Lecture 4. Higher-order functions

Functional Programming

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Goal of typed purely functional programming

Keep programs easy to reason about by

- data-flow only through function arguments and return values
  - no hidden data-flow through mutable variables/state

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- (almost) unique types
  - no inheritance hell

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- function call and return as only control-flow primitive
  - no loops, `break`, `continue`, `goto`
- (almost) unique types
  - no inheritance hell
- high-level declarative data-structures
  - no explicit reference-based data structures

Goal of typed purely functional programming

Keep programs easy to reason about by

- function call and return as only control-flow primitive
  - no loops, `break`, `continue`, `goto`
  - instead: higher-order functions (functions which use other functions)
  - extra pay-off: huge abstraction power -> more code reuse!

The remaining two: this Thursday!

Goals of today

- Define and use higher-order functions
  - Functions which use other functions
  - In particular, `map`, `filter`, `foldr` and `foldl`
  - vs general recursion
- Use anonymous functions
- Understand function composition
- Understand partial application

Chapter 7 and 4.5-4.6 from Hutton's book

Higher-order functions vs curried functions

- Curried functions (of multiple arguments):

  ```
  f :: a -> b -> c
  ```

  read

  ```
  f :: a -> (b -> c)
  ```

- Higher-order functions:

  ```
  f :: (a -> b) -> c
  ```

- Exercise: come up with some examples from high school mathematics

What can higher-order functions do?

- How can we use argument-functions?
- Can we pattern match on them?
- Can we inspect their source code from a higher-order function?

What can higher-order functions do?

- How can we use argument-functions?
  - By applying them! That's it!
- Can we pattern match on them?
  - No! But we can feed them inputs and pattern match on the results!
- Can we inspect their source code from a higher-order function?
  - No! Only their input-output behaviour!

Usage of `map`

From the previous lectures...

- `map` applies a function uniformly over a list
  - The function to apply is an argument to `map`

    ```
    map :: (a -> b) -> [a] -> [b]
    ```

  ```
  > map length ["a", "abc", "ab"]
  [1,3,2]
  ```

- It is very similar to a list comprehension

  ```
  > [length s | s <- ["a", "abc", "ab"]]
  [1,3,2]
  ```

Cooking `map`

1. Define the type

```
map :: _
```

2. Enumerate the cases
   - We cannot pattern match on functions

```
map f []     = _
map f (x:xs) = _
```

Try it yourself!

Cooking `map`

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map :: (a -> b) -> [a] -> [b]
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2. Enumerate the cases
   - We cannot pattern match on functions

```
map f []     = _
map f (x:xs) = _
```

3. Define the simple (base) cases

```
map f []     = []
```

Cooking `map`

4. Define the other (recursive) cases
   - The current element needs to be transformed by `f`
   - The rest are transformed uniformly by `map`

```
map f (x:xs) = f x : map f xs
```

It makes no difference whether the function we use is global or is an argument

Usage of `filter`

`filter p xs` leaves only the elements in `xs` which satisfy the predicate `p`

- A predicate is a function which returns `True` or `False`
- In other words, `p` must return `Bool`

```
> even x = x `mod` 2 == 0
> filter even [1 .. 4]
[2,4]
> largerThan10 x = x > 10
> filter largerThan10 [1 .. 4]
[]
```

Cooking `filter`

1. Define the type

```
filter :: _
```

2. Enumerate the cases

```
filter p []     = _
filter p (x:xs) = _
```

Try it yourself!

Cooking `filter`

1. Define the type

```
filter :: (a -> Bool) -> [a] -> [a]
```

2. Enumerate the cases

```
filter p []     = _
filter p (x:xs) = _
```

3. Define the simple (base) cases

```
filter p []     = []
```

Cooking `filter`

4. Define the other (recursive) cases
   - We have to distinguish whether the predicate holds
   - Version 1, using conditionals

```
filter p (x:xs) = if p x
                     then x : filter p xs
                     else     filter p xs
```

   - Version 2, using guards

```
filter p (x:xs) | p x       = x : filter p xs
                | otherwise =     filter p xs
```

Alternative definitions using comprehensions

`map` and `filter` can be easily defined using comprehensions

```
map    f xs = [f x | x <- xs]

filter p xs = [x   | x <- xs, p x]
```

The recursive definitions are better to reason about code

(Ab)use of local definitions

Suppose we want to double the numbers in a list

- We can define a `double` function and apply it to the list

```
double n = 2 * n
doubleList xs = map double xs
```

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doubleList xs = map double xs
  where double n = 2 * n
```

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- We can define a `double` function and apply it to the list

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- This pollutes the code, so we can put it in a `where`

```
doubleList xs = map double xs
  where double n = 2 * n
```

- But we are still using too much code for such a simple and small function!
  - Each call to `map` or `filter` may require one of those

Anonymous functions

\ arguments -> code

Haskell allows you to define functions without a name

```
doubleList xs = map (\x -> 2 * x) xs
```

- They are called anonymous functions or (lambda) abstractions
- The \ symbol resembles a Greek λ

Anonymous functions

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- The \ symbol resembles a Greek λ

Historical note: the theoretical basis for functional programming is called λ-calculus and was introduced in the 1930s by the American mathematician Alonzo Church

Anonymous functions are just functions

- They have a type, which is always a function type

```
> :t \x -> 2 * x
\x -> 2 * x :: Num a => a -> a
```

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```

- You can use it everywhere you need a function

```
> (\x -> 2 * x) 3
6
> filter (\x -> x > 10) [1 .. 20]
[11,12,13,14,15,16,17,18,19,20]
```

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> (\x -> 2 * x) 3
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[11,12,13,14,15,16,17,18,19,20]
```

- Even when you define a function

```
double = \x -> 2 * x
```

Functions which return functions

```
flip :: (a -> b -> c) -> (b -> a -> c)
flip f = _
```

Functions which return functions

```
flip :: (a -> b -> c) -> (b -> a -> c)
flip f = \y x -> f x y
```

- This function is called a combinator
  - It creates a function from another function
- The resulting function may get more arguments
  - They appear in reverse order from the original

```
> flip map [1,2,3] (\x -> 2 * x)
[2,4,6]
```

Functions are curried

- In Haskell, functions take one argument at a time
  - The result might be another function

```
map :: (a -> b) ->  [a] -> [b]
map :: (a -> b) -> ([a] -> [b])
```

  - We say functions in Haskell are curried
- A two-argument function is actually a one-argument function which returns yet another function which takes the next argument and produces a result

Different ways to write

Take a function with three arguments

```
addThree :: Int -> Int -> Int -> Int
addThree x y z = x + y + z
```

Parentheses in functions associate to the right

```
addThree :: Int -> (Int -> (Int -> Int))
```

Different ways to write

Take a function with three arguments

```
addThree :: Int -> Int -> Int -> Int
addThree x y z = x + y + z
```

Parentheses in functions associate to the right

```
addThree :: Int -> (Int -> (Int -> Int))
```

We can define the function in these other ways

```
addThree x y =             \z -> x + y + z
addThree x   =       \y -> \z -> x + y + z
addThree     = \x -> \y -> \z -> x + y + z
addThree     = \x     y     z -> x + y + z
```

Partial application

- Since Haskell functions take one argument at a time, we can provide less than the ones stated in the signature
  - The result is yet another function
  - We say the function has been partially appplied

```
> :t map (\x -> 2 * x)
map (\x -> 2 * x) :: ???
```

Partial application

- Since Haskell functions take one argument at a time, we can provide less than the ones stated in the signature
  - The result is yet another function
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```
> :t map (\x -> 2 * x)
map (\x -> 2 * x) :: Num b => [b] -> [b]

> :{
| let doubleList = map (\x -> 2 * x)
| in doubleList [1,2,3]
| :}
[2,4,6]
```

Definition by partial application

Instead of writing out all the arguments

```
doubleList xs = map (\x -> 2 * x) xs
```

Haskells make use of partial application if possible

```
doubleList    = map (\x -> 2 * x)
```

Note that `xs` has been dropped from both sides

Definition by partial application

Instead of writing out all the arguments

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doubleList xs = map (\x -> 2 * x) xs
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Haskells make use of partial application if possible

```
doubleList    = map (\x -> 2 * x)
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Note that `xs` has been dropped from both sides

Technical note: this is called η (eta) reduction

Sections

Sections are shorthand for partial application of operators

```
(x #) = \y -> x # y   -- Application of 1st arg.
(# y) = \x -> x # y   -- Application of 2nd arg.
```

They help us remove even more clutter

```
doubleList    = map (2 *)
largerThan10 = filter (> 10)
```

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They help us remove even more clutter

```
doubleList    = map (2 *)
largerThan10 = filter (> 10)
```

Warning! Order matters in sections

```
> filter (> 10) [1 .. 20]
[11,12,13,14,15,16,17,18,19,20]
> filter (10 >) [1 .. 20]
[1,2,3,4,5,6,7,8,9]
```

Example: working with a list of functions

Apply a list of functions in order to a starting argument

```
> applyAll [(+ 1), (* 2), (\x -> x - 3)] 3
5  -- ((3 + 1) * 2) - 3
```

- Define the function
- What is the type of `applyAll`?

Try it yourself!

Example: working with a list of functions

```
applyAll [f]      x = f x
applyAll (f : fs) x = applyAll fs (f x)
```

Let's think harder about the base case!

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```

Let's think harder about the base case!

```
applyAll []       x = x
applyAll (f : fs) x = applyAll fs (f x)
```

```
> :t applyAll
applyAll :: [a -> a] -> a -> a
```


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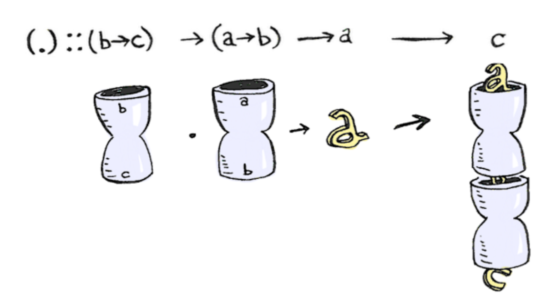
Function composition

Another example of function combinator

- g composed with `f`, or g after `f`

```
(.) :: (b -> c) -> (a -> b) -> (a -> c)
g . f = _
```



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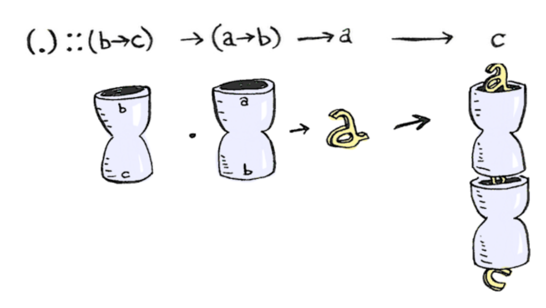

Examples of function composition

```haskell
not  :: Bool -> Bool
even :: Int  -> Bool

odd x = not (even x)
odd   = not . even  -- Better

-- Remove all elements which satisfy the predicate
filterNot :: (a -> Bool) -> [a] -> [a]
```

Try it yourself!

Examples of function composition

```haskell
not  :: Bool -> Bool
even :: Int  -> Bool

odd x = not (even x)
odd   = not . even  -- Better

-- Remove all elements which satisfy the predicate
filterNot :: (a -> Bool) -> [a] -> [a]
filterNot p xs = filter (\x -> not (p x)) xs
filterNot p xs = filter (not . p) xs  -- Better
filterNot p    = filter (not . p)     -- Even better
```

Function pipelines

You can define many functions as a pipeline

- Sequence of functions composed one after the other
- This style of coding is called point-free
  - Even though it actually has more point symbols!

```
maxAverage :: [[Float]] -> Float
maxAverage
  = maximum . map average . filter (not . null)
  where average xs
          = sum xs / fromIntegral (length xs)
```

Point-free craziness

You can go even further in this point-free style by using more combinators

```
where average = (/) <$> sum
                    <*> (fromIntegral . length)
```

```
(<$>) ::      (a -> b) -> (c -> a) -> (c -> b)
(<*>) :: (c -> a -> b) -> (c -> a) -> (c -> b)
```

Warning! Don't overdo it!

- This definition of `average` is less readable

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Question

Write `applyAll` in point-free style

```
applyAll []       x = x
applyAll (f : fs) x = applyAll fs (f x)
```

Hint: for the first case remember that `id x = x`

Question

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Hint: for the first case remember that `id x = x`

```
applyAll []       = id
applyAll (f : fs) = applyAll fs . f
```

Folds

Similar functions

```
sum []     = 0
sum (x:xs) = x + sum xs

product []     = 1
product (x:xs) = x * product xs

and []     = True
and (x:xs) = x && and xs
```

Similar functions

```
sum []     = 0
sum (x:xs) = x + sum xs

product []     = 1
product (x:xs) = x * product xs

and []     = True
and (x:xs) = x && and xs
```

- The three return a value in the `[]` case
- For the `x:xs` case, they combine the head with the result for the rest of the list
  - `(+)` for `sum`, `(*)` for `product`, `(&&)` for `and`

Avoid duplication, abstract!

```haskell
sum []     = 0
sum (x:xs) = x + sum xs
```

Let's replace the moving parts with arguments `f` and `v`

- First-class functions are key for abstraction

```haskell
foldr :: (a -> b -> b) -> b -> [a] -> b
foldr _ v []     = v
foldr f v (x:xs) = f x (foldr f v xs)
                 = x `f` foldr f v xs  -- Infix
```

Avoid duplication, abstract!

- The previous definitions become much shorter
- The use of `foldr` conveys an intention
  - They all compute a result by iteratively applying a function over all the elements in the list

```
sum     = foldr (+)  0
product = foldr (*)  1
and     = foldr (&&) True
```

`foldr` is for "fold right"

```
foldr (+) 0 (x : y : z : [])
=
x + foldr (+) 0 (y : z : [])
=
x + (y + foldr (+) 0 (z : []))
=
x + (y + (z + foldr 0 []))
=
x + (y + (z + 0))
```

- `foldr` introduces parentheses "to the right"
- Initial value is in innermost parentheses

Another view of `foldr`

```
foldr (+) 0 [x, y, z]
=
foldr (+) 0 (x : (y : (z : [ ])))
                 |     |     |   |
                 |     |     |   |
                 ↓     ↓     ↓   ↓
            (x + (y + (z +  0 )))
```

- `(:)` is replaced by the combination function
- `[]` is replaced by the initial value

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`length` as a right fold

```
length []     = 0
length (_:xs) = 1 + length xs

foldr _ v []     = v
foldr f v (x:xs) = f x (foldr f v xs)
```

We want to find `f` and `v` such that

$$length = foldr\ f\ v$$

Try it yourself!

`length` as a right fold

- Case of empty list, `[]`

```
length [] = 0
          = v = foldr f v []
```

`length` as a right fold

- Case of empty list, `[]`

```
length [] = 0
          = v = foldr f v []
```

- Case of cons, `x:xs`

```
length (x:xs) = 1 + length xs
              = f x (foldr f v xs)
              = -- Assuming we know it for xs
                f x (length xs)
```

  - We need to have a function such that

```
f x (length xs) = 1 + length xs
===> f x y = 1 + y
===> f     = \x y -> 1 + y
```


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`length` as a right fold

In conclusion,

```
length = foldr (\_ y -> 1 + y) 0

length [1,2,3]
= -- definition of length
foldr (\_ y -> 1 + y) [1,2,3]
= -- application of foldr
1 + (1 + (1 + 0))
= -- perform addition
3
```


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Left folds

```
foldr (+) 0 [x,y,z]
= (x + (y + (z + 0)))
```

Is it possible to have a "mirror" function `foldl`?

```
foldl (+) 0 [x,y,z]
= (((0 + x) + y) + z)
```

- Parenthesis associate to the left
- Initial value still in the innermost position

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Calculating `foldl`

- The case for empty lists is the same as `foldr`

```
foldl f v [] = v
```

Calculating `foldl`

- The case for empty lists is the same as `foldr`

```
foldl f v [] = v
```

- For the general case, notice this fact:

```
  foldl (+)    0                    [x,y,z]
= foldl (+)   (0 + x)               [y,z]
= foldl (+)  ((0 + x) + y)          [z]
= foldl (+) (((0 + x) + y) + z)     []
```

  - The second argument works as an accumulator

```
foldl f v (x:xs) = foldl f (f v x) xs
```


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foldr versus foldl

```
  foldr (+) 0 [1, 2, ..., n]
= 1 + foldr (+) 0 [2, ..., n]
= ... = 1 + (2 + (... + (n + 0)))
      = 1 + (2 + (... + n)) = ...

  foldl (+) 0 [1, 2, ..., n]
= foldl (+) (0 + 1) [2, ..., n]
= ... = foldl (+) (((0 + 1) + ...) + n) []
= (((0 + 1) + ...) + n)
= ((1 + ...) + n)  = ...
```

- With `foldr` and `foldl` you wait until the end to start combining

`foldr` versus `foldl`

```
  foldl' (+) 0 [1, 2, ..., n]
= foldl' (+) (0 + 1) [2, ..., n]
= foldl' (+) 1 [2, ..., n]  -- (!)
= foldl' (+) (1 + 2) [..., n]
= foldl' (+) 3 [..., n]     -- (!)
```

- With `foldr` and `foldl` you wait until the end to start combining
- With `foldl'` you compute the value “on the go”
  - `foldl'` is usually more efficient

`foldr` versus `foldl`

In the case of `(+)`, the result is the same

```
> foldr (+) 0 [1,2,3]
6
> foldl (+) 0 [1,2,3]
6
```

This is not the case for every function

```
> foldr (-) 0 [1,2,3]
2
> foldl (-) 0 [1,2,3]
-6
```

Monoids

One possible set of properties which ensure that the direction of folding does not matter

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One possible set of properties which ensure that the direction of folding does not matter

1. The initial value does not affect the outcome

   `f v x = x = f v x`   `0 + x = x = x + 0`

   - We say that `v` is an identity for `f`

Monoids

One possible set of properties which ensure that the direction of folding does not matter

1. The initial value does not affect the outcome

   `f v x = x = f v x` `0 + x = x = x + 0`

   - We say that `v` is an identity for `f`

2. The way we parenthesize does not affect the outcome

   `f (f x y) z = f x (f y z)`

   `(x + y) + z = x + (y + z)`

   - We say that the operation `f` is associative

A data type with such an operation is called a monoid

Avoid explicit recursion

- `map`, `filter`, `foldr` and `foldl` abstract common recursion patterns over lists
  - Most functions can be written as a combination of those
- Good style: prefer using those functions over recursion
  - The intention of the code is clearer
  - Less code written means less code to debug
  - Complex recursion suggest that you might be doing too much in one function
  - Primitive rather than general recursion: always terminates!

Avoid explicit recursion, example

`count p xs` counts how many elements in `xs` satisfy `p`

```
count :: (a -> Bool) -> [a] -> Int
count _ []          = 0
count p (x:xs) | p x       = 1 + count p xs
               | otherwise =     count p xs
```

Try it yourself!

Avoid explicit recursion, example

`count p xs` counts how many elements in `xs` satisfy `p`

```
count :: (a -> Bool) -> [a] -> Int
count _ []      = 0
count p (x:xs) | p x       = 1 + count p xs
               | otherwise =     count p xs
```

```
count p xs = length (filter p xs)
```

```
count p    = length . filter p
```

`applyAll` as a left fold

```
applyAll []       x = x
applyAll (f : fs) x = applyAll fs (f x)
```

Is `applyAll` as a right or a left fold?

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```
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```

Is `applyAll` as a right or a left fold?

```
> applyAll [f1,f2,f3] x
f3 (f2 (f1 x))  -- start from the left value
```

applyAll as a left fold

```
applyAll []        x = x
applyAll (f : fs) x = applyAll fs (f x)
```

Is applyAll as a right or a left fold?

```
> applyAll [f1,f2,f3] x
f3 (f2 (f1 x))  -- start from the left value

-- Solution 1
applyAll fs x = foldl (\y f -> f y) x fs
```

applyAll as a right fold

```
applyAll []       = id
applyAll (f : fs) = applyAll fs . f
```

We can also see it as a series of compositions

```
> applyAll [f1,f2,f3]
id . (f3 . (f2 . f1))
```

applyAll as a right fold

```
applyAll []       = id
applyAll (f : fs) = applyAll fs . f
```

We can also see it as a series of compositions

```
> applyAll [f1,f2,f3]
id . (f3 . (f2 . f1))
```

```
-- Solution 2
applyAll fs = foldr (\r f -> f . r) id fs
```

Can we make it look better?

applyAll as a fold

```
applyAll fs = foldr (\r f -> f  .  r)   id fs
-- Drop the argument in both sides
applyAll    = foldr (\r f -> f  .  r)   id
-- Use "normal" application order for (.)
applyAll    = foldr (\r f -> (.) f r) id
-- Use the flip combinator
applyAll    = foldr (flip (.))          id
-- "flip (.)" has a name for itself
applyAll    = foldr (>>>)               id
```

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- `map`, `filter`, `foldr` and `foldl` describe common recursion patterns over lists

Acknowledgements

Function composition image taken from
`adit.io/posts/2013-07-22-lenses-in-pictures.html`

A type inference question

Given a list of numbers, let's create a list of "adders", each of them adding this number to another given one

```
adders = map (\n -> \x -> n + x)
       = -- eta reducation
         map (\n -> (n +))
       = -- eta reduction
         map (+)
```

```
> [a 5 | a <- adders [1,2,3]]
[6,7,8]
```

A type inference question

Let us look at the types of the functions involved

```
(+) :: Int -> (Int -> Int)

-- Generalized type
map :: (a -> b) -> [a] -> [b]

-- In our case a      = Int
--              a -> b = Int -> (Int -> Int)
--        Thus,      b =        Int -> Int
map :: (Int -> Int -> Int)
    -> [Int] -> [Int -> Int]
```