Purely Functional Data structures

Functional Programming

Utrecht University

Goals

- Know the difference between persistent (purely functional) and ephemeral data structures,
- Be able to use persistent data structures,
- Define and work with custom data types

Data Structures in Memory

- What does `x:xs` look like in memory?

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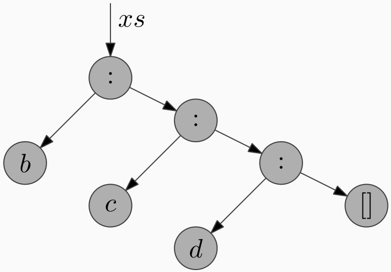

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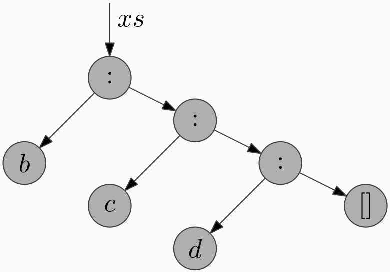

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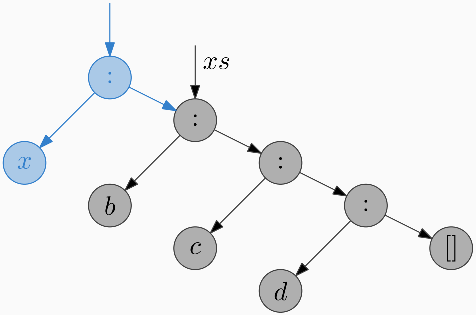

Data Structures in Memory

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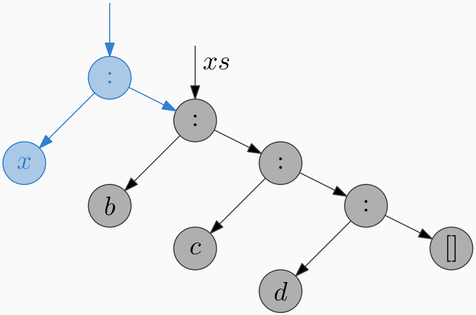

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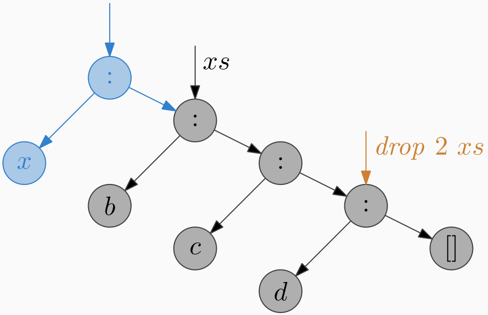

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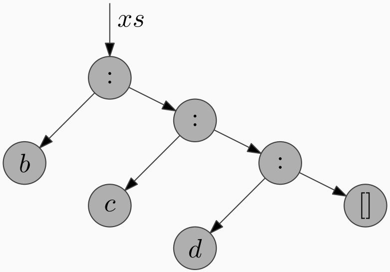

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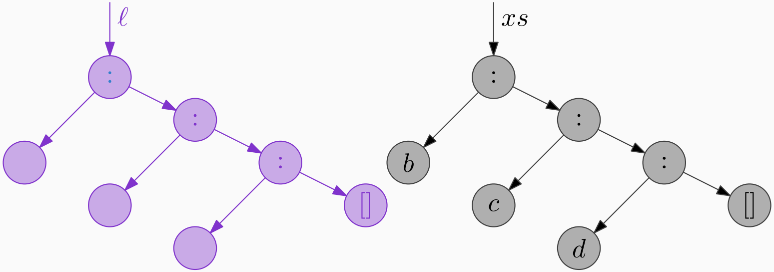

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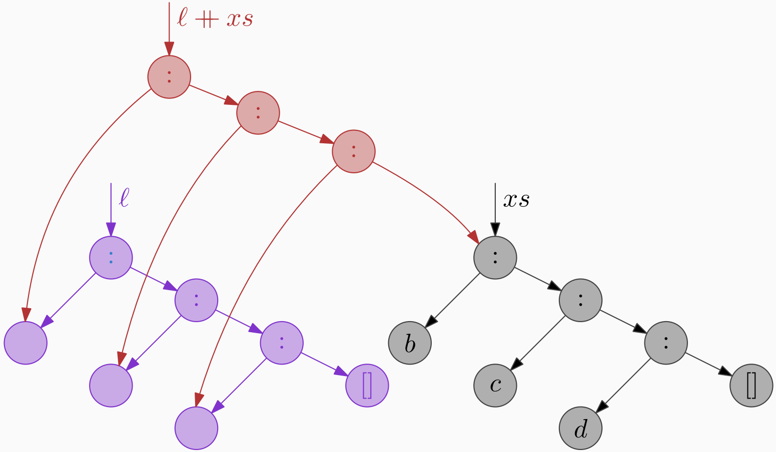

Persistent vs Ephemeral

- Data structures in which old versions are available are *persistent* data structures.
- Traditional data structures are *ephemeral*.

Persistent vs Ephemeral

- Advantages of persistent data structures:
    - Convenient to have both old and new:
        - Separation of concerns;
        - Compute subexpressions independently
    - Output may contain old versions (i.e. tails)

Can we get this for other data structures?

Yes*!

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[*] for a lot of them

Successor/Ordered Set Data Structure SuccDS

- Store an set $S \subseteq U$ of ordered elements s.t. we can efficiently find successor of a query $q \in U$.
- The successor of q is the smallest element in S larger or equal to q.

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- Store an set $S \subseteq U$ of ordered elements s.t. we can efficiently find successor of a query $q \in U$.
- The successor of q is the smallest element in S larger or equal to q.
- Example: $S = \{1, 4, 5, 8, 9, 20\}$, successor of $q = 6$ is 8.

Implementing a Successor DS `SuccDS a`

- Store the elements of type a in a data structure of type `SuccDS a`
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```
succOf :: Ord a => a -> SuccDS a -> Maybe a
```

Implementing a Successor DS: Try 1, Lists

- Idea: Use an (unordered) list

```
type SuccDS a = [a]
```

Implementing a Successor DS: Try 1, Lists

```haskell
succOf      :: Ord a => a -> SuccDS a -> Maybe a
succOf q s  = minimum' [ x | x <- s, x >= q]
  where
    minimum' [] = Nothing
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- Running time: $O(n)$

Implementing a Successor DS: Try 2, Ordered Lists

- Idea: Use an *ordered* list.

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succOf q []                 = Nothing
succOf q (x:s) | x < q      = succOf q s
               | otherwise  = Just x
```

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- Idea: Use an *ordered* list.

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- Does not really help: running time is still $O(n)$.
- We need a better data structure.

Implementing a Successor DS: Try 3, BSTs

- Idea: Use a binary search tree (BST).

```haskell
data Tree a = Leaf
            | Node (Tree a) a (Tree a)
            deriving (Show,Eq)

type SuccDS a = Tree a
```

Implementing a Successor DS: Try 3, BSTs

- Idea: Use a binary search tree (BST).

```haskell
data Tree a = Leaf
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type SuccDS a = Tree a
```

- Can we list all elements in a `Tree a`?
- Can we test if a `t :: Tree a` is a BST?

Warmup: Listing The elements of a Tree

```haskell
elems                :: Tree a -> [a]
elems Leaf           = []
elems (Node l x r) = elems l ++ [x] ++ elems r
```

Warmup: Testing if a Tree is a BST?

```
isBST              :: Ord a => Tree a -> Bool
isBST Leaf         = True
isBST (Node l x r) =  all (<= x) (elems l)
                   && all (>= x) (elems r)
                   && isBST l && isBST r
```

- This implementation uses $O(n^2)$ time.
- Exercise: write an implementation that runs in $O(n)$ time.

Implementing a Successor DS: Queries

```haskell
succOf q Leaf                        = Nothing
succOf q (Node l x r) | x < q        = succOf q r
                      | otherwise = case succOf q l of
                                        Nothing -> Just x
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Nice if the input tree happens to be balanced, i.e. of height $O(\log n)$

Making Balanced Trees

- Suppose that the input is a sorted list, how to build a balanced tree?

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```
buildBalanced    :: [a] -> Tree a
buildBalanced [] = Leaf
buildBalanced xs = Node l x r
  where
    m = length xs `div` 2
    (ls,x:rs) = splitAt m xs

    l = buildBalanced ls
    r = buildBalanced rs
```

- Running time: $O(n \log n)$.

Dynamic Successor: Insert

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```haskell
insert             :: Ord a => a -> Tree a -> Tree a
insert x Leaf      = Node Leaf x Leaf
insert x t@(Node l y r)
    | x < y        = Node (insert x l) y r
    | x == y       = t
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```
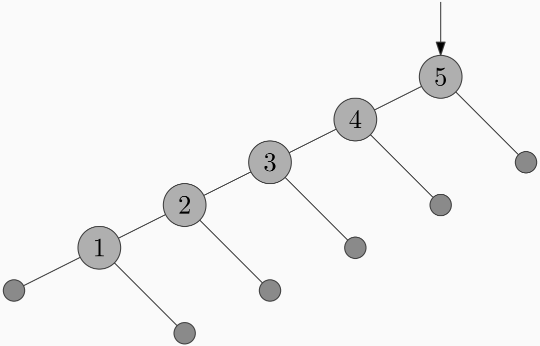

- Not just `insert x l`!
- Note that we are building new trees!

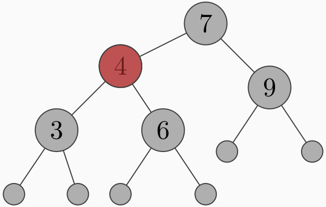
May unbalance the tree

- Repeatedly inserting elements unbalances the tree

```
> foldr insert Leaf [1..5]
Node (Node (Node (Node (Node Leaf 1 Leaf) 2 Leaf) 3 Leaf) 4 Leaf) 5 Leaf
```

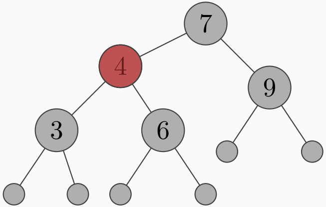


Self balancing trees: Red Black Trees



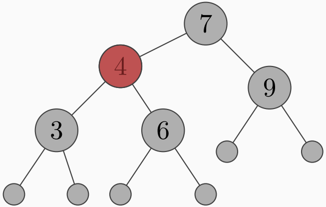
- Properties:
  1) leaves are black
  2) root is black
  3) red nodes have black children
  4) for any node, all paths to leaves have the same number of black children.

Self balancing trees: Red Black Trees



- Properties:
  1) leaves are black
  2) root is black
  3) red nodes have black children
  4) for any node, both children have the same *blackheight*
- blackHeight of a node = number of black children on any path from that node to its leaves.

Self balancing trees: Red Black Trees



- Properties:
  1) leaves are black
  2) root is black
  3) red nodes have black children
  4) for any node, both children have the same *blackheight*
- Support queries and updates in $O(\log n)$ time.

Red Black Trees in Haskell

```haskell
data Color = Red | Black deriving (Show,Eq)

data RBTree a = Leaf
              | Node Color (RBTree a) a (RBTree a)
              deriving (Show,Eq)
```

- Enforces property 1. Other properties are more difficult to enforce in the type.

Implementing Queries and Inserts

- `succOf` more or less the same as before.
- Insert:
    - Make sure black heights remain ok by replacing a black leaf by a red node.
    - The only issue is red,red violations.
    - Allow red,red violations with the root, but not below that.
    - Recolor the root black at the end.

Implementing Insert

```haskell
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blackenRoot                :: RBTree a -> RBTree a
blackenRoot Leaf           = Leaf
blackenRoot (Node _ l y r) = Node Black l y r

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As before, this creates an unbalanced tree. So, what’s left is to rebalance the newly created trees.

Implementing Insert’

```haskell
insert'          :: Ord a => a -> RBTree a -> RBTree a
insert' x Leaf  = Node Red Leaf x Leaf

insert' x t@(Node c l y r)
    | x < y      = balance c (insert' x l) y r
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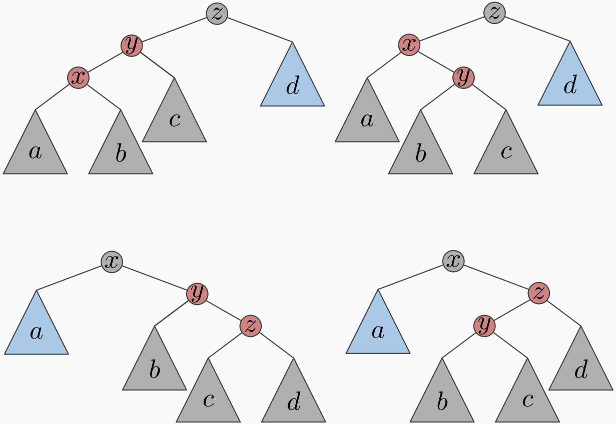
balance :: Color -> RBTree a -> a -> RBTree a -> RBTree a
```

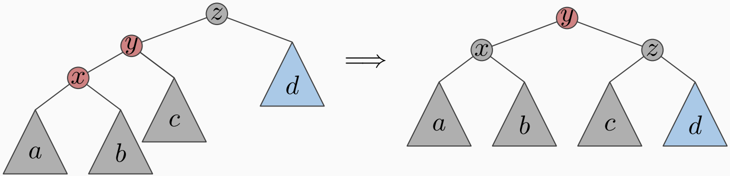
Rebalancing

- The only potential issue is two red nodes near the root.
- There are only four configurations:



Rebalancing

- Make the root red, and its children black:



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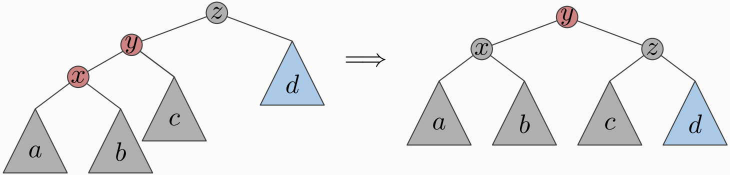


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balance Black (Node Red (Node Red a x b) y c) z d =
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Rebalancing code

- Other cases are symmetric:

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balance c l x r                                   =
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```

Deleting

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- What if we also want to remove elements from S?
- Possible in $O(\log n)$ time with Red-Black trees, but a bit more messy.

Data structures in the Haskell Standard Library

- Self balancing BST Implementation available in `Data.Set`
- Often useful to store additional information: `Data.Map`.

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lookup :: Ord k => k -> Map k v -> Maybe v
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- All these data structures are persistent.

Example Application: Point Location

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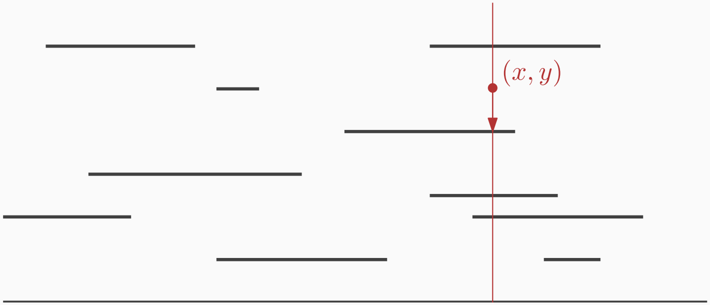

- Easy if we had the platforms intersecting the vertical line at x in top-to-bottom order in a `Set` or `Map`: find successor of y.

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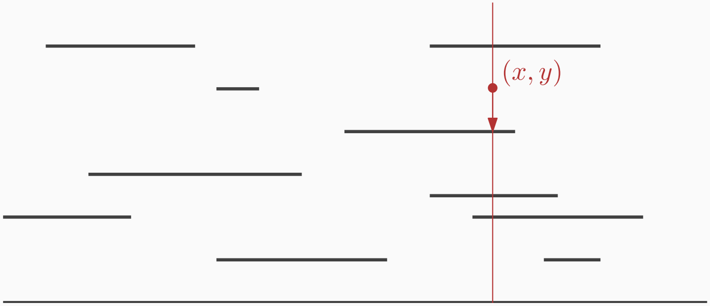

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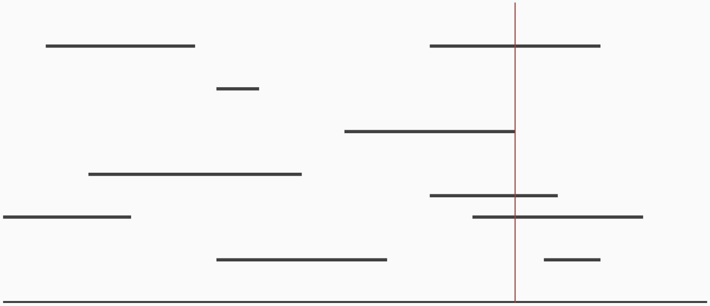

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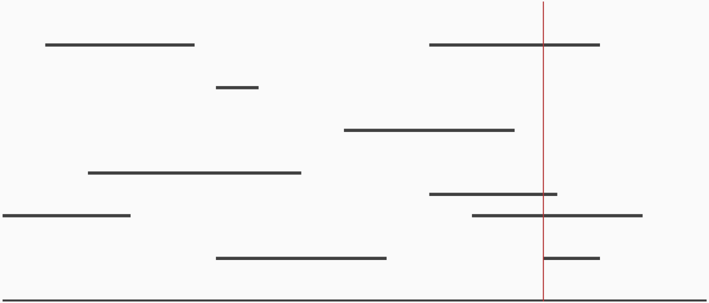

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- What happens when vertical line starts/stops to intersect a platform?
- Add or remove a platform from the `Set`
- Since `Set` is persistent, old versions remain in tact. Store them in a `Map`.
- To answer a query: go to the version at time x using a successor query, and find successor of y.

Homework: Verifying Red-Black Tree Properties

- Write a function `validRBTree :: RBTree a -> Bool` that checks if a given `RBTree a` satisfies all red-black tree properties.