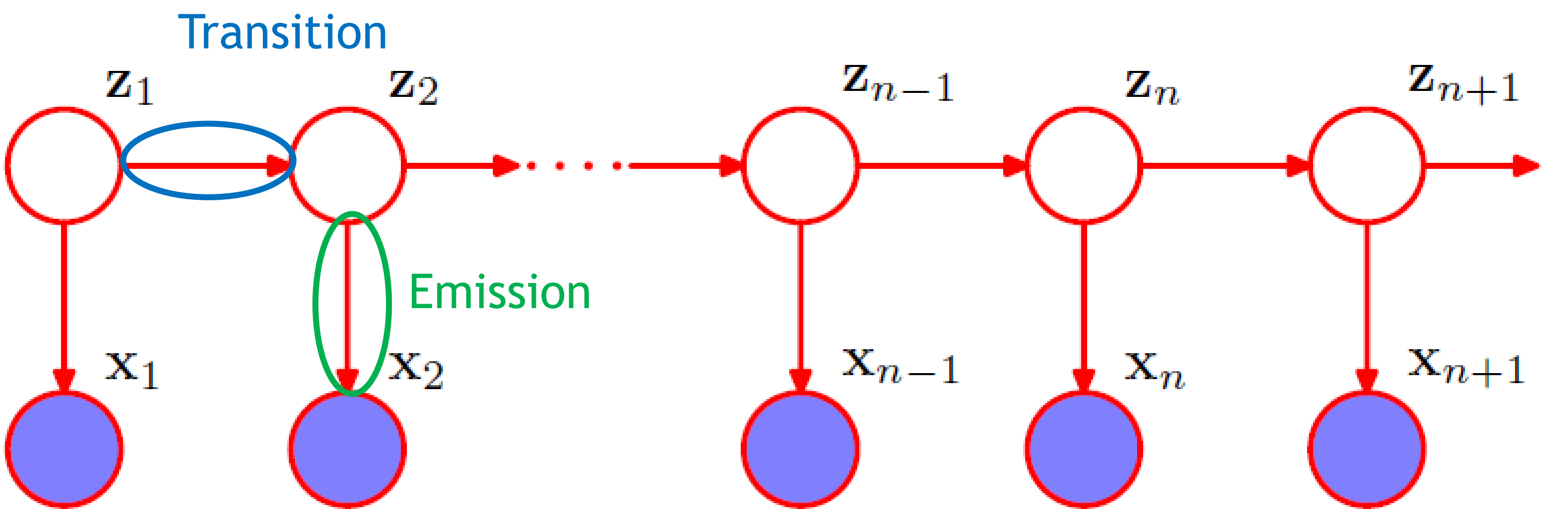


$$\mathbf{z}_{n+1} \perp\!\!\!\perp \mathbf{z}_{n-1} \mid \mathbf{z}_n.$$

$$p(\mathbf{x}_1, \ldots, \mathbf{x}_N, \mathbf{z}_1, \ldots, \mathbf{z}_N) = p(\mathbf{z}_1) \left[\prod_{n=2}^{N} p(\mathbf{z}_n | \mathbf{z}_{n-1}) \right] \prod_{n=1}^{N} p(\mathbf{x}_n | \mathbf{z}_n).$$

State Space Model :

Markov
+ latent factors



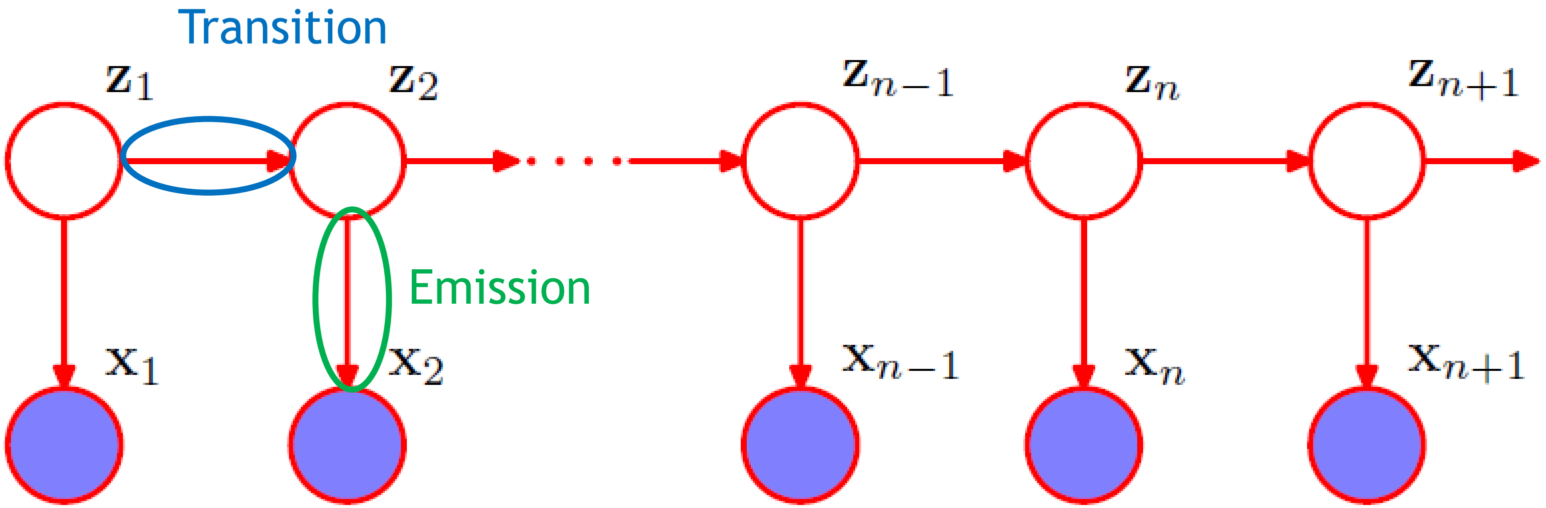
If Z is discrete – Hidden Markov Model(HMM)

If Z, X are Gaussian, with linear transition and emission

- **Linear Dynamic System(LDA)**

State Space Model：

HMM & LDA



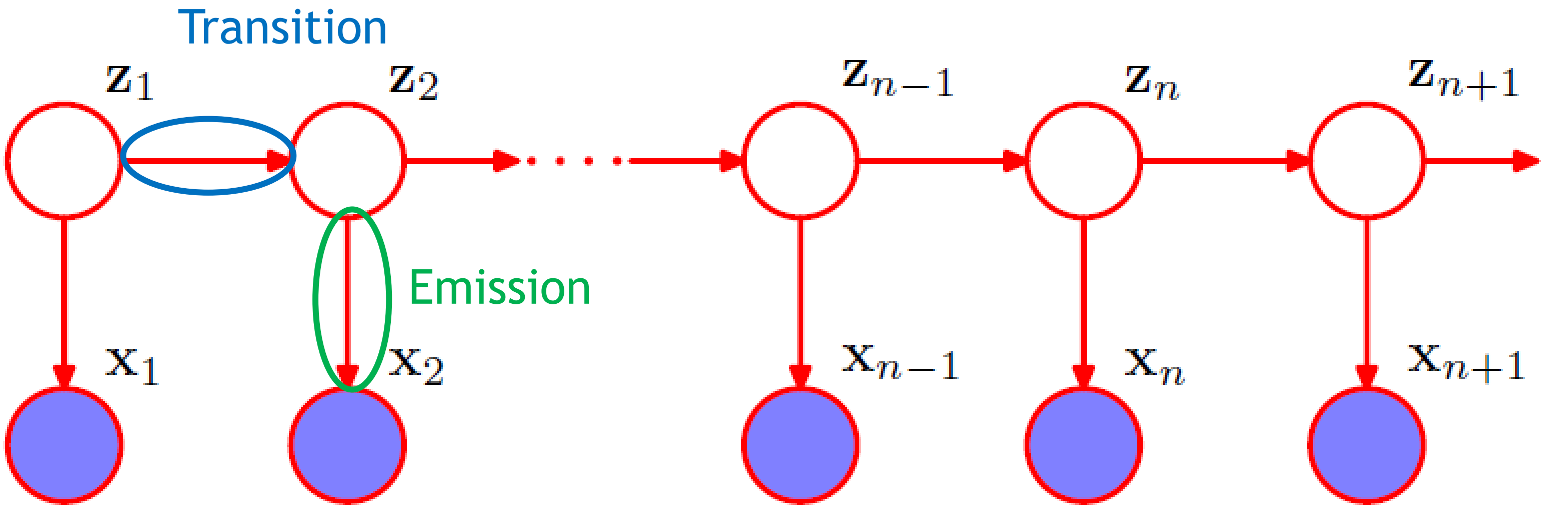
$$p(\mathbf{z}_1) = \mathcal{N}(\mathbf{z}_1 | \boldsymbol{\mu}_0, \mathbf{V}_0)$$

Transition $p(\mathbf{z}_n|\mathbf{z}_{n-1}) = \mathcal{N}(\mathbf{z}_n|\mathbf{A}\mathbf{z}_{n-1}, \boldsymbol{\Gamma})$

Emission $p(\mathbf{x}_n|\mathbf{z}_n) = \mathcal{N}(\mathbf{x}_n|\mathbf{C}\mathbf{z}_n, \boldsymbol{\Sigma}).$

LDS:

Linear Transition & Emission of Gaussian



Alternative form

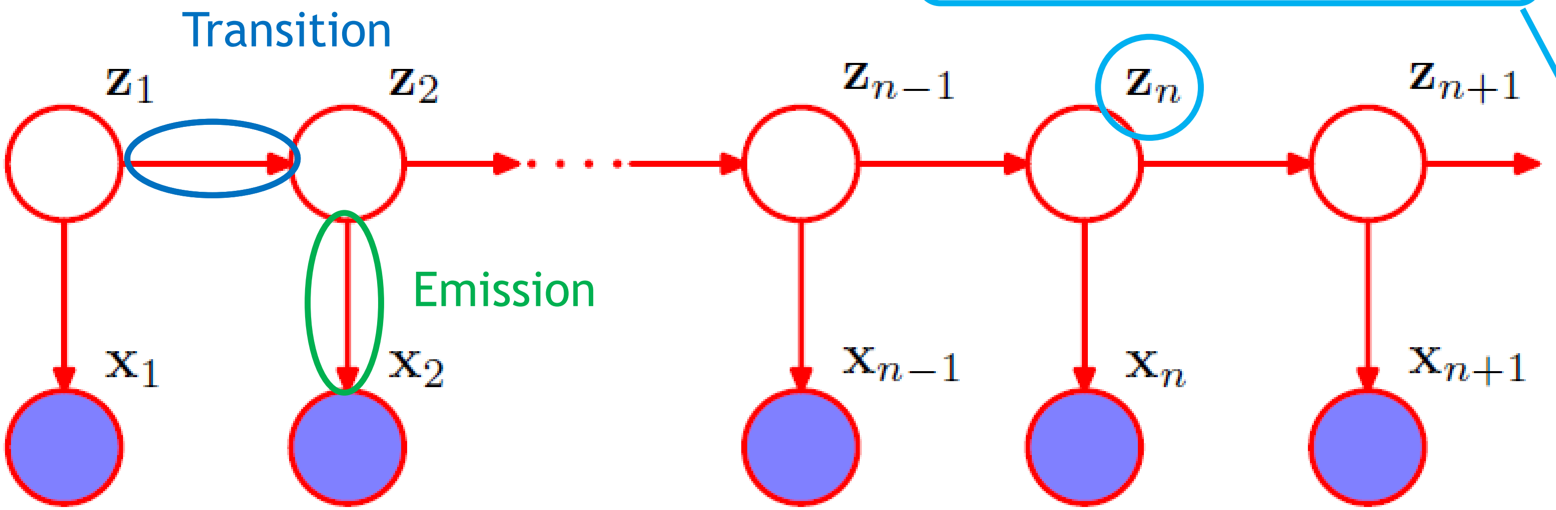
$$
\begin{aligned}
&\text{Transition} \quad & \mathbf{z}_n &= \mathbf{A}\mathbf{z}_{n-1} + \mathbf{w}_n \\
&\text{Emission} \quad & \mathbf{x}_n &= \mathbf{C}\mathbf{z}_n + \mathbf{v}_n \\
& & \mathbf{z}_1 &= \boldsymbol{\mu}_0 + \mathbf{u}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{w} &\sim \mathcal{N}(\mathbf{w}|\mathbf{0}, \boldsymbol{\Gamma}) \\
\mathbf{v} &\sim \mathcal{N}(\mathbf{v}|\mathbf{0}, \boldsymbol{\Sigma}) \\
\mathbf{u} &\sim \mathcal{N}(\mathbf{u}|\mathbf{0}, \mathbf{V}_0).
\end{aligned}
$$

LDS:

Linear Transition & Emission of Gaussian

$$\widehat{\alpha}(\mathbf{z}_n) = \mathcal{N}(\mathbf{z}_n|\boldsymbol{\mu}_n, \mathbf{V}_n).$$



$$c_n\widehat{\alpha}(\mathbf{z}_n) = p(\mathbf{x}_n|\mathbf{z}_n)\int \widehat{\alpha}(\mathbf{z}_{n-1})p(\mathbf{z}_n|\mathbf{z}_{n-1})\,\mathrm{d}\mathbf{z}_{n-1}.$$

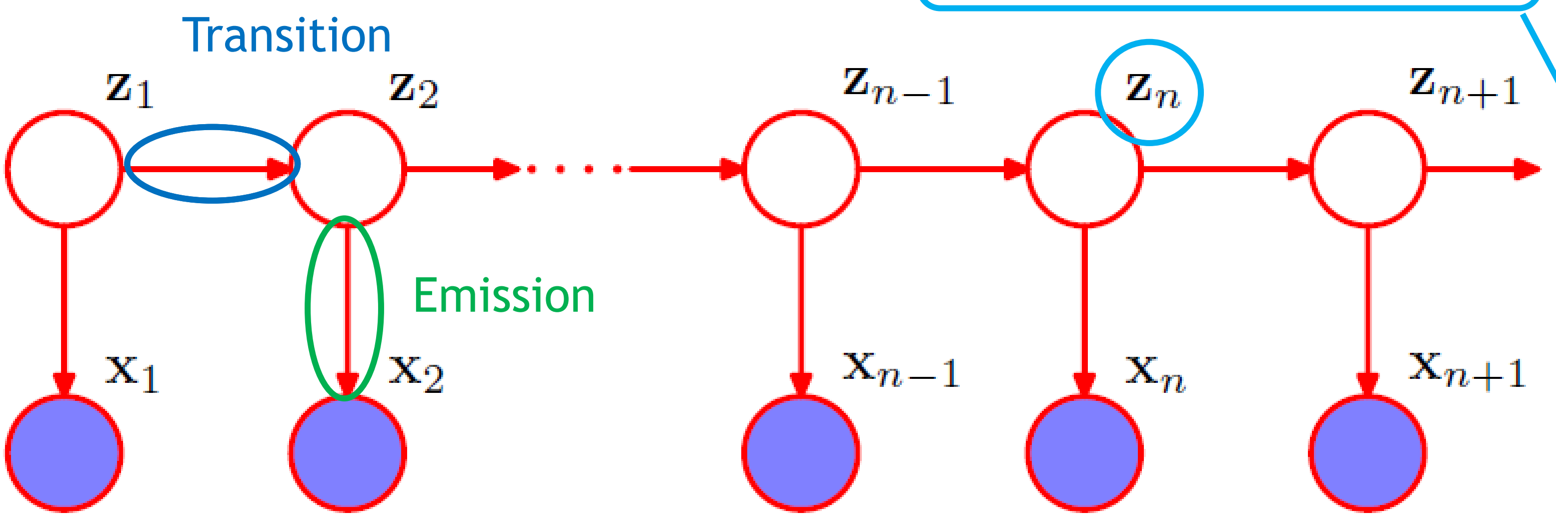
$$c_n\mathcal{N}(\mathbf{z}_n|\boldsymbol{\mu}_n, \mathbf{V}_n) = \mathcal{N}(\mathbf{x}_n|\mathbf{C}\mathbf{z}_n, \boldsymbol{\Sigma})$$
$$\int \mathcal{N}(\mathbf{z}_n|\mathbf{A}\mathbf{z}_{n-1}, \boldsymbol{\Gamma})\mathcal{N}(\mathbf{z}_{n-1}|\boldsymbol{\mu}_{n-1}, \mathbf{V}_{n-1})\,\mathrm{d}\mathbf{z}_{n-1}.$$

Goal of "Prediction"

Compute posterior:

P(Z_n|Xn,X_(n-1)..X_1

$$\widehat{\alpha}(\mathbf{z}_n) = \mathcal{N}(\mathbf{z}_n|\boldsymbol{\mu}_n, \mathbf{V}_n).$$



$$c_n\widehat{\alpha}(\mathbf{z}_n) = p(\mathbf{x}_n|\mathbf{z}_n)\int \widehat{\alpha}(\mathbf{z}_{n-1})p(\mathbf{z}_n|\mathbf{z}_{n-1})\,\mathrm{d}\mathbf{z}_{n-1}.$$

$$c_n\mathcal{N}(\mathbf{z}_n|\boldsymbol{\mu}_n, \mathbf{V}_n) = \boxed{\mathcal{N}(\mathbf{x}_n|\mathbf{C}\mathbf{z}_n, \boldsymbol{\Sigma})}$$
$$\boxed{\int \mathcal{N}(\mathbf{z}_n|\mathbf{A}\mathbf{z}_{n-1}, \boldsymbol{\Gamma})\mathcal{N}(\mathbf{z}_{n-1}|\boldsymbol{\mu}_{n-1}, \mathbf{V}_{n-1})\,\mathrm{d}\mathbf{z}_{n-1}.}$$

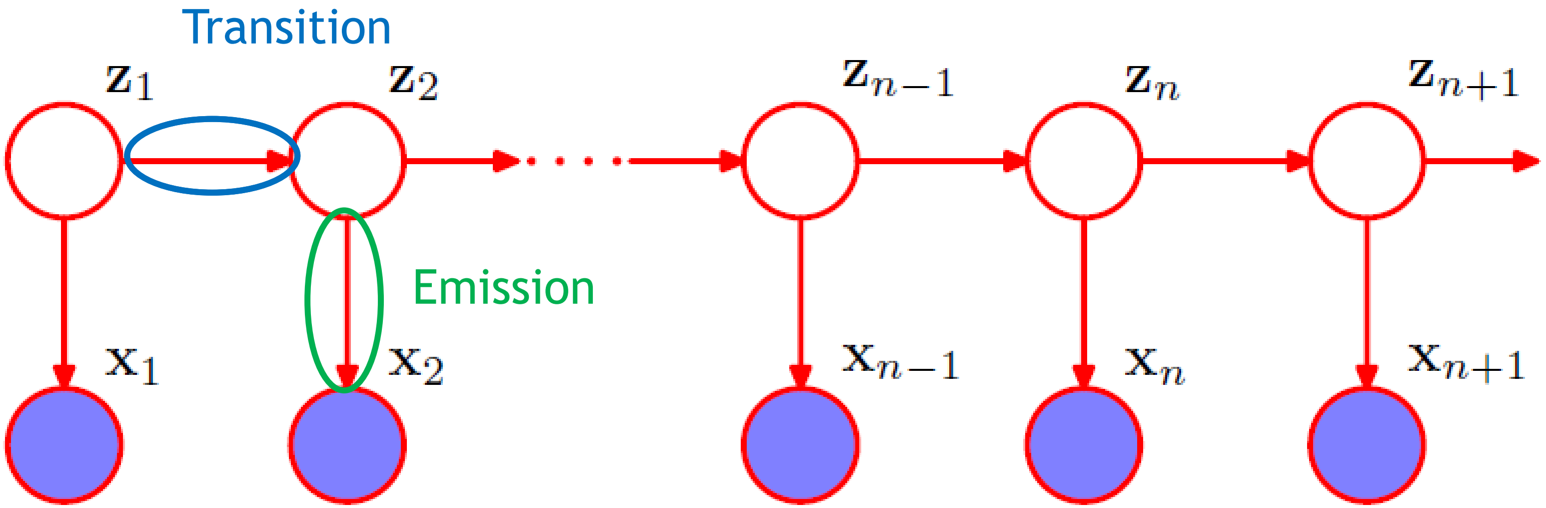
Everything is Gaussian here..!

Goal of "Prediction"

Compute posterior:

P(Z_n|Xn,X_(n-1)..X_1

$$\widehat{\alpha}(\mathbf{z}_n) = \mathcal{N}(\mathbf{z}_n|\boldsymbol{\mu}_n, \mathbf{V}_n).$$



$$\begin{aligned}
\boldsymbol{\mu}_n &= \mathbf{A}\boldsymbol{\mu}_{n-1} + \mathbf{K}_n(\mathbf{x}_n - \mathbf{C}\mathbf{A}\boldsymbol{\mu}_{n-1}) \\
\mathbf{V}_n &= (\mathbf{I} - \mathbf{K}_n\mathbf{C})\mathbf{P}_{n-1} \\
c_n &= \mathcal{N}(\mathbf{x}_n|\mathbf{C}\mathbf{A}\boldsymbol{\mu}_{n-1}, \mathbf{C}\mathbf{P}_{n-1}\mathbf{C}^{\mathrm{T}} + \boldsymbol{\Sigma}).
\end{aligned}$$

$$\boxed{\mathbf{K}_n} = \mathbf{P}_{n-1}\mathbf{C}^{\mathrm{T}}\left(\mathbf{C}\mathbf{P}_{n-1}\mathbf{C}^{\mathrm{T}} + \boldsymbol{\Sigma}\right)^{-1}.$$

Kalman gain matrix

Goal of "Prediction"

Compute posterior:

P(Z_n|Xn,X_(n-1)..X_1

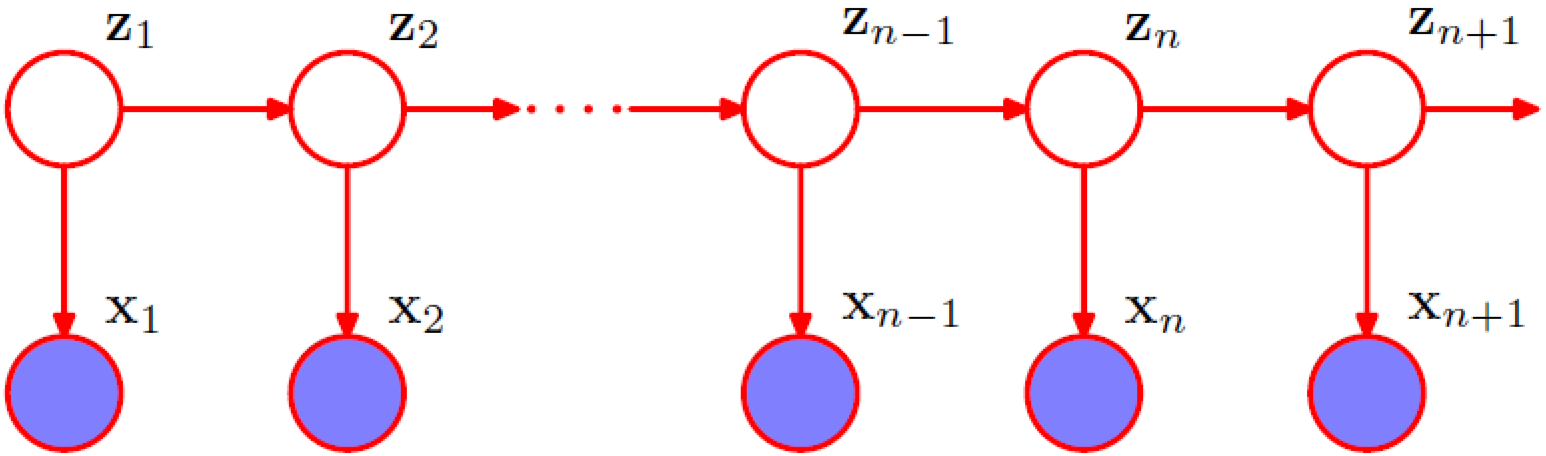
Close-form + Recursive

update posterior

---We got Kalman Filter

Forward passing of msg

$$\gamma(\mathbf{z}_n) = \boxed{\widehat{\alpha}(\mathbf{z}_n)}\boxed{\widehat{\beta}(\mathbf{z}_n)} = \mathcal{N}(\mathbf{z}_n|\widehat{\boldsymbol{\mu}}_n, \widehat{\mathbf{V}}_n).$$



$$c_{n+1}\widehat{\beta}(\mathbf{z}_n) = \int \widehat{\beta}(\mathbf{z}_{n+1})p(\mathbf{x}_{n+1}|\mathbf{z}_{n+1})p(\mathbf{z}_{n+1}|\mathbf{z}_n)\,\mathrm{d}\mathbf{z}_{n+1}.$$

$$\begin{aligned} \widehat{\boldsymbol{\mu}}_n &= \boldsymbol{\mu}_n + \mathbf{J}_n\left(\widehat{\boldsymbol{\mu}}_{n+1} - \mathbf{A}\boldsymbol{\mu}_N\right) \\ \widehat{\mathbf{V}}_n &= \mathbf{V}_n + \mathbf{J}_n\left(\widehat{\mathbf{V}}_{n+1} - \mathbf{P}_n\right)\mathbf{J}_n^{\mathrm{T}} \end{aligned}$$

where we have defined

$$\mathbf{J}_n = \mathbf{V}_n\mathbf{A}^{\mathrm{T}}\left(\mathbf{P}_n\right)^{-1}$$

How about "backward"

Compute posterior:

P(Z_n|X1,X2..X_N)

Similar things:

Close-form + Recursive update posterior

---Kalman Smoother

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Now we are here

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(stochastic differential equation)

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Ito integration

Done!

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What is a stochastic differential equation (SDE)?

- At first, we have an ordinary differential equation (ODE):

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t).$$

- Then we add white noise to the right hand side:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t) + \mathbf{w}(t).$$

- Generalize a bit by adding a multiplier matrix on the right:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t) + \mathbf{L}(\mathbf{x}, t)\,\mathbf{w}(t).$$

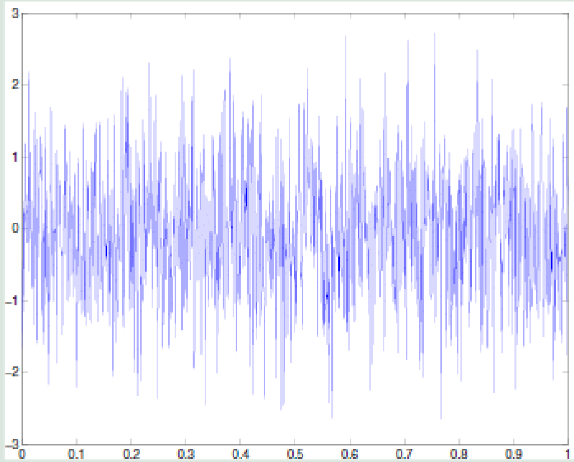
- Now we have a stochastic differential equation (SDE).
- $\mathbf{f}(\mathbf{x}, t)$ is the drift function and $\mathbf{L}(\mathbf{x}, t)$ is the dispersion matrix.

White noise

White noise

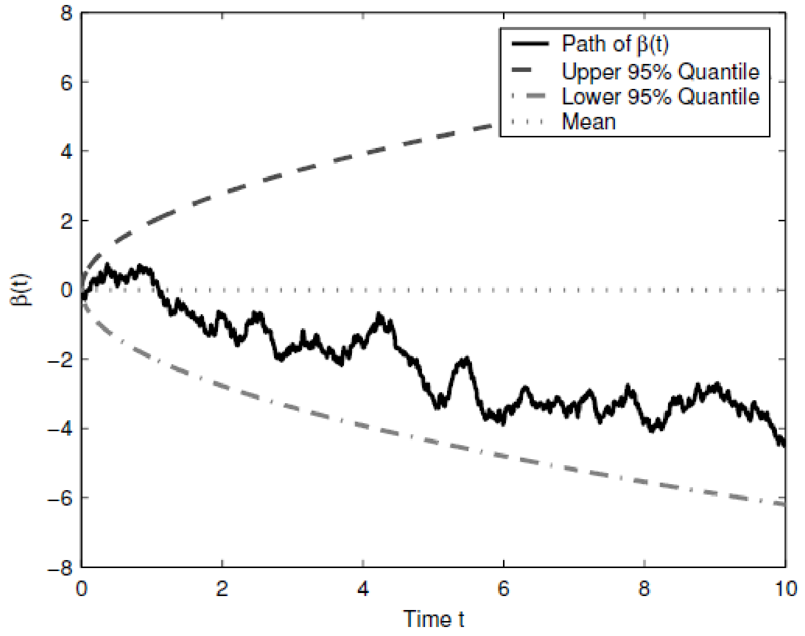
1. $\mathbf{w}(t_1)$ and $\mathbf{w}(t_2)$ are independent if $t_1 \neq t_2$.
2. $t \mapsto \mathbf{w}(t)$ is a Gaussian process with the mean and covariance:

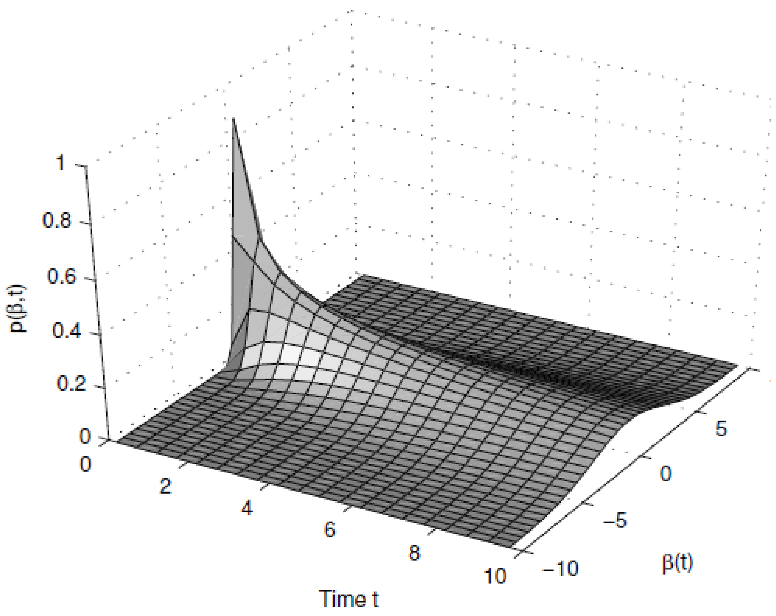
$$\mathrm{E}[\mathbf{w}(t)] = \mathbf{0}$$
$$\mathrm{E}[\mathbf{w}(t)\,\mathbf{w}^{\mathsf{T}}(s)] = \delta(t-s)\,\mathbf{Q}.$$



- $\mathbf{Q}$ is the spectral density of the process.
- The sample path $t \mapsto \mathbf{w}(t)$ is discontinuous almost everywhere.
- White noise is unbounded and it takes arbitrarily large positive and negative values at any finite interval.

What does a solution of SDE look like?





$$B_{t+\Delta t} = B_t + N(0, \Delta t)$$

$$B_t - B_s \sim N(0, t - s)$$

- *Left*: Path of a Brownian motion which is solution to stochastic differential equation

$$\frac{dx}{dt} = w(t)$$

- *Right*: Evolution of probability density of Brownian motion.

Mathematical Meaning and Notation of SDE [1/2]

- What is Itô stochastic calculus then?
- Let's take a look at the scalar equation

$$\frac{dx(t)}{dt} = f(x(t)) + L(x(t))\, w(t).$$

- Integrating from s to t gives

$$x(t) - x(s) = \int_s^t f(x(t))\, dt + \boxed{\int_s^t L(x(t))\, w(t)\, dt.}$$

- White noise is unbounded and discontinuous almost everywhere – the second integral cannot be defined as Riemann, Stieltjes, or Lebesgue integral!

Mathematical Meaning and Notation of SDE [2/2]

- Itô's idea: define $d\beta(t) = w(t)\,dt$, where $\beta(t)$ is the Wiener/Brownian process:

$$x(t) - x(s) = \int_s^t f(x(t))\,dt + \int_s^t L(x(t))\,d\beta(t).$$

- Commonly used shorthand notation for the above:

$$dx(t) = f(x(t))\,dt + L(x(t))\,d\beta(t).$$

- In stochastics literature you see this in form:

$$dX_t(\omega) = f(X_t(\omega))\,dt + L(X_t(\omega))\,d\beta_t(\omega).$$

Itô Integral

$$E(\int_T g(X_s, s)dB_s) = 0$$

- The Itô integral is defined as the limit of the expression

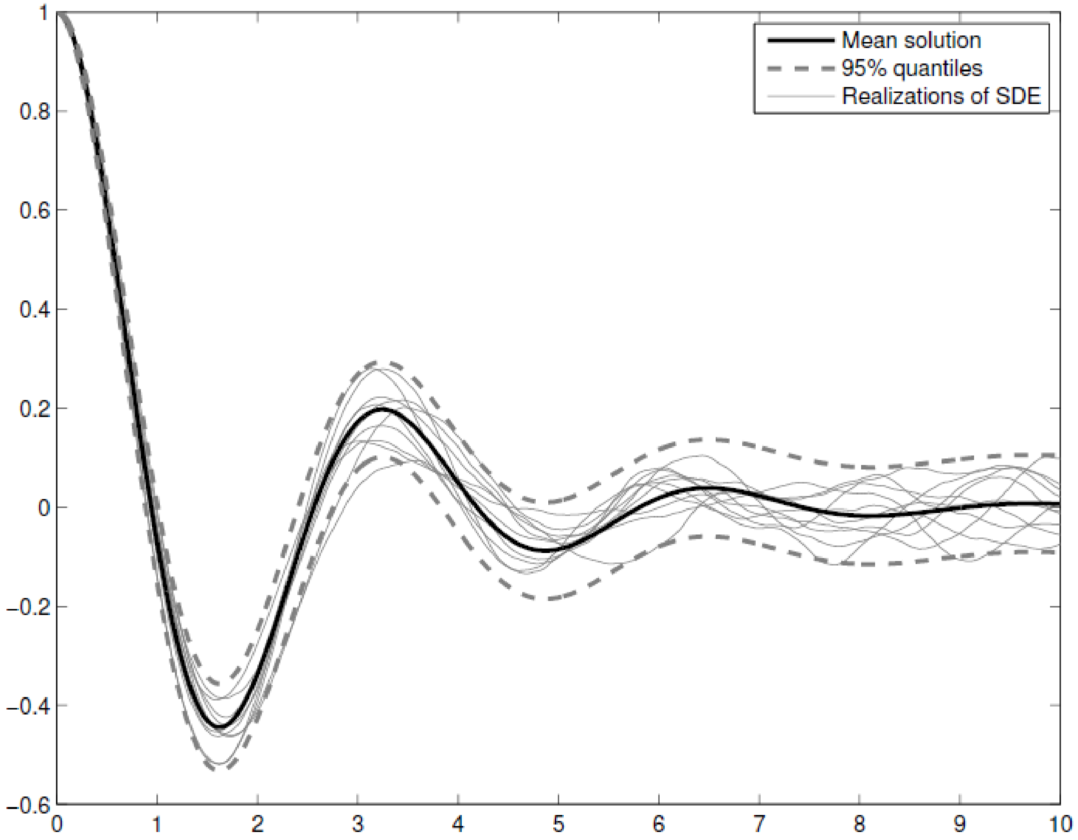
$$\begin{aligned}\int_s^t L(x(t))\, d\beta(t) = & \; L(x(t_1))\, [\beta(t_2) - \beta(t_1)] \\ & + L(x(t_2))\, [\beta(t_3) - \beta(t_2)] \\ & + \dots \\ & + L(x(t_{n-1}))\, [\beta(t_n) - \beta(t_{n-1})]\end{aligned}$$

$$E(\int_T f(X_s, s)dB_s \cdot \int_T g(X_s, s)dB_s) = \int_T E[f \cdot g]dt$$

$$(dB_t)^2 = dt$$

- The key issue is that b is evaluated at the beginning of interval, that is, we have $L(x(t_1))\, [\beta(t_2) - \beta(t_1)]$ instead of, say, $L(x(t_2))\, [\beta(t_2) - \beta(t_1)]$.
- In Riemann, Stieltjes, or Lebesgue integral the result should be independent of the evaluation point.
- The resulting calculus is called Itô calculus or the stochastic calculus

What does a solution of SDE look like? (cont.)



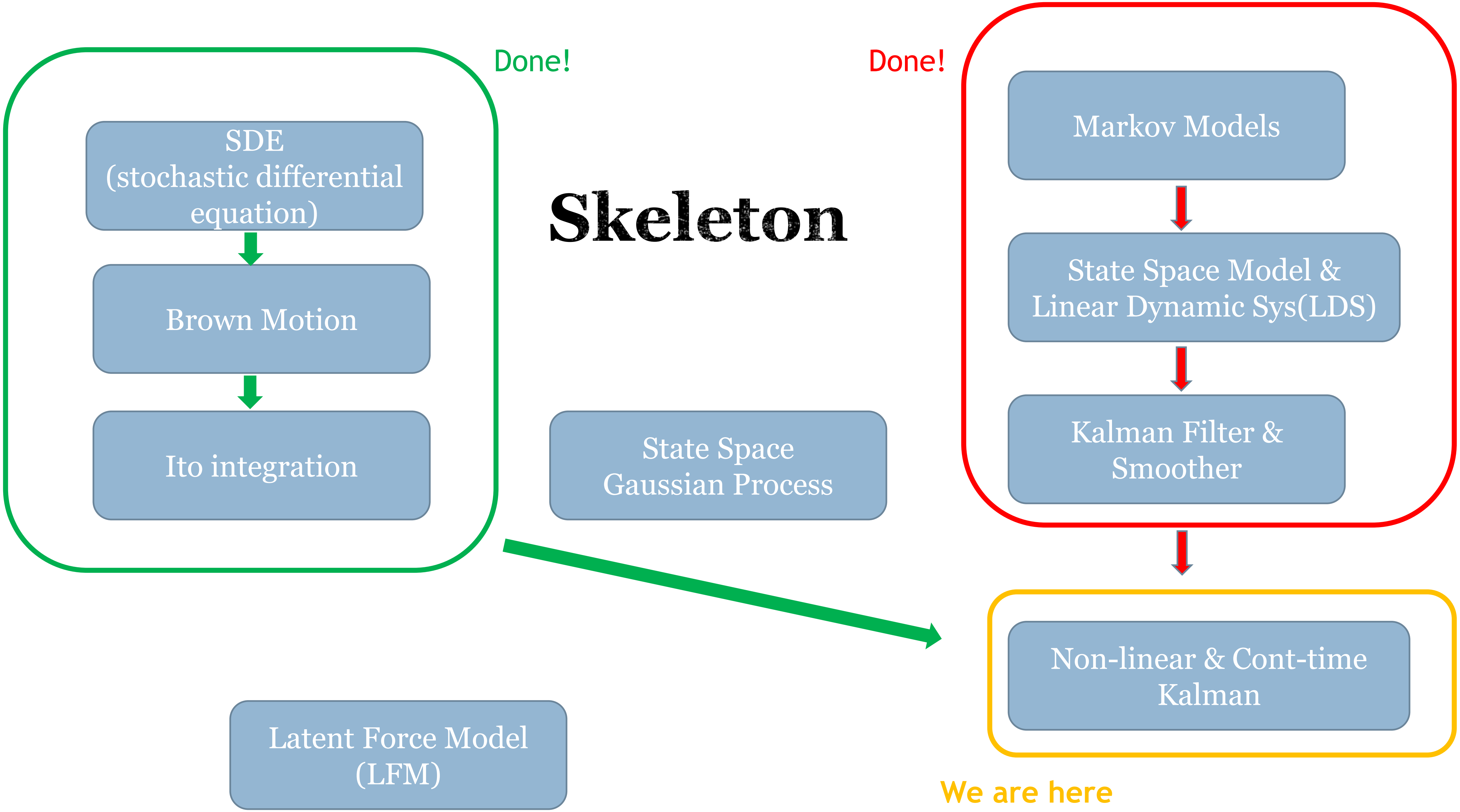
Paths of stochastic spring model

$$\frac{d^2x(t)}{dt^2} + \gamma\,\frac{dx(t)}{dt} + \nu^2\,x(t) = w(t).$$

What kind of solutions do SDEs have?

- Path of solution: Draw random path $\mathbf{w}(t)$ (or $\beta(t)$) and solve the equation using it as the input.
    - Monte Carlo simulation of SDE solutions.
    - Used in particle filtering and smoothing methods.
- Distribution of solution: Given many random $\mathbf{w}(t)$'s, what is the distribution of the state $p(\mathbf{x}(t))$?
    - Solution is given by the Fokker-Planck-Kolmogorov PDE.
    - Used in grid based and basis function methods (FEM, BEM).
- Moments: What are the mean and covariance of $\mathbf{x}(t)$?
    - Ordinary differential equations for the mean and covariance.
    - Used in non-linear Kalman (Gaussian) filters and smoothers.

Skeleton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Mathematical Problem Formulation

Classical Kalman:
Linear + discrete

- Mathematical model is (the special case considered here):

Nonlinear +
Continue (trans) +
Discrete (emission)

$$\boxed{\begin{aligned} d\mathbf{x} &= \mathbf{f}(\mathbf{x})\, dt + \mathbf{L}\, d\beta(t) \\ \mathbf{y}_k &= \mathbf{H}\, \mathbf{x}(t_k) + \mathbf{r}_k . \end{aligned}}$$

Transition
Emission

$$\boxed{\begin{aligned} \mathbf{z}_n &= \mathbf{A}\mathbf{z}_{n-1} + \mathbf{w}_n \\ \mathbf{x}_n &= \mathbf{C}\mathbf{z}_n + \mathbf{v}_n \end{aligned}}$$

- The dynamics of state $\mathbf{x}(t) \in \mathbb{R}^n$ are modeled as Itô-type stochastic differential equations (SDE, Itô diffusion).
- $\beta(t) \in \mathbb{R}^s$ is a vector of Brownian motions (Wiener processes) with diffusion matrix $\mathbf{Q}$ and dimension $s \leq n$.
- $\mathbf{r}_k \in \mathbb{R}^d$ is a Gaussian random variable $\mathbf{r}_k \sim \mathrm{N}(0, \mathbf{R})$.
- We can think SDE as white noise driven differential equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) + \mathbf{L}\, \mathbf{w}(t),$$

where the white noise is defined as $\mathbf{w}(t) = d\beta(t)/dt$.

Bayesian Filtering and Smoothing Solution

- We don't aim to compute the full (infinite-dimensional) posterior of the state, but instead only its time-marginals.
- Filtering/prediction solutions: Compute the posterior distribution(s)

$$p(\mathbf{x}(t) \mid \mathbf{y}_1, \ldots, \mathbf{y}_k), \qquad t \in [t_k, t_{k+1}).$$

- Smoothing solution: Compute the posterior distribution(s)

$$p(\mathbf{x}(t) \mid \mathbf{y}_1, \ldots, \mathbf{y}_T), \qquad t \in [t_0, t_T].$$

- If we could solve the transition density $p(\mathbf{x}(t_k) \mid \mathbf{x}(t_{k-1}))$, the model would reduce to a discrete-time model:

$$\begin{aligned}\mathbf{x}(t_k) &\sim p(\mathbf{x}(t_k) \mid \mathbf{x}(t_{k-1})) \\ \mathbf{y}_k &\sim p(\mathbf{y}_k \mid \mathbf{x}(t_k)).\end{aligned}$$

Continuous-Discrete Non-Linear Kalman Filtering [1/2]

- The current special case of the model is:

$$
\begin{aligned}
d\mathbf{x} &= \mathbf{f}(\mathbf{x})\,dt + \mathbf{L}\,d\beta(t) \\
\mathbf{y}_k &= \mathbf{H}\,\mathbf{x}(t_k) + \mathbf{r}_k\,.
\end{aligned}
$$

- We can now apply Gaussian (process) approximation to the posteric of the process $\mathbf{x}(t)$ – when combined with approximate Bayesian filter, leads to non-linear Kalman filters.
- Note that we can easily generalize to non-linear measurement mode $\mathbf{H}\,\mathbf{x}(t_k) \to \mathbf{h}(\mathbf{x}(t_k))$.
- The resulting approximation is of the form

$$
p(\mathbf{x}(t) \mid \mathbf{y}_{1:k}) \approx \mathrm{N}(\mathbf{x}(t) \mid \mathbf{m}(t), \mathbf{P}(t)), \quad t \in [t_k, t_{k+1}),
$$

  where $\mathbf{m}(t)$ and $\mathbf{P}(t)$ are computed by the non-linear Kalman filter.
- Different brands: EKF, UKF, CKF, GHKF, etc.

Continuous-Discrete Non-Linear Kalman Filtering [2/2]

Continuous-Discrete Non-Linear Kalman Filter

1. Prediction step: Integrate the following time t_{k-1} to t_k^-:

$$\frac{d\mathbf{m}}{dt} = \mathrm{E}[\mathbf{f}(\mathbf{x})]$$
$$\frac{d\mathbf{P}}{dt} = \mathrm{E}[(\mathbf{x} - \mathbf{m}_k)\, \mathbf{f}^T(\mathbf{x})] + \mathrm{E}[\mathbf{f}(\mathbf{x})\,(\mathbf{x} - \mathbf{m})^T] + \mathrm{E}[\mathbf{L}(\mathbf{x})\, \mathbf{Q}\, \mathbf{L}^T(\mathbf{x})].$$

2. Update step: Update step is the linear Kalman filter update:

$$\begin{aligned}
\mathbf{S}_k &= \mathbf{H}\, \mathbf{P}(t_k^-)\, \mathbf{H}^T + \mathbf{R}_k \\
\mathbf{K}_k &= \mathbf{P}(t_k^-)\, \mathbf{H}_k^T\, \mathbf{S}_k^{-1} \\
\mathbf{m}(t_k) &= \mathbf{m}(t_k^-) + \mathbf{K}_k\, [\mathbf{y}_k - \mathbf{H}\, \mathbf{m}(t_k^-)] \\
\mathbf{P}(t_k) &= \mathbf{P}(t_k^-) - \mathbf{K}_k\, \mathbf{S}_k\, \mathbf{K}_k^T.
\end{aligned}$$

Continuous-time extended Kalman filter [edit]

Model

$$\boxed{\begin{aligned}\dot{\mathbf{x}}(t) &= f\bigl(\mathbf{x}(t), \mathbf{u}(t)\bigr) + \mathbf{w}(t) & \mathbf{w}(t) &\sim \mathcal{N}\bigl(\mathbf{0}, \mathbf{Q}(t)\bigr)\\ \mathbf{z}(t) &= h\bigl(\mathbf{x}(t)\bigr) + \mathbf{v}(t) & \mathbf{v}(t) &\sim \mathcal{N}\bigl(\mathbf{0}, \mathbf{R}(t)\bigr)\end{aligned}}$$

State transition become a First-order SDE

Initialize

$$\hat{\mathbf{x}}(t_0) = E\bigl[\mathbf{x}(t_0)\bigr], \mathbf{P}(t_0) = Var\bigl[\mathbf{x}(t_0)\bigr]$$

Predict-Update

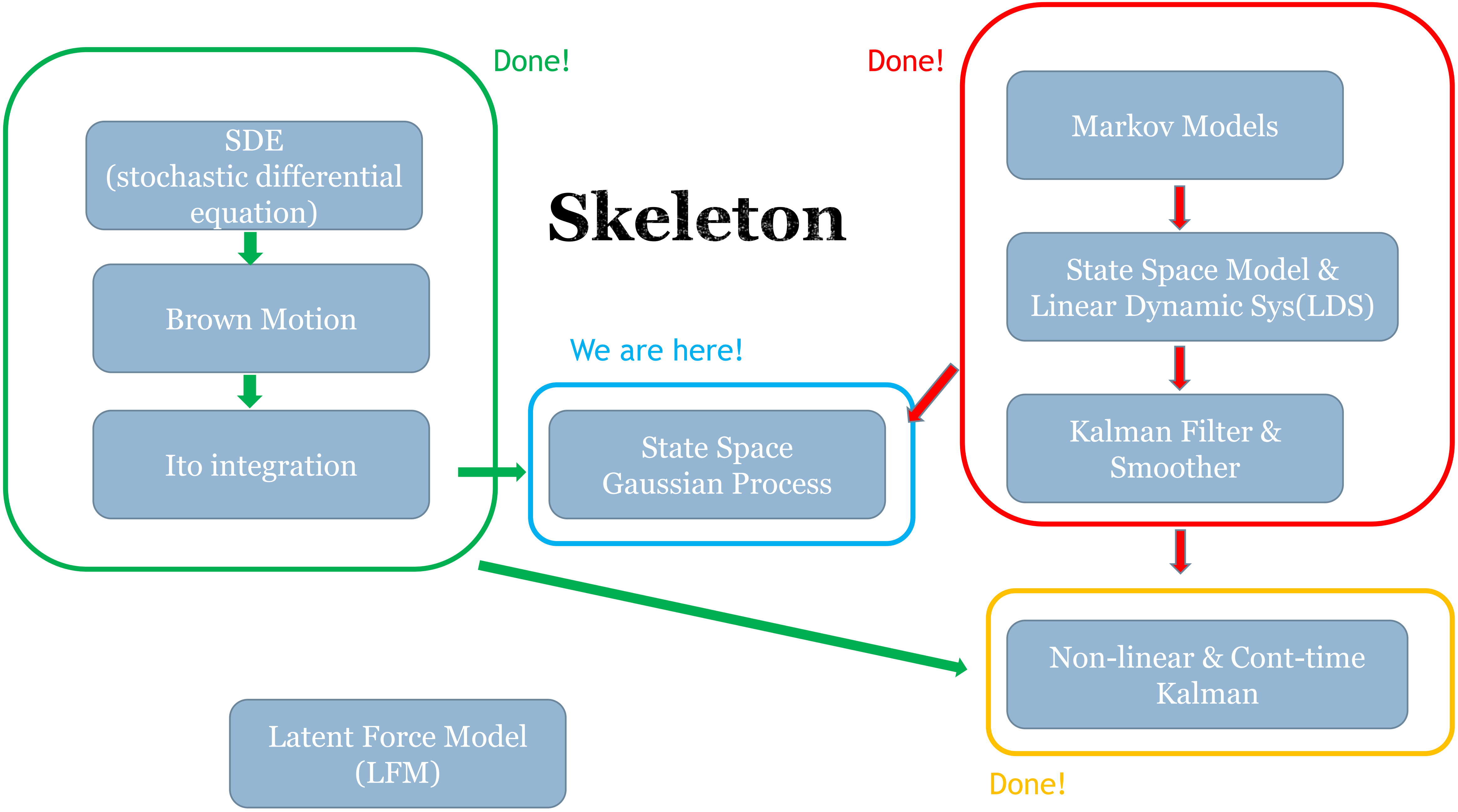
$$\boxed{\begin{aligned}\dot{\hat{\mathbf{x}}}(t) &= f\bigl(\hat{\mathbf{x}}(t), \mathbf{u}(t)\bigr) + \mathbf{K}(t)\Bigl(\mathbf{z}(t) - h\bigl(\hat{\mathbf{x}}(t)\bigr)\Bigr)\\ \dot{\mathbf{P}}(t) &= \mathbf{F}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}(t)^{\top} - \mathbf{K}(t)\mathbf{H}(t)\mathbf{P}(t) + \mathbf{Q}(t)\end{aligned}}$$

$$\mathbf{K}(t) = \mathbf{P}(t)\mathbf{H}(t)^{\top}\mathbf{R}(t)^{-1}$$

$$\mathbf{F}(t) = \left.\frac{\partial f}{\partial \mathbf{x}}\right|_{\hat{\mathbf{x}}(t), \mathbf{u}(t)}$$

$$\mathbf{H}(t) = \left.\frac{\partial h}{\partial \mathbf{x}}\right|_{\hat{\mathbf{x}}(t)}$$

Moment of state can be computed by solving ODE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Latent Force Model
(LFM)

State Space GP

- GPs with certain stationary covariance functions (e.g.Matern) can be represented as state space models.

- Consider a Gaussian process regression problem

$$f(x) \sim \mathrm{GP}(0, \sigma^2 \exp(-\lambda|x - x'|))$$
$$y_k = f(x_k) + \varepsilon_k$$

- This is equivalent to the state-space model

$$\frac{df(t)}{dt} = -\lambda f(t) + w(t)$$
$$y_k = f(t_k) + \varepsilon_k$$

that is, with $f_k = f(t_k)$ we have a Gauss-Markov model

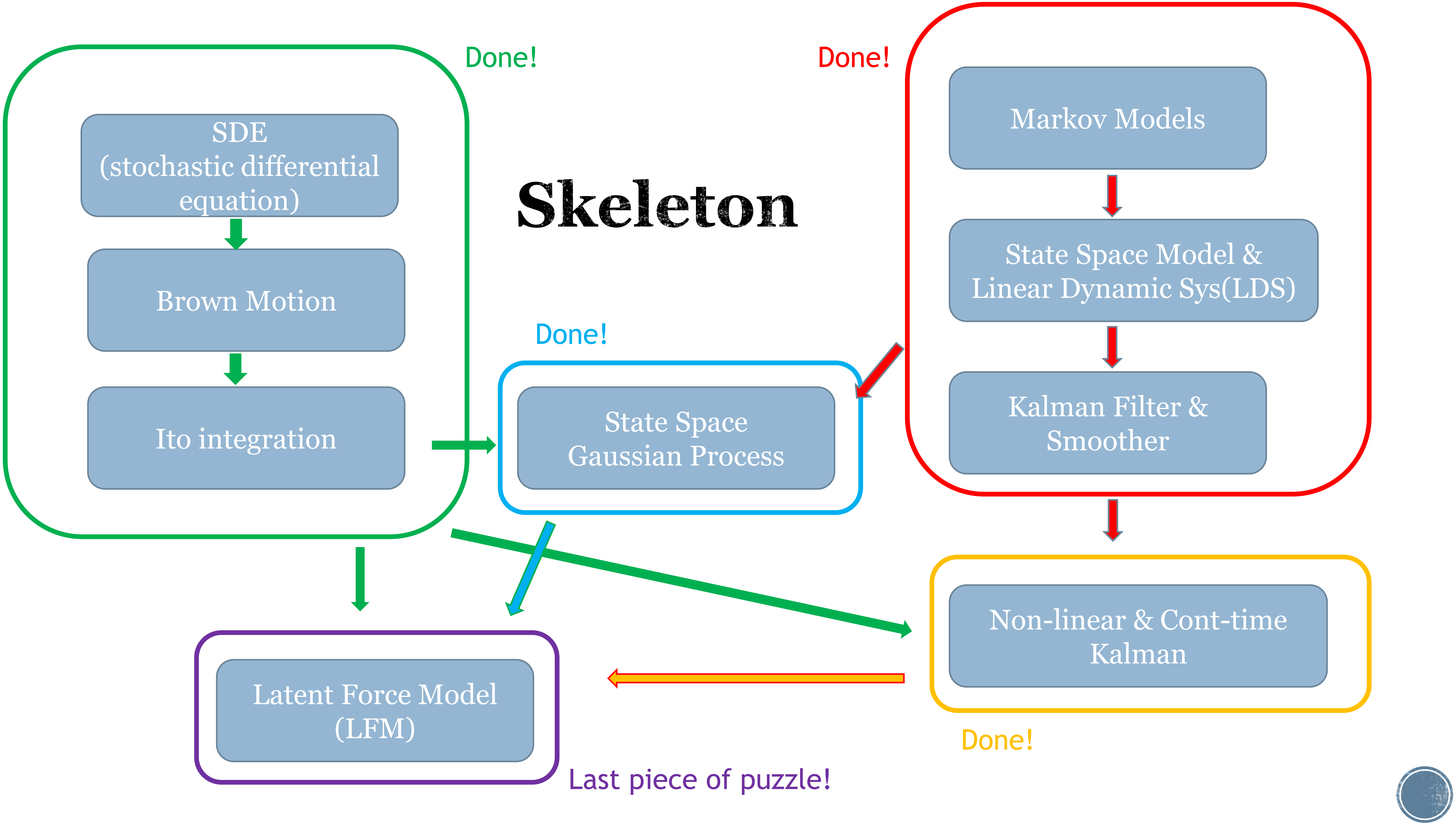
$$f_{k+1} \sim p(f_{k+1} \mid f_k)$$
$$y_k \sim p(y_k \mid f_k)$$

- Solvable in $O(n)$ time using Kalman filter/smoother.

State Space GP

GPs with certain stationary covariance functions (e.g.Matern) can be represented as state space models.

GP model $\mathbf{x} \in \mathbb{R}^d, t \in \mathbb{R}$	Equivalent S(P)DE model
Spatial $k(\mathbf{x}, \mathbf{x}')$	SPDE model ($\mathcal{L}$ is an operator) $$\mathcal{L}\, f(\mathbf{x}) = w(\mathbf{x})$$
Temporal $k(t, t')$	State-space/SDE model $$\frac{d\mathbf{f}(t)}{dt} = \mathbf{A}\,\mathbf{f}(t) + \mathbf{L}\,w(t)$$
Spatio-temporal $k(\mathbf{x}, t; \mathbf{x}', t')$	Stochastic evolution equation $$\frac{\partial}{\partial t}\mathbf{f}(\mathbf{x}, t) = \mathcal{A}_x\,\mathbf{f}(\mathbf{x}, t) + \mathbf{L}\,w(\mathbf{x}, t)$$



Skeleton
Done!
SDE
(stochastic differential equation)
Brown Motion
Ito integration
Done!
Markov Models
State Space Model &
Linear Dynamic Sys(LDS)
Kalman Filter &
Smoother
Done!
State Space
Gaussian Process
Non-linear & Cont-time
Kalman
Done!
Latent Force Model
(LFM)
Last piece of puzzle!

The Basic Idea of State-Space Representation

- Assume that our latent force model is of the form

$$\boxed{\frac{dx_f(t)}{dt} = g(x_f(t)) + u(t),}$$

where $u(t)$ is the latent force.

- We measure the system at discrete instants of time:

$$y_k = x_f(t_k) + r_k$$

- Let's now model $u(t)$ as a Gaussian process of Matern type

$$C(\tau) = \sigma^2 \frac{2^{1-\nu}}{\Gamma(\nu)} \left(\sqrt{2\nu}\,\frac{\tau}{l}\right)^{\nu} K_{\nu}\left(\sqrt{2\nu}\,\frac{\tau}{l}\right)$$

- Recall that if, for example, $\nu = 1/2$ then the GP can be expressed as the solution of the stochastic differential equation (SDE)

$$\boxed{\frac{du(t)}{dt} = -\lambda\, u(t) + w(t)}$$

The Basic Idea of State-Space Representation (cont.)

- If we define $\mathbf{x} = (x_f, u)$, we get a two-dimensional SDE

$$\frac{d\mathbf{x}}{dt} = \underbrace{\begin{pmatrix} g(x_1(t)) + x_2(t) \\ -\lambda\, x_2(t) \end{pmatrix}}_{\mathbf{f}(\mathbf{x})} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\mathbf{L}} w(t)$$

- We can now rewrite the measurement model as

$$y_k = \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{\mathbf{H}} \mathbf{x}(t_k) + r_k$$

- Thus the result is a model of the generic form

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) + \mathbf{L}\,\mathbf{w}(t)$$
$$\mathbf{y}_k = \mathbf{H}\,\mathbf{x}(t_k) + \mathbf{r}_k.$$

- This model can now be efficiently tackled with non-linear Kalman filtering and smoothing.

SDE View of Latent Force Models [1/4]

- Let's now take a look at the non-linear state-space LFM methodology presented in Hartikainen and Särkkä (2012).
- Consider the latent force model (Lawrence et al., 2006)

$$\frac{dx_j(t)}{dt} = B_j + \sum_{r=1}^{R} S_{j,r} g_j(u_r(t)) - D_j x_j(t), \quad j = 1, \ldots, N$$

- We can now use independent Gaussian process (GP) priors

$$u_r(t) \sim \mathrm{GP}(m(t), k_{u_r}(t, t')), \quad r = 1, \ldots, R$$

where $m(t)$ and $k_{u_r}(t, t')$ were suitably chosen mean and covariance functions.

- That is, we can formulate the GP priors on the components of $\mathbf{u}(t) = (u_1(t) \ldots u_R(t))^T$ as multivariate space space models (SDEs) of form

$$d\mathbf{z}_r(t) = \boxed{\mathbf{F}_{z,r}} \mathbf{z}_r(t)\, dt + \boxed{\mathbf{L}_{z,r}} d\beta_{z,r}(t)$$

Companion form matrix

where

$$\mathbf{z}_r(t) = \begin{pmatrix} u_r(t) & \frac{du_r(t)}{dt} & \cdots & \frac{d^{d_r-1}u_r(t)}{dt^{d_r-1}} \end{pmatrix}^T$$

and

$$\mathbf{F}_{z,r} = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -a_r^0 & \cdots & -a_r^{p_r-2} & -a_r^{p_r-1} \end{pmatrix}, \quad \mathbf{L}_{z,r} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ q_r \end{pmatrix}.$$

General LFM in SDE view

Summary

- Non-linear LFMs can be converted into state-space form by:
  1. Converting the latent GPs into state-space form.
  2. Forming an augmented state space model.
- Bayesian filtering and smoothing, in principle, provide the full solution to the problem.
- In practice, formal solution is intractable – involves, e.g., solutions to particle differential equations.
- Approximate inference in non-linear LFMs can be implemented with non-linear Kalman filters and smoothers.

LFM in SDE view:

One trivial application

Single-latent-force + learnable transition & emission

q(f,\theta) can be further inferenced by SVI, sampling..

Black-Box Inference for Non-Linear Latent Force Models, AISTAT 2020

$$\alpha_0(\boldsymbol{x}, u, t; \boldsymbol{\theta})\boldsymbol{x}(t) + \sum_{i=1}^{n} \alpha_i(\boldsymbol{x}, u, t; \boldsymbol{\theta}) \frac{\mathrm{d}^i}{\mathrm{d}t^i}\boldsymbol{x}(t) = u(t),$$

$$\boldsymbol{y}_j \sim \pi(\boldsymbol{h}(\boldsymbol{x}(\tau_j); \boldsymbol{\theta}))$$

where $u(t) \sim \mathcal{GP}(0, k(t, t'))$

define a joint state vector $\boldsymbol{f}(t) = [\boldsymbol{x}(t), \mathrm{d}\boldsymbol{x}/\mathrm{d}t, \ldots, u(t_k), \mathrm{d}u/\mathrm{d}t, \ldots]^\top$

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{f}(t) = \boxed{\boldsymbol{D}}(\boldsymbol{f}, t; \boldsymbol{\theta}) + \boxed{\mathbf{L}}w(t),$$

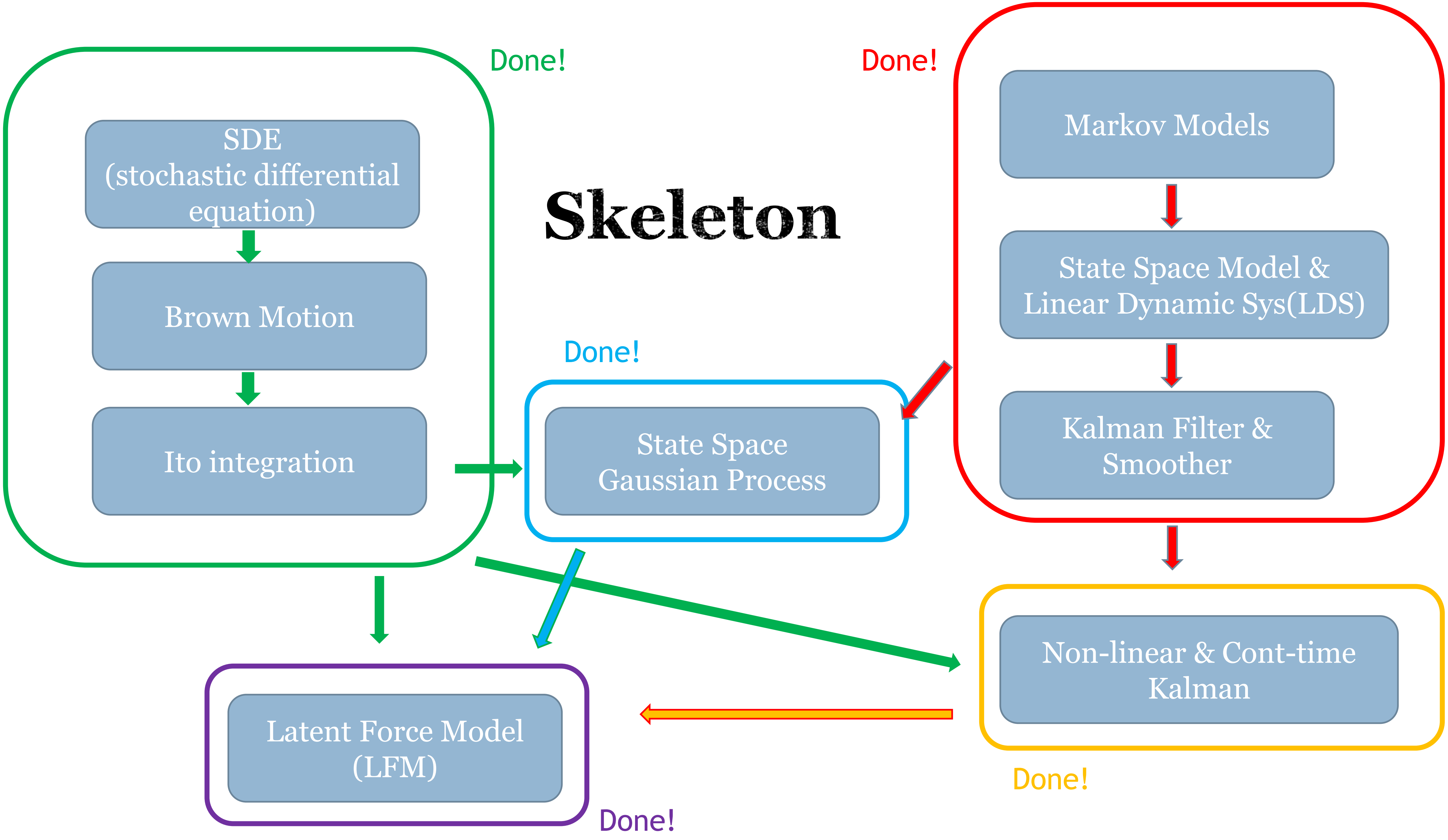
Companion form sys

$$p(\boldsymbol{x}_{0:T}, u_{0:T}, \boldsymbol{\theta} \mid \boldsymbol{y}) \propto$$

$$p(\boldsymbol{\theta})p(\boldsymbol{f}_0 \mid \boldsymbol{\theta}) \prod_{k=0}^{T-1} \boxed{p(\boldsymbol{f}_{k+1} \mid \boldsymbol{f}_k, \boldsymbol{\theta})} \prod_{j=1}^{N} \boxed{p(\boldsymbol{y}_j \mid \boldsymbol{f}(\tau_j), \boldsymbol{\theta})},$$

Transition density from non-linear Kalman

Observed/llk function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Done!

Simo Särkkä

FOLLOW

Associate Professor, Aalto University
Verified email at aalto.fi - Homepage

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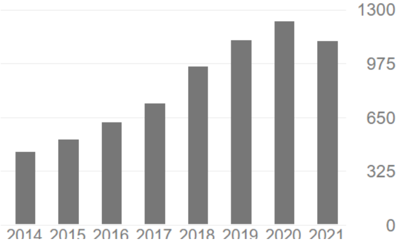

Arno Solin

FOLLOW

Assistant Professor in Machine Learning, Aalto University
Verified email at aalto.fi - Homepage

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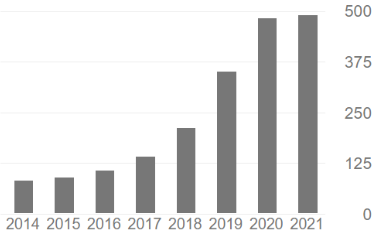

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Simo Särkkä
Associate Professor, Aalto Unive…

Aalto University, Espoo, Finland