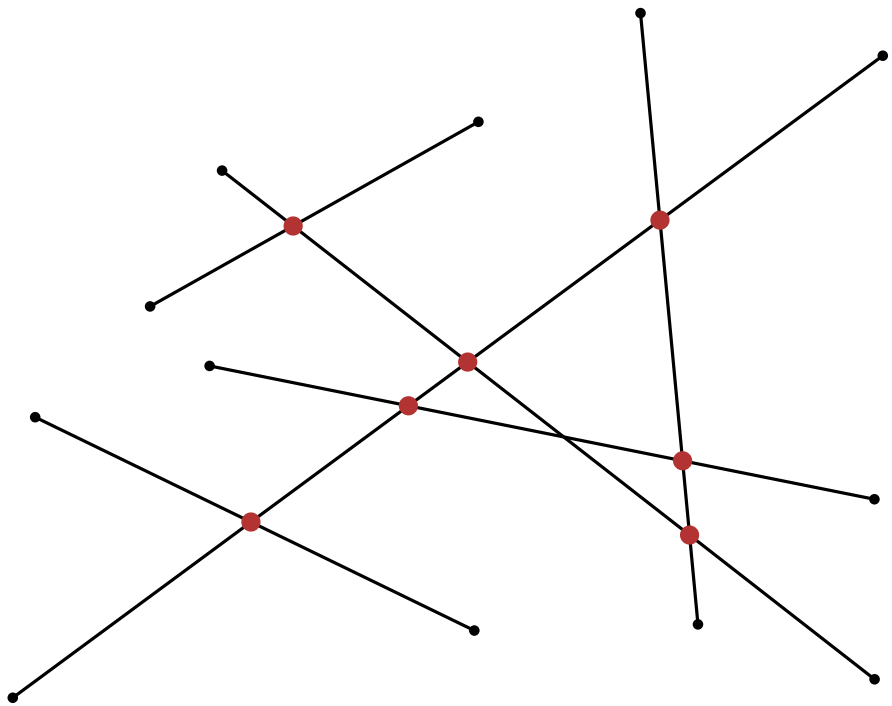
Linesegment Intersection

Problem: Given a set of n line segments in $\mathbb{R}^2$, compute all intersection points.



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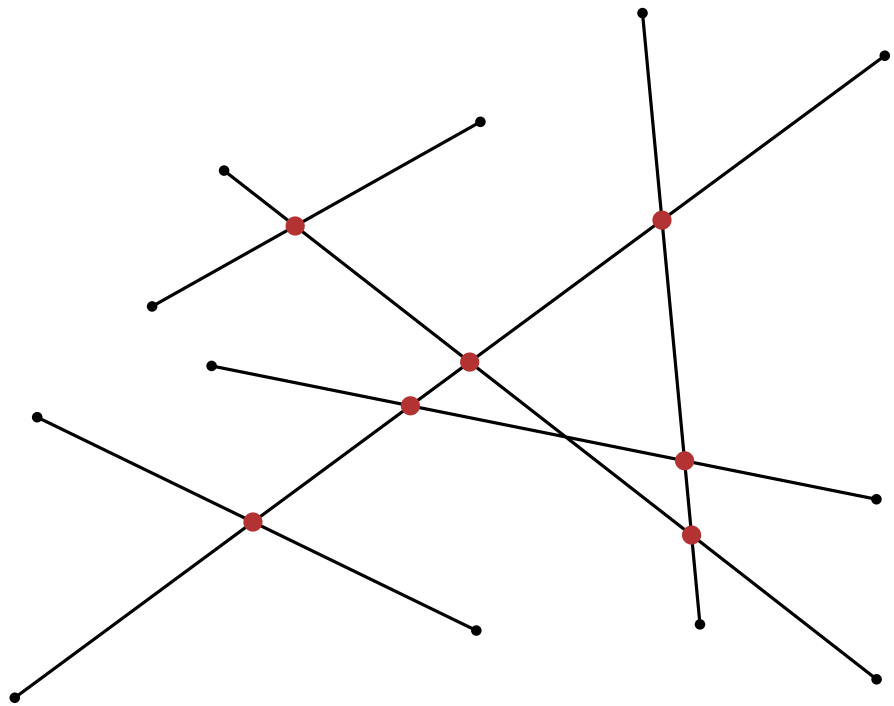

Easy Algorithm:

For each pair of segments check if they intersect

Running time: $O(n^2)$

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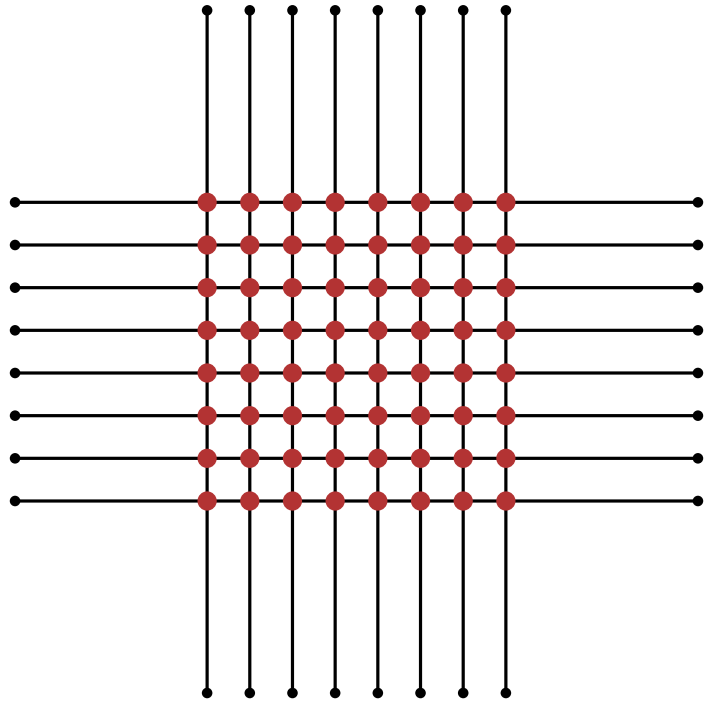


Easy Algorithm:

For each pair of segments check if they intersect

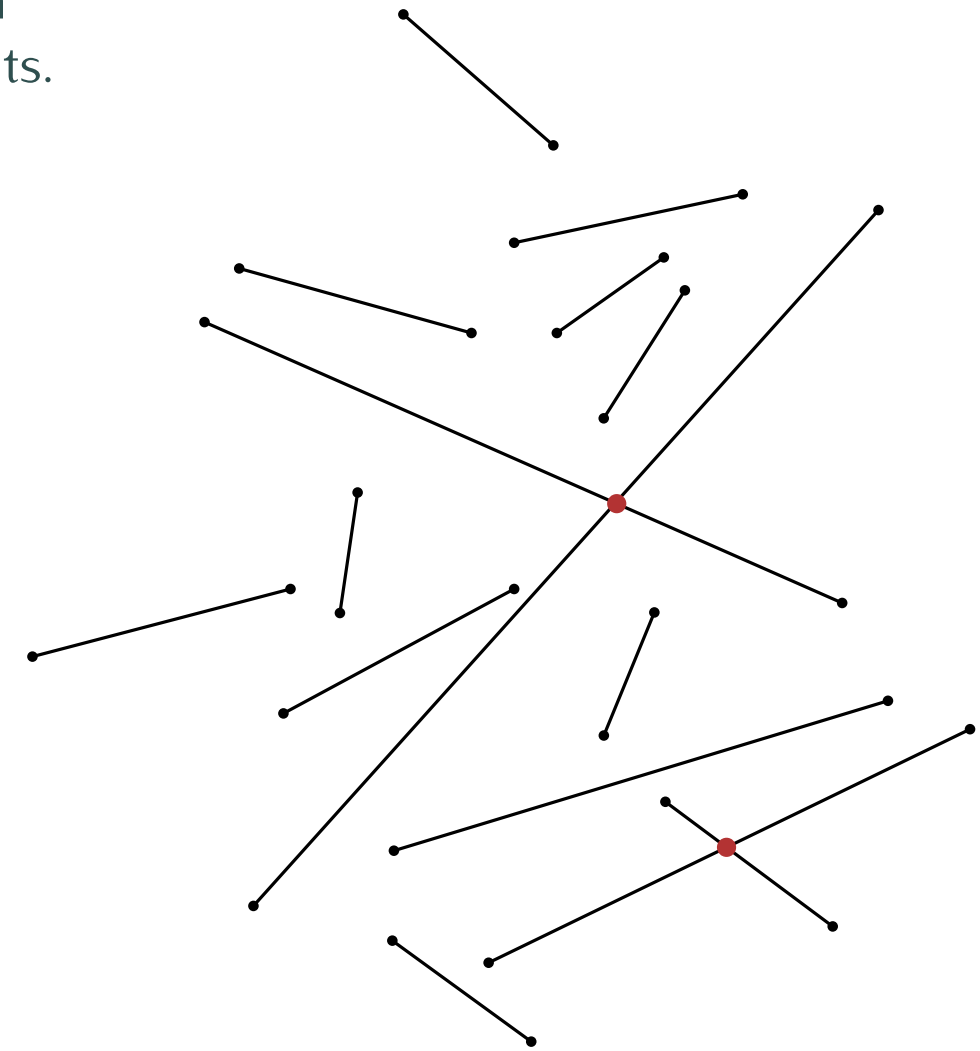
Running time: $O(n^2)$

Optimal in the worst case

Linesegment Intersection

Problem: Given a set of n line segments in $\mathbb{R}^2$, compute all intersection points.

What if there are only few, say k, intersections?



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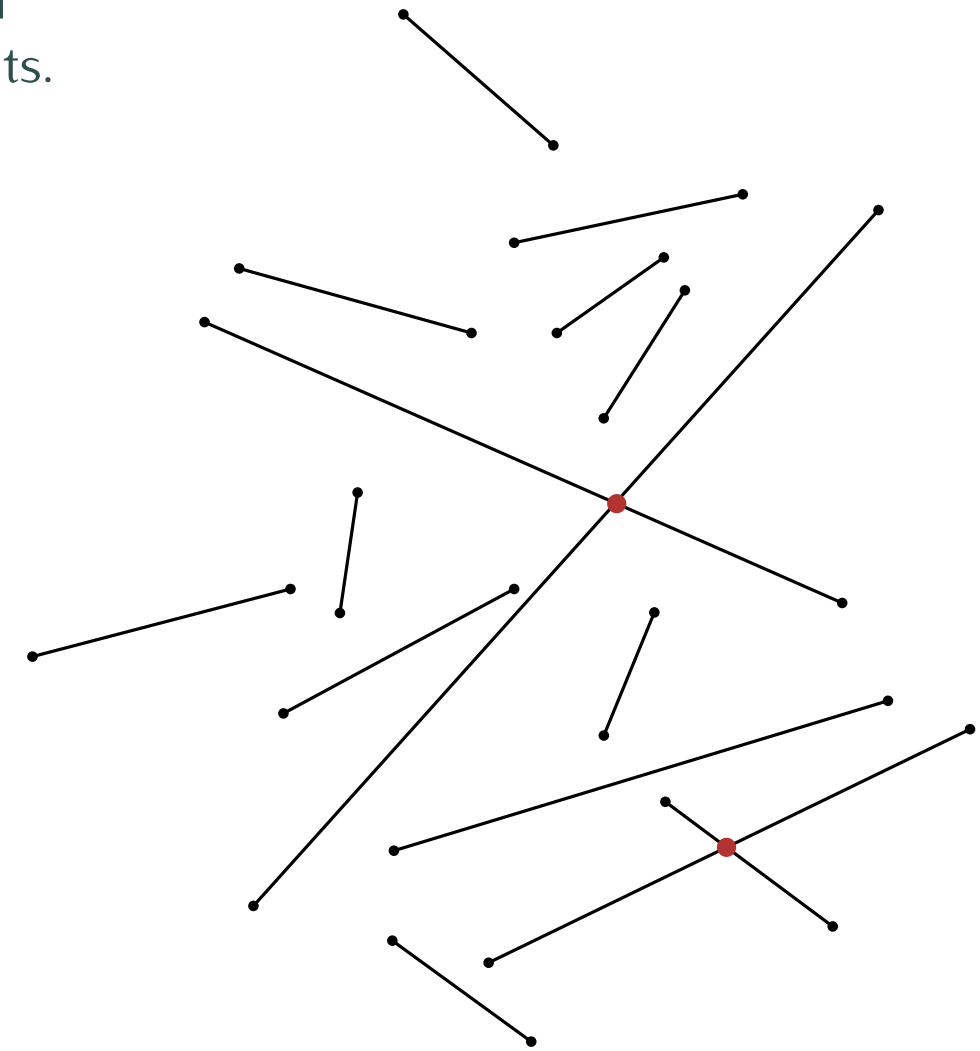
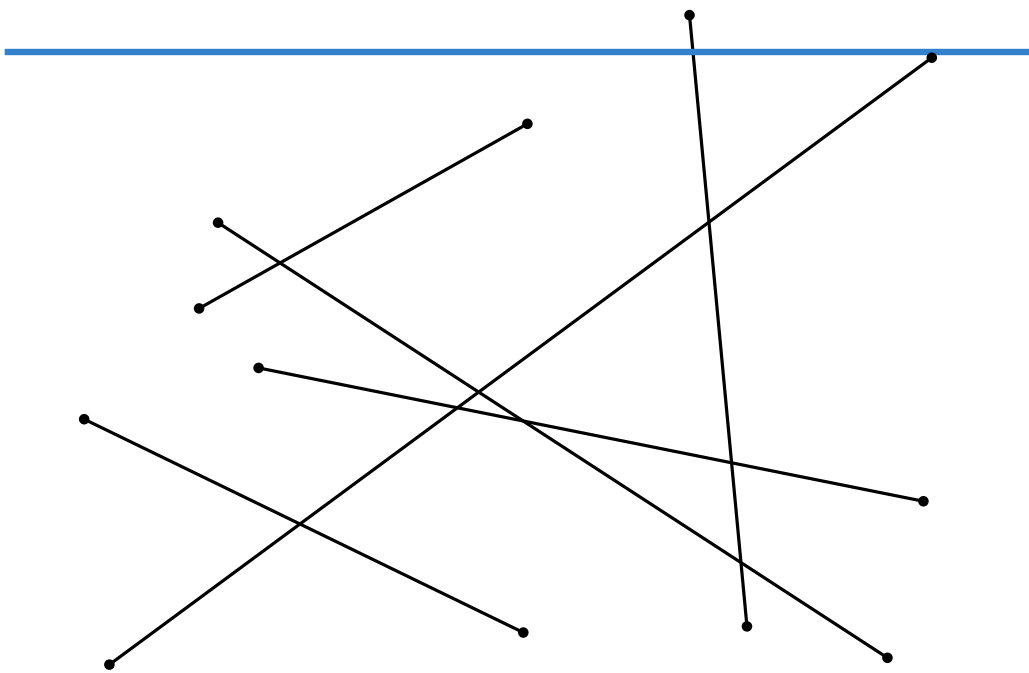


What if there are only few, say k, intersections?

We can compute all k intersections in $O((n+k)\log n)$ time, using a sweep-line algorithm.

Sweep line paradigm

Idea: Sweep a (horizontal) line ℓ over the plane, make sure that all intersection points above the sweep line have been reported



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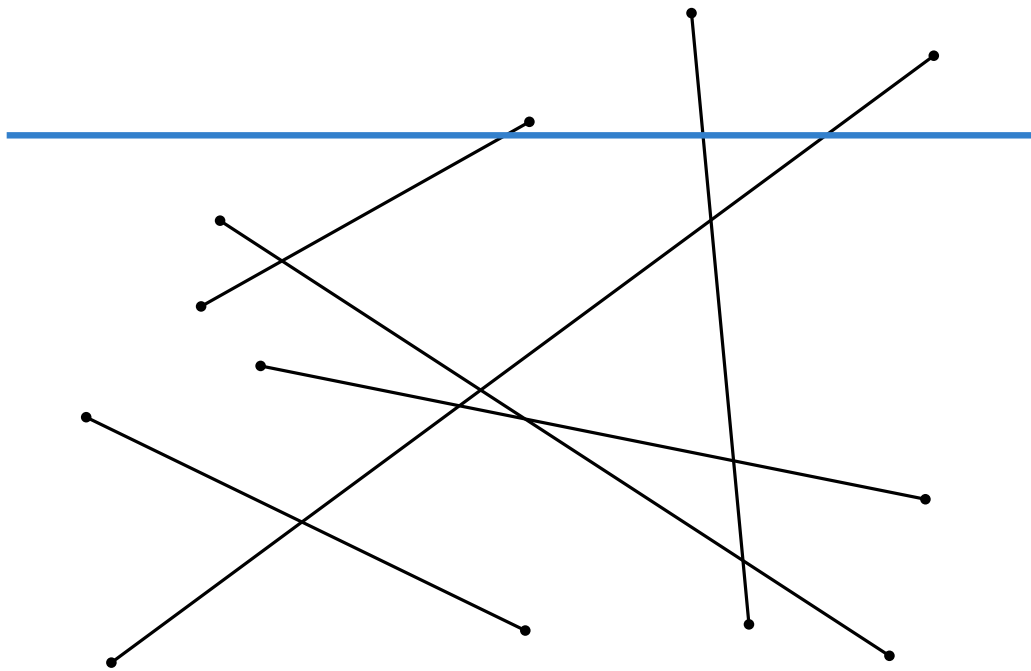

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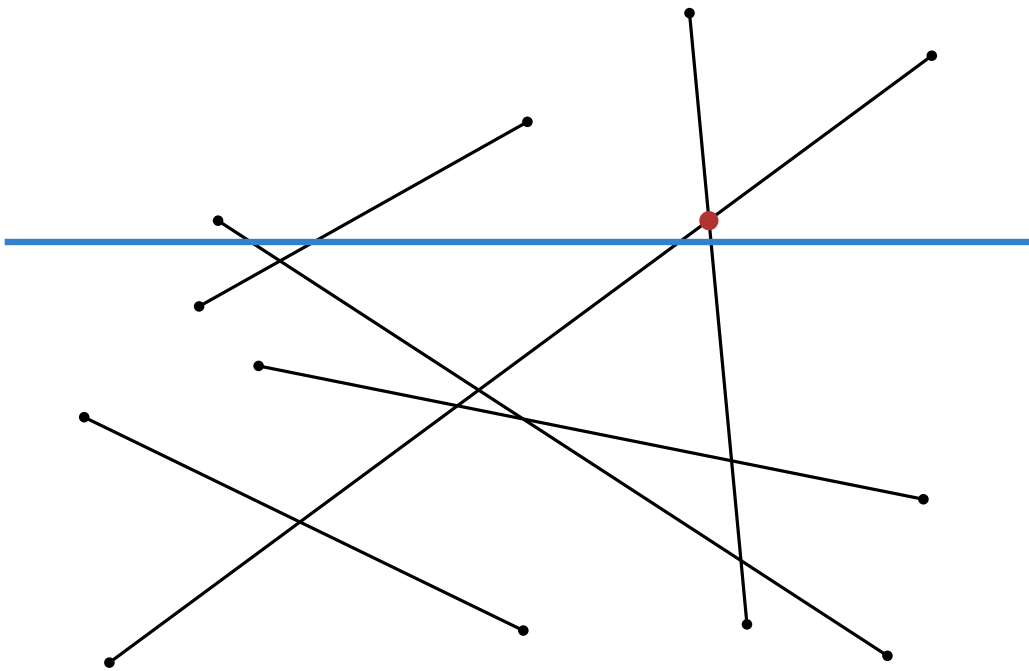

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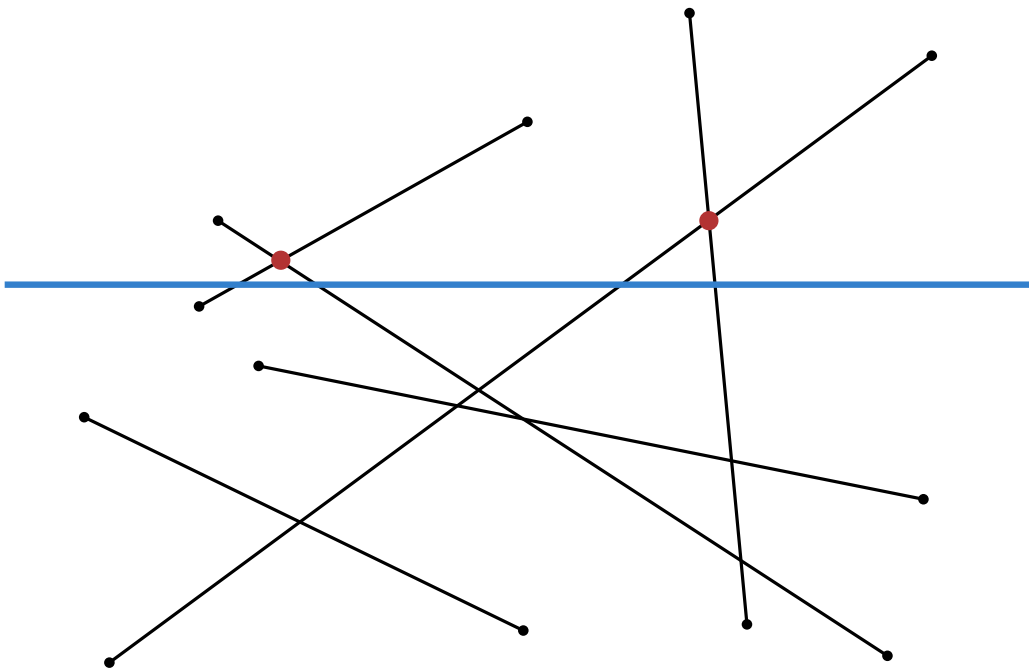

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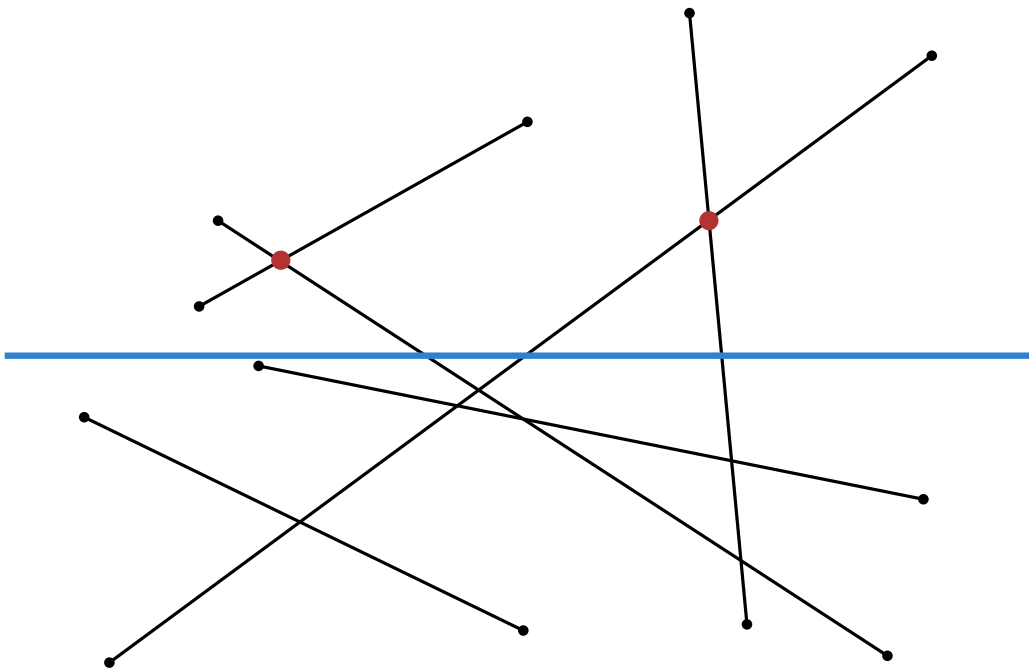

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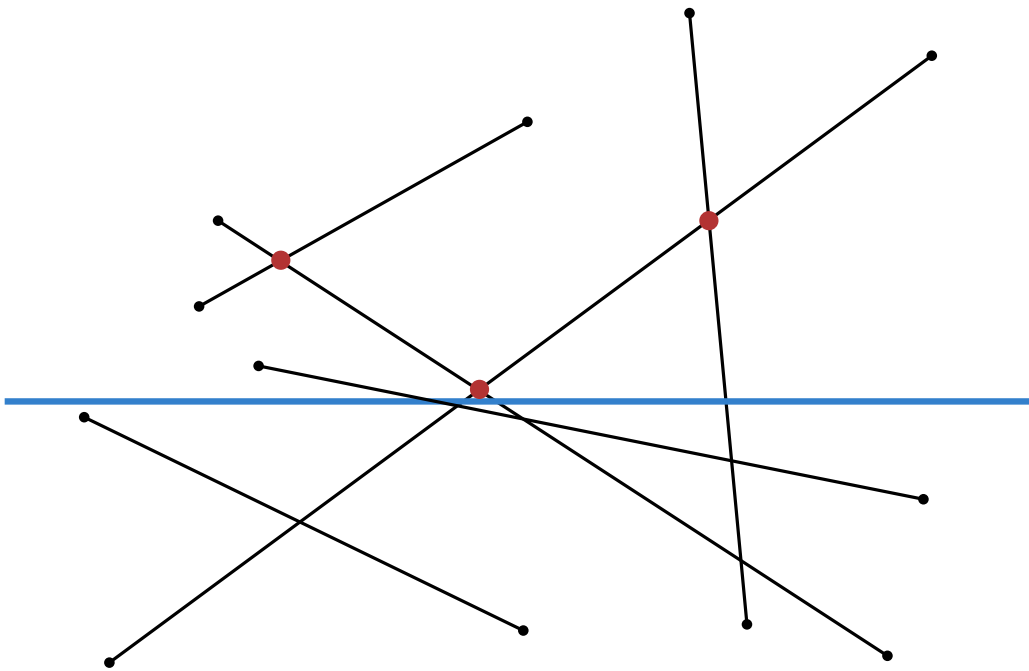

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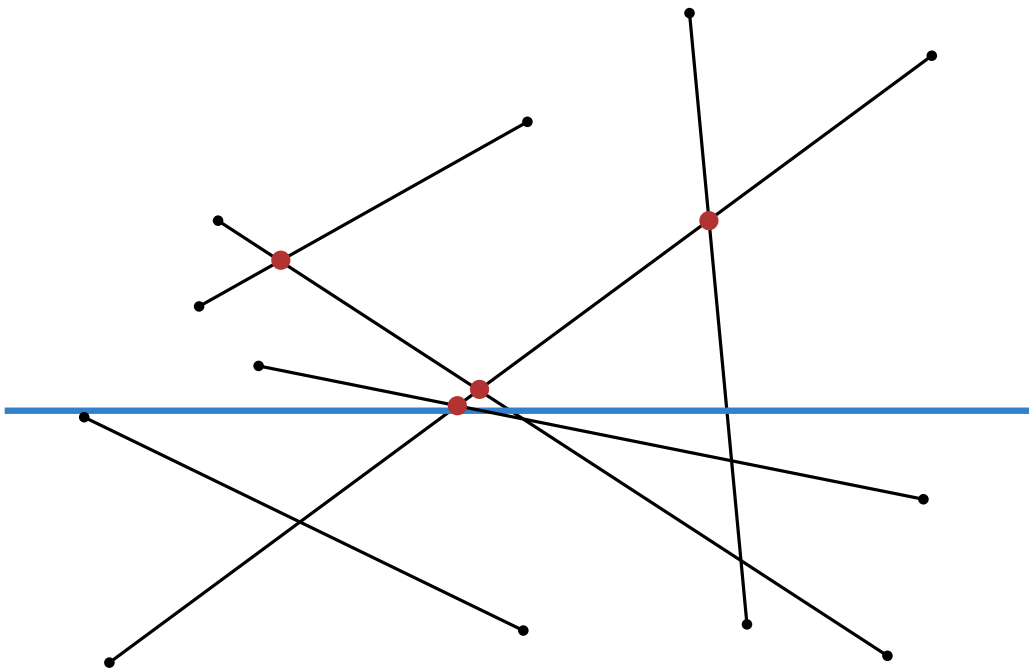

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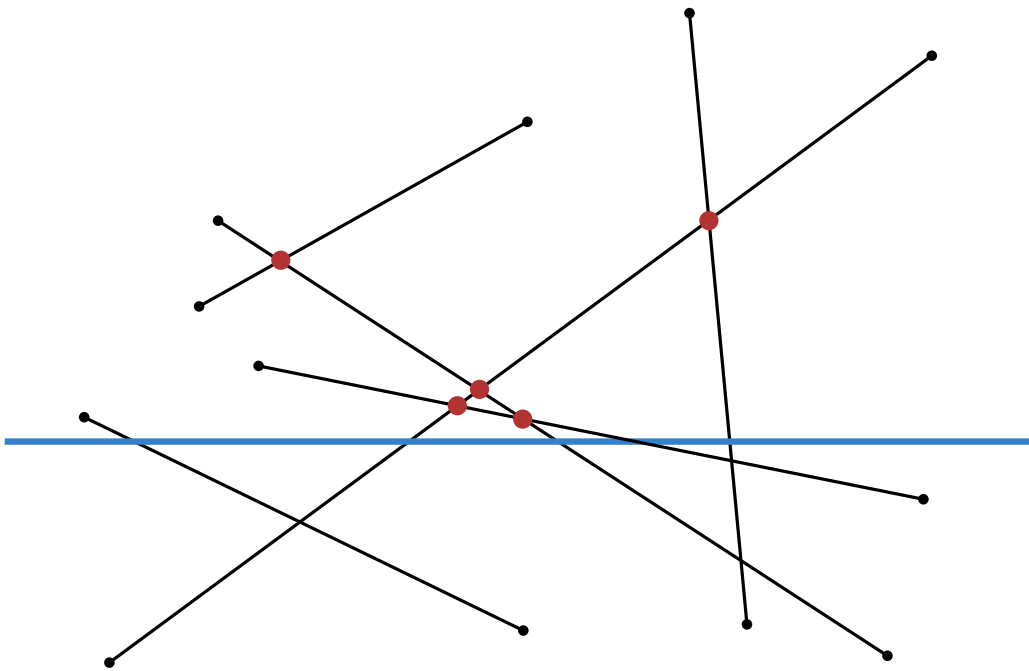

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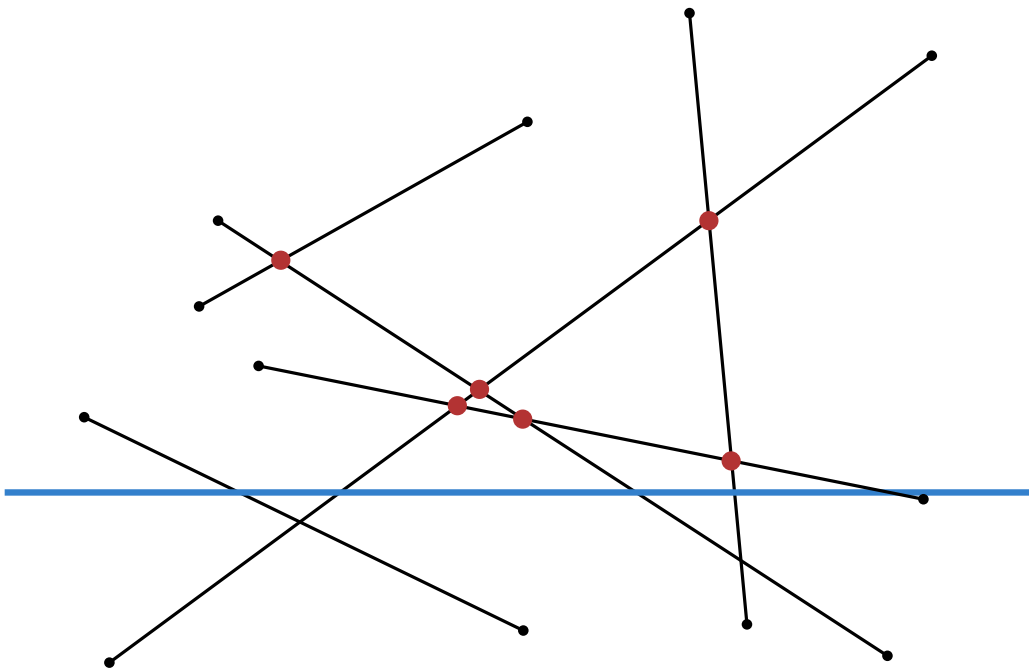

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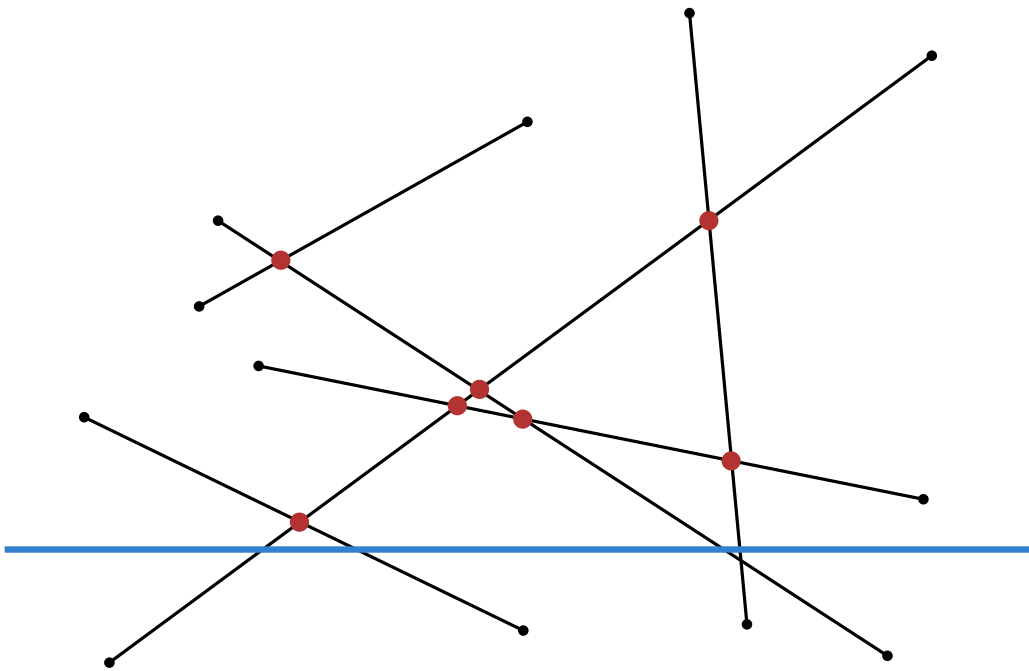

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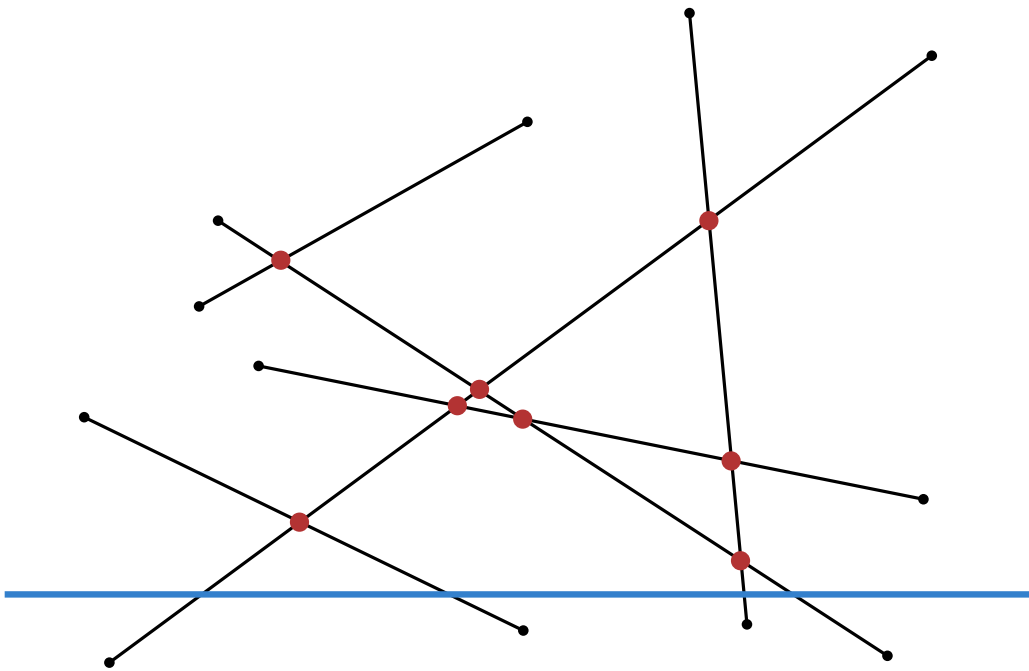

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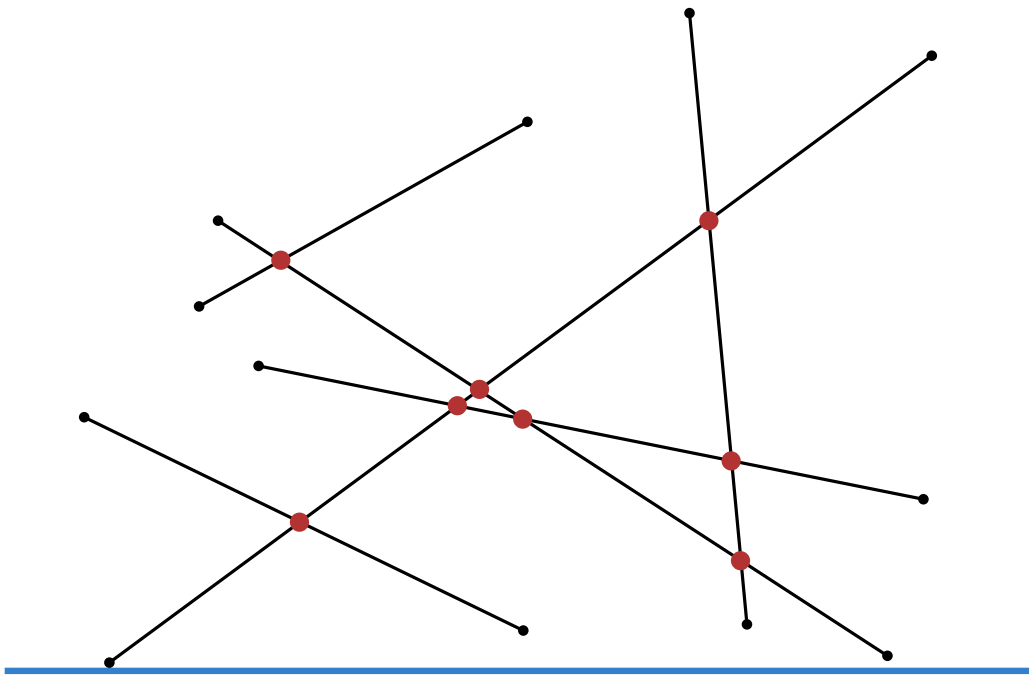

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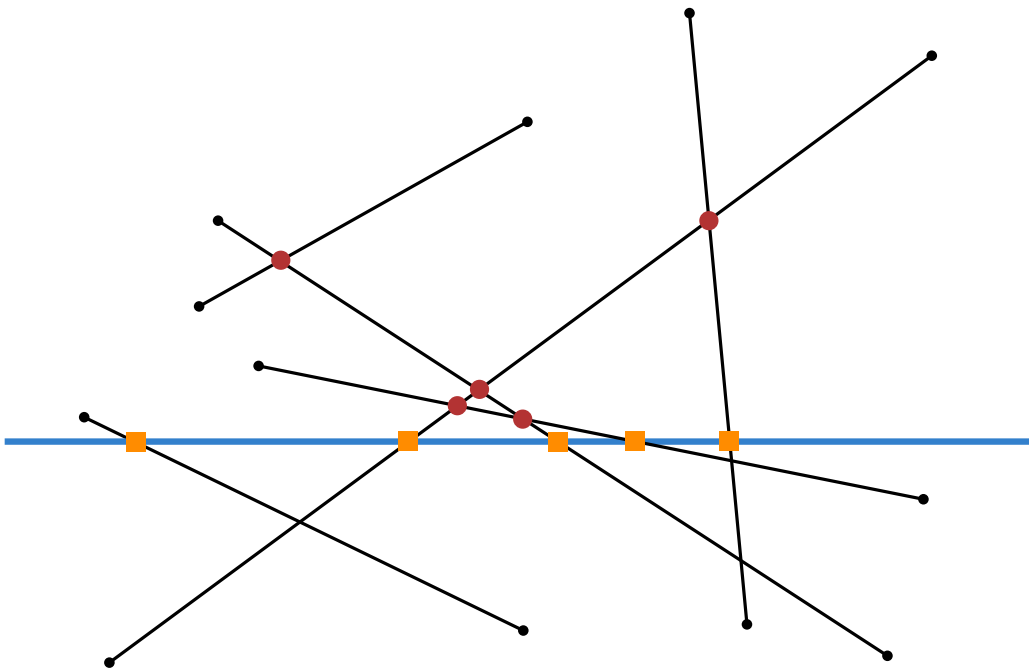
While sweeping, maintain the order in which the segments intersect ℓ.



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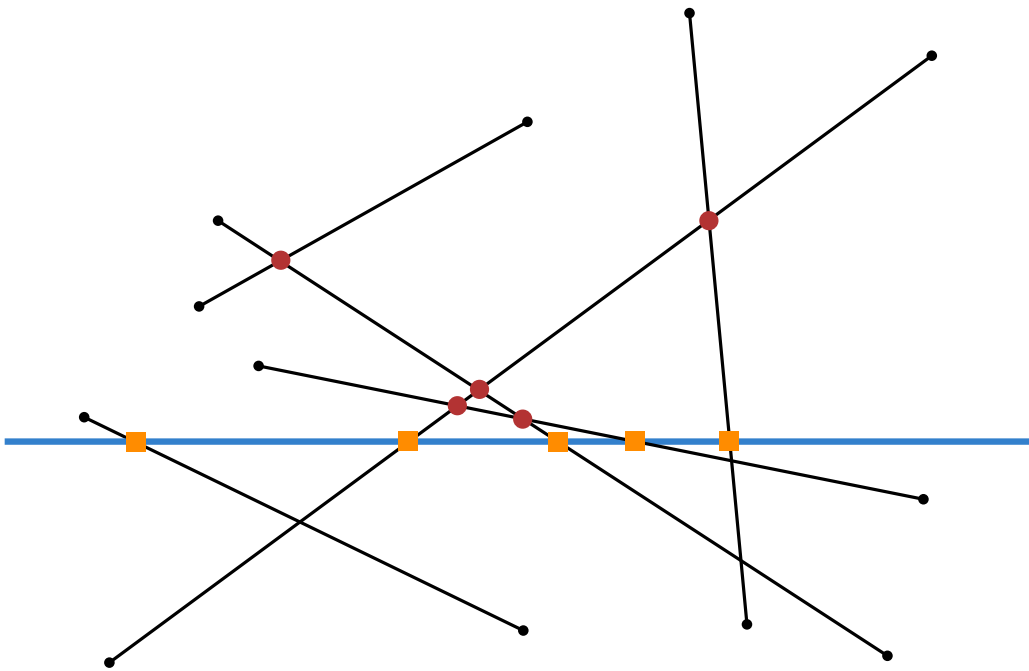
Idea: Sweep a (horizontal) line ℓ over the plane, make sure that all intersection points above the sweep line have been reported

While sweeping, maintain the order in which the segments intersect ℓ.

Only compare segments when they become neighbors on the sweep line.



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Store these segments in the status structure.

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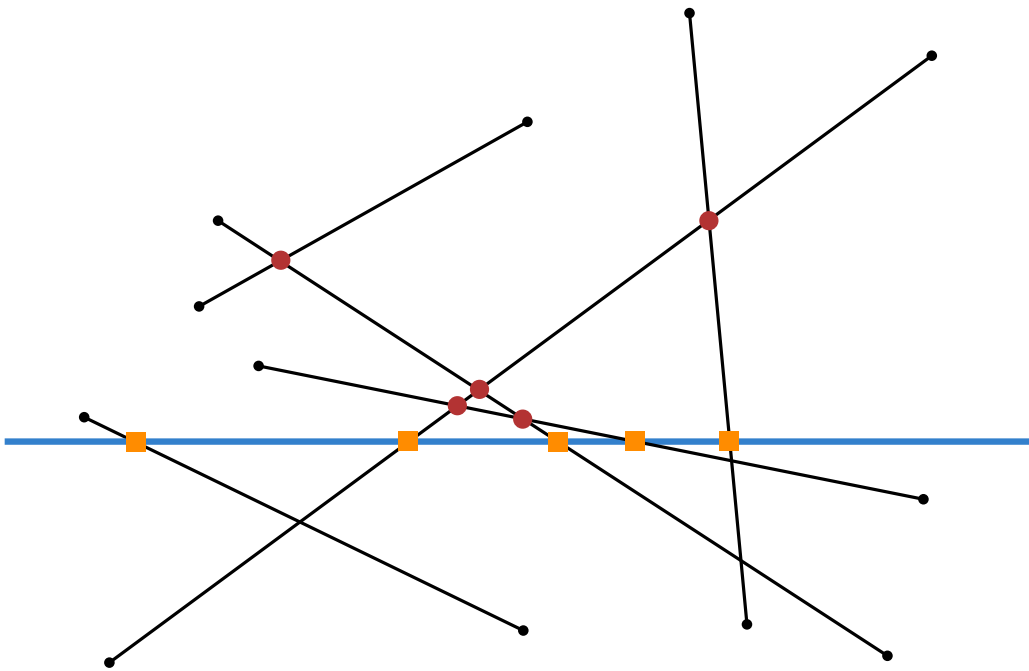

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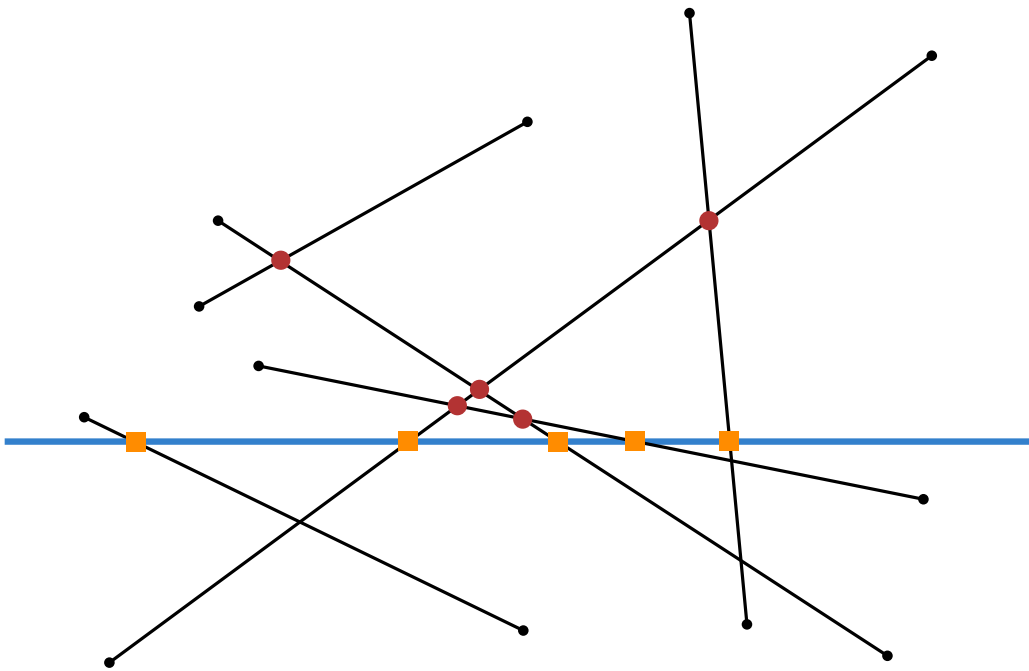

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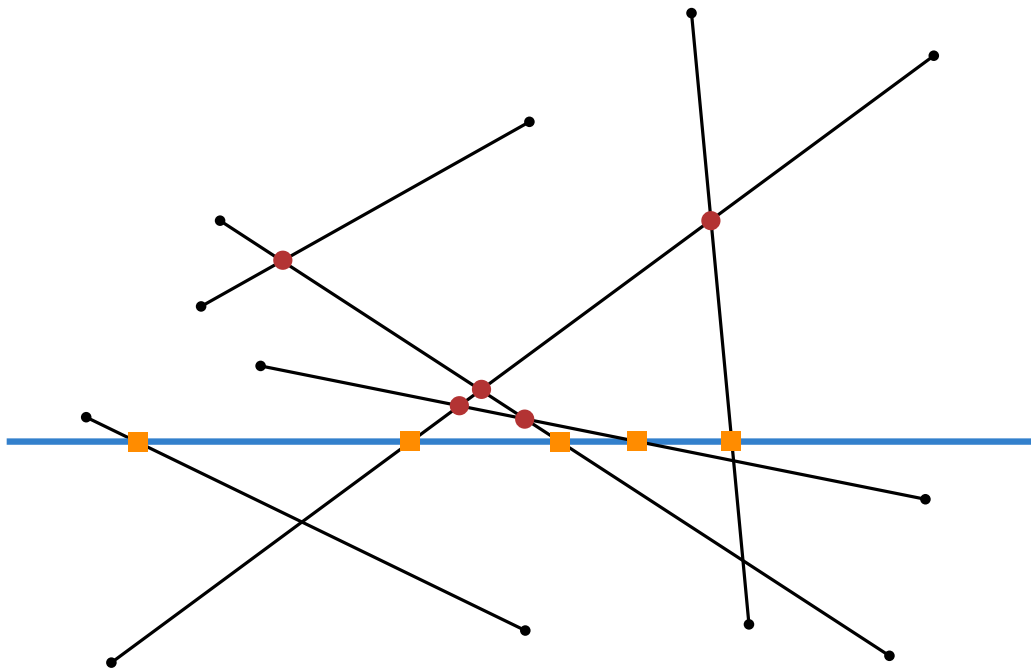
We use a balanced BST as status structure.

Only compare segments when they become neighbors in the status structure.



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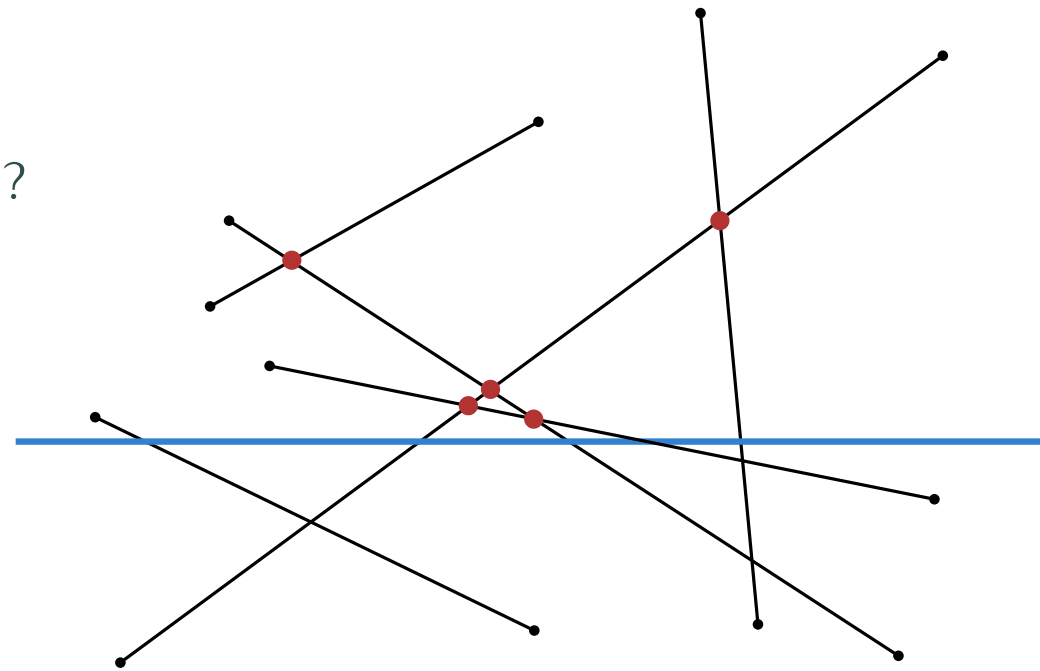
When does the status structure change?



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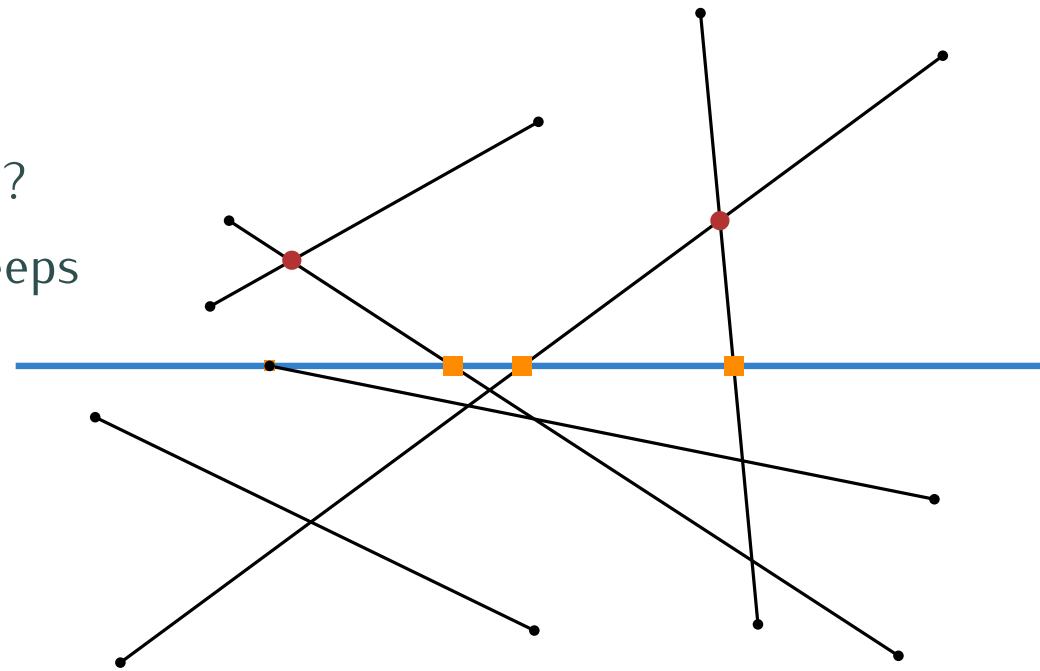
When does the status structure change?

At discerete events, i.e. when ℓ sweeps over

- the upper endpoint of a segment



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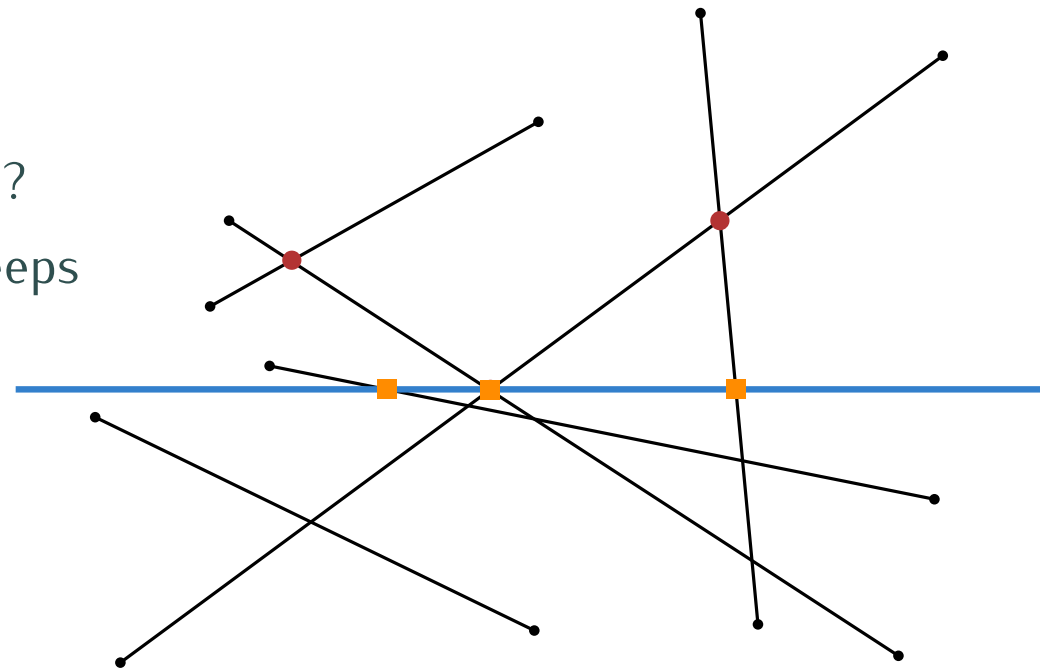

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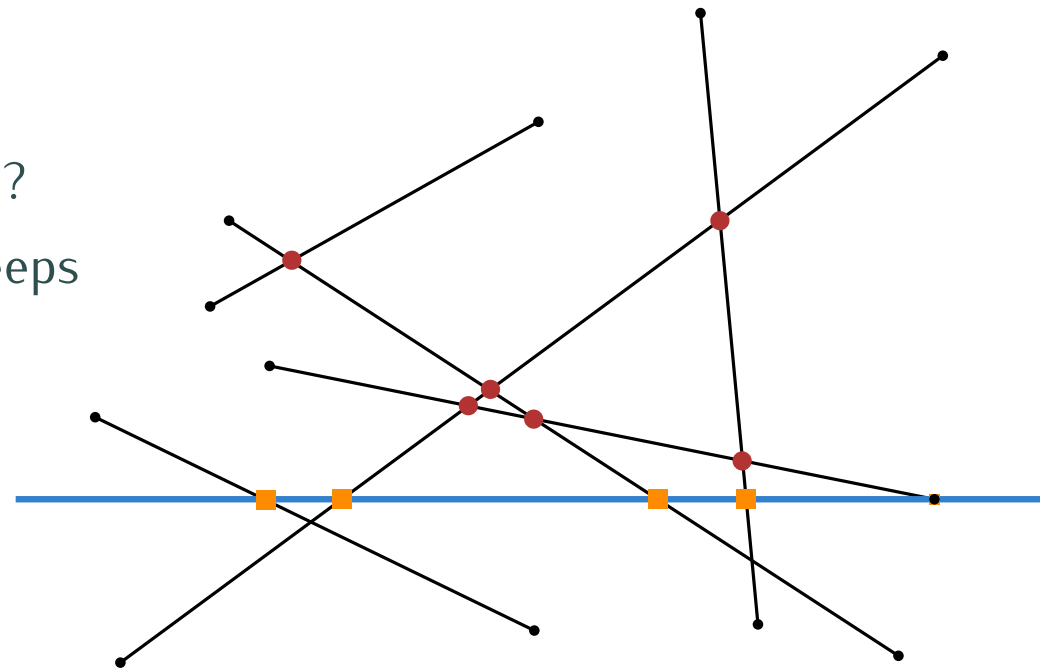

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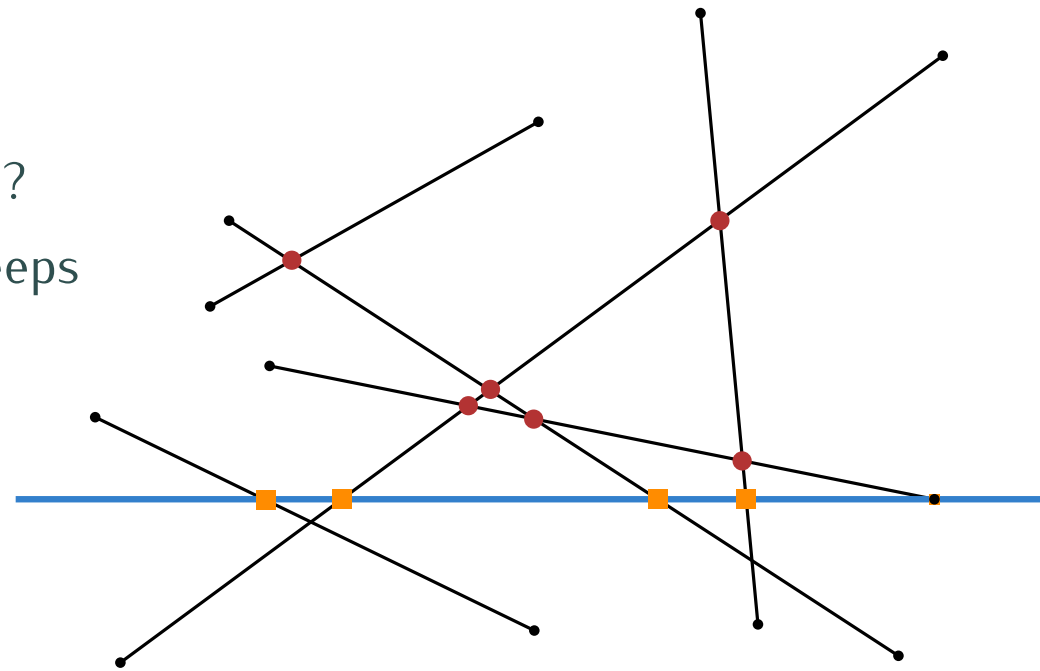
Maintain the events in an event queue.



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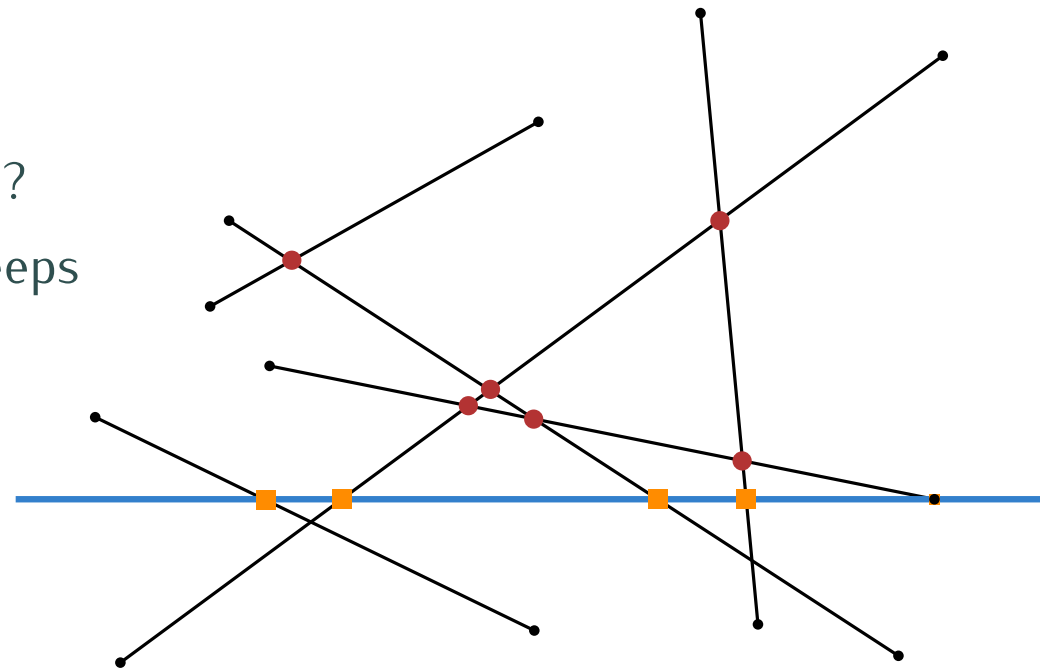
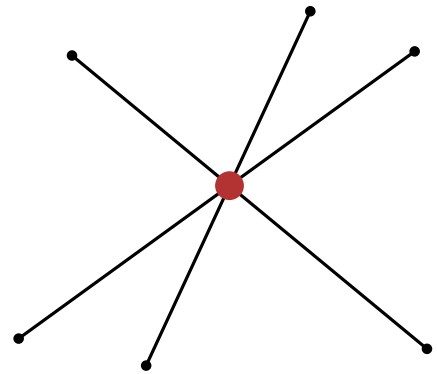


General position

Our algorithm works for segments in general position, i.e. if there are

- no three segments that intersect in a single point



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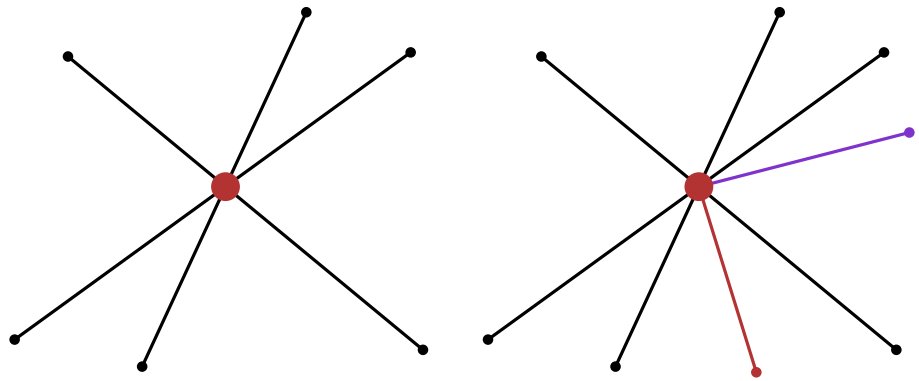

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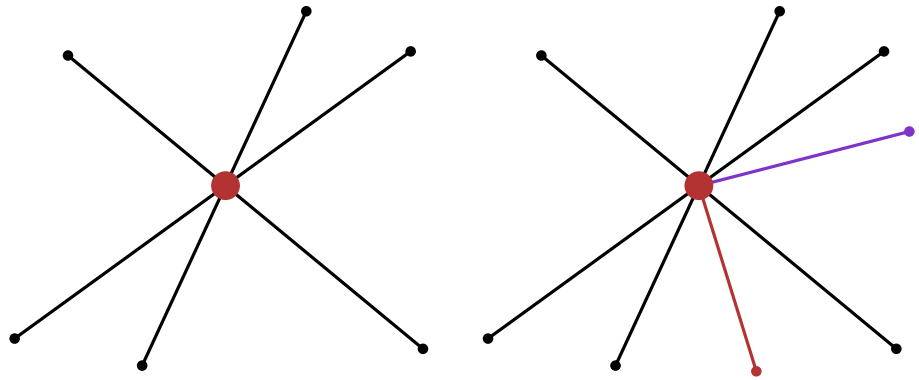

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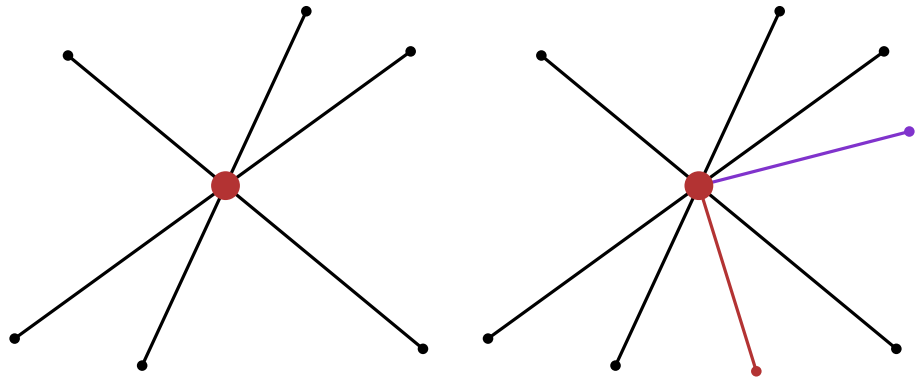

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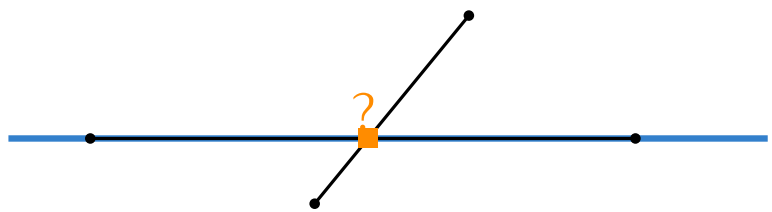

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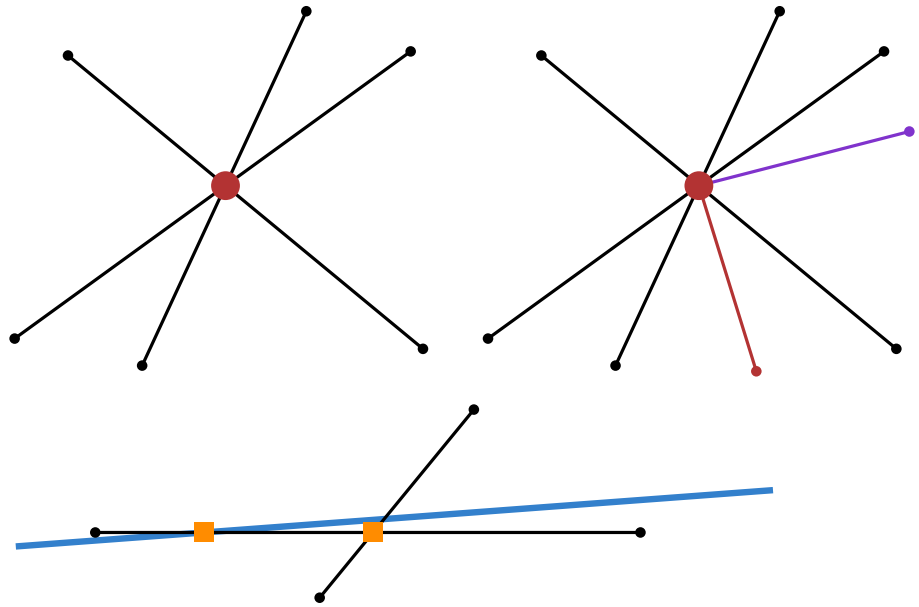
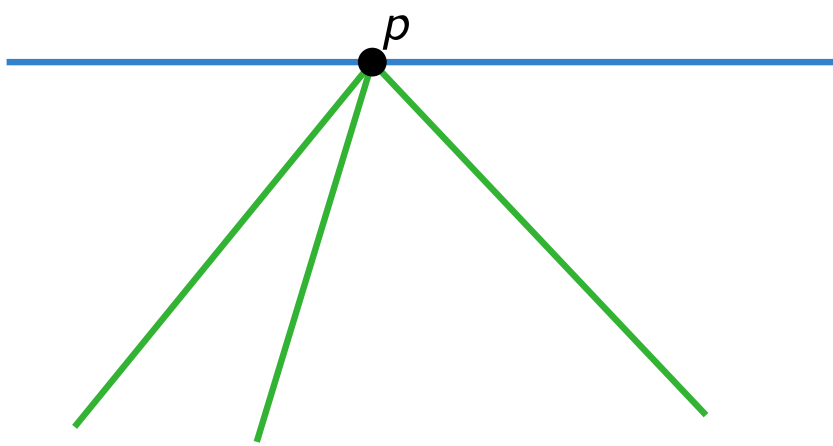


- no horizontal segments

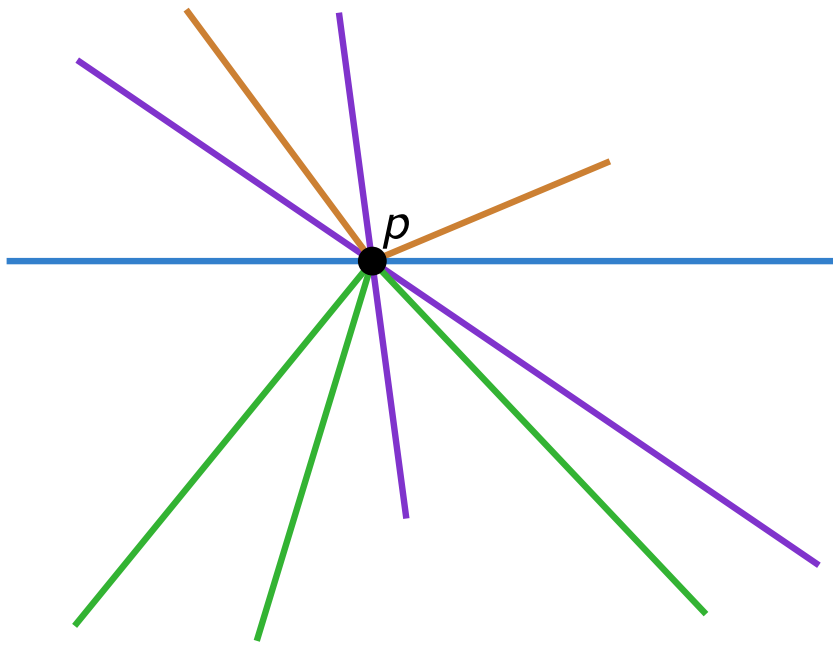
Handling an Eventpoint p

1. Let $U(p)$ be the segments whose upper endpoint is p.



Handling an Eventpoint p

1. Let $U(p)$ be the segments whose upper endpoint is p.
2. Find the segments in the status structure $\mathcal{T}$ that contain p; let $L(p)$ be the segments whose lower endpoint is p, and let $C(p)$ be the segments that contain p in their interior.



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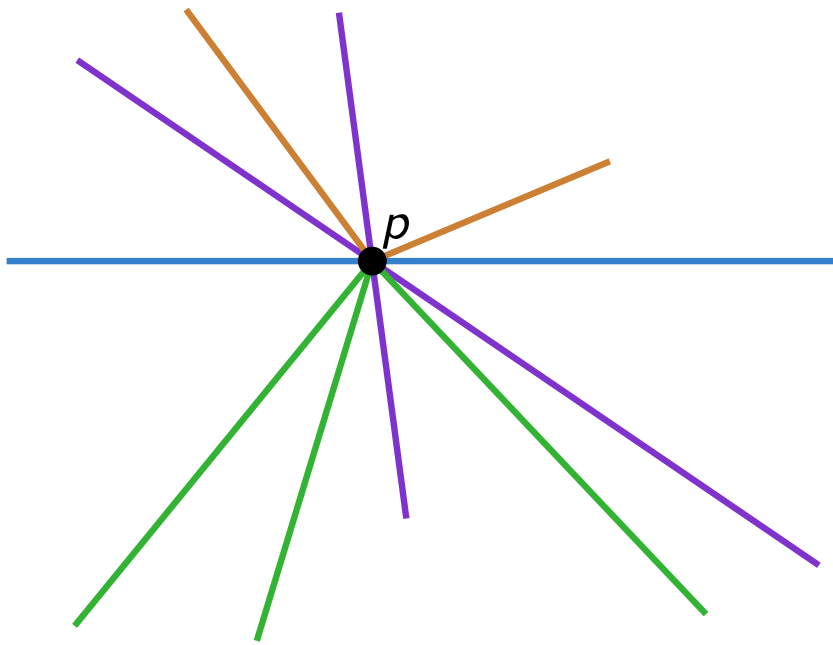
3. **if** $|U(p) \cup L(p) \cup C(p)| \geq 2$ **then**
   Report p and all segments in $U(p) \cup L(p) \cup C(p)$



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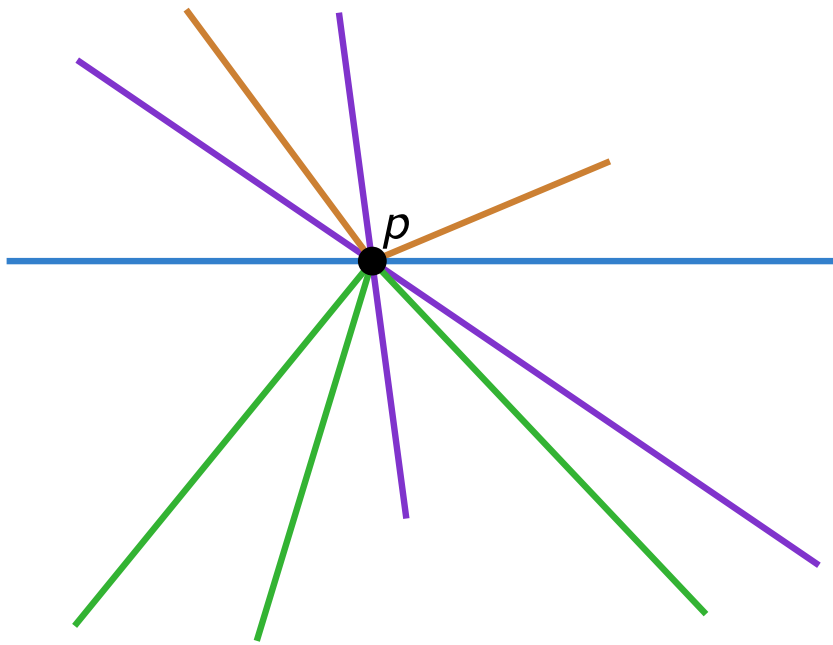
4. Delete the segments in $L(p)$ and $C(p)$ from $\mathcal{T}$.
5. Insert the segments in $U(p)$ and $C(p)$ into $\mathcal{T}$.



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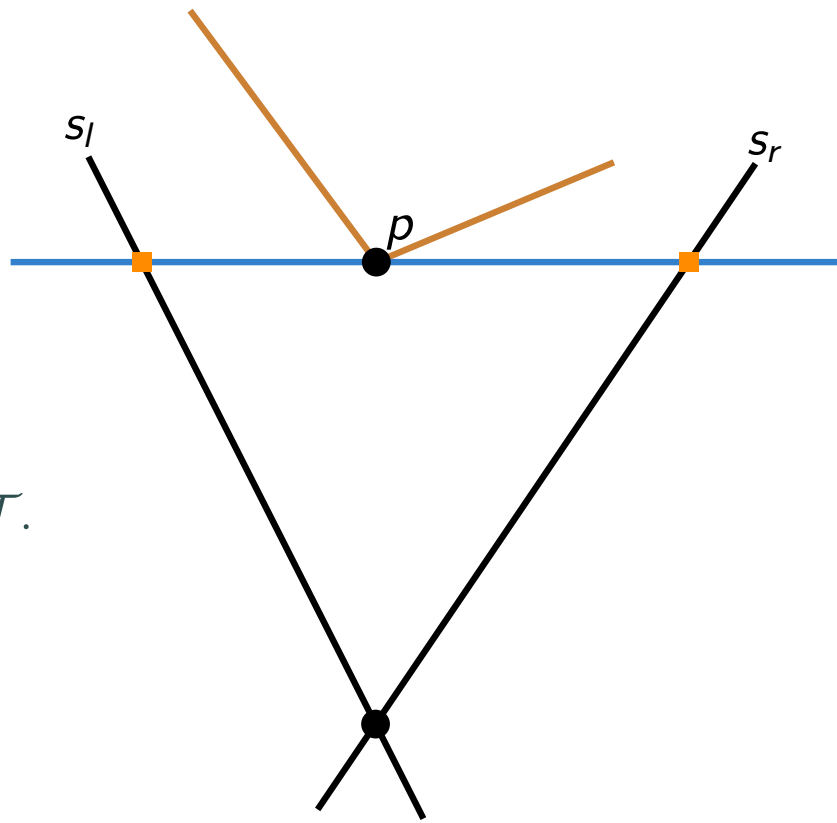
6. **if** $U(p) \cup C(p) = \emptyset$ **then**
7. Find the left and right neighbors s_l and s_r of p in $\mathcal{T}$.
8. FindNewEvent(s_l, s_r, p)



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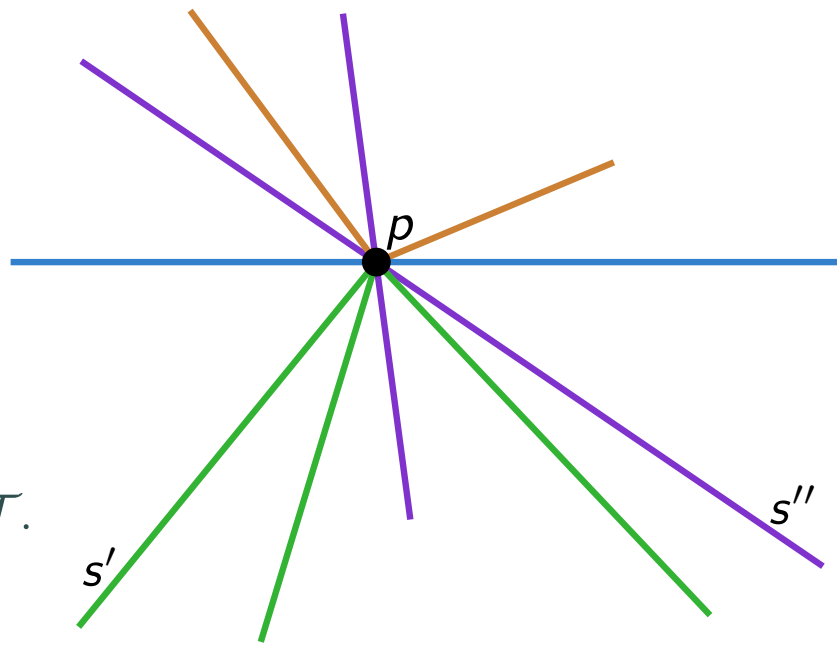
9. **else**
   Let s' and s'' be the leftmost and rightmost segment in $U(p) \cup C(p)$



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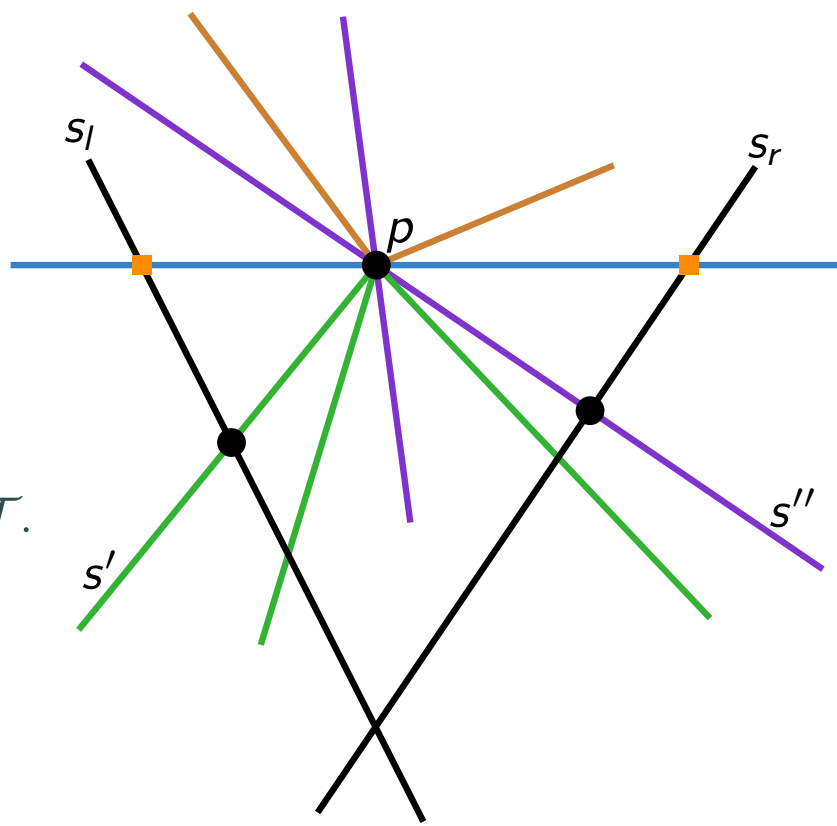
   Let s' and s'' be the leftmost and rightmost segment in $U(p) \cup C(p)$
10. Let s_l be the left neighbor of s' and s_r the right neighbor of s''.
11. FindNewEvent(s_ℓ, s', p)
12. FindNewEvent(s'', s_r, p)



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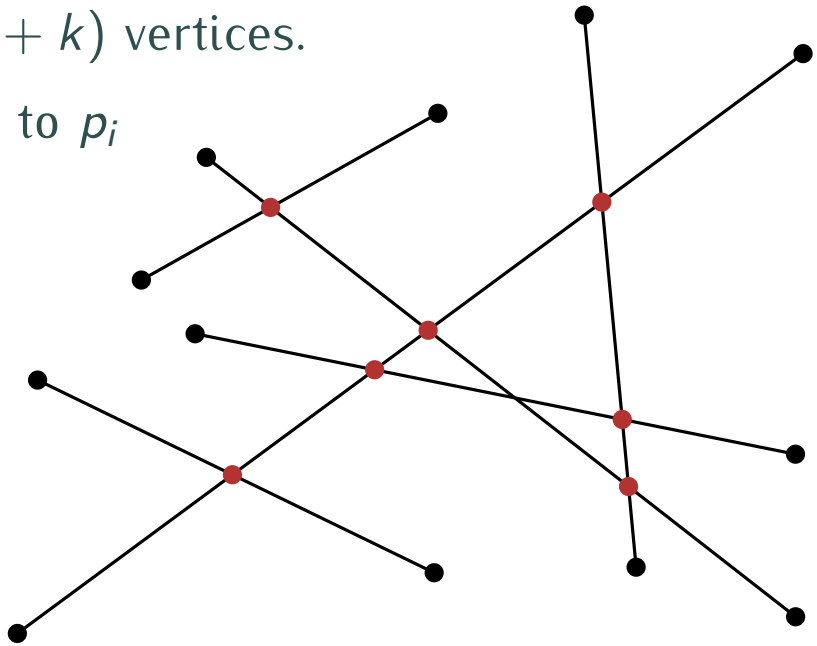

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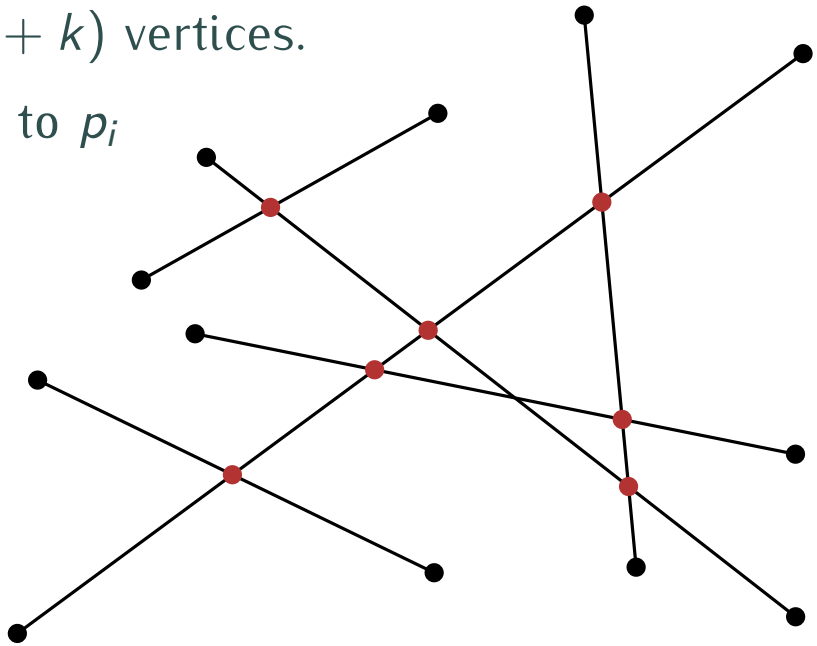

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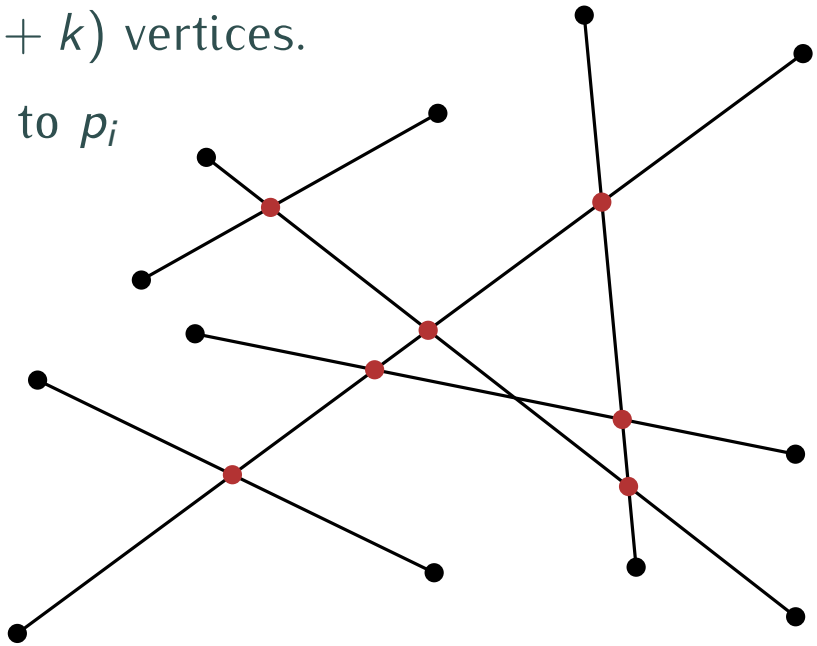

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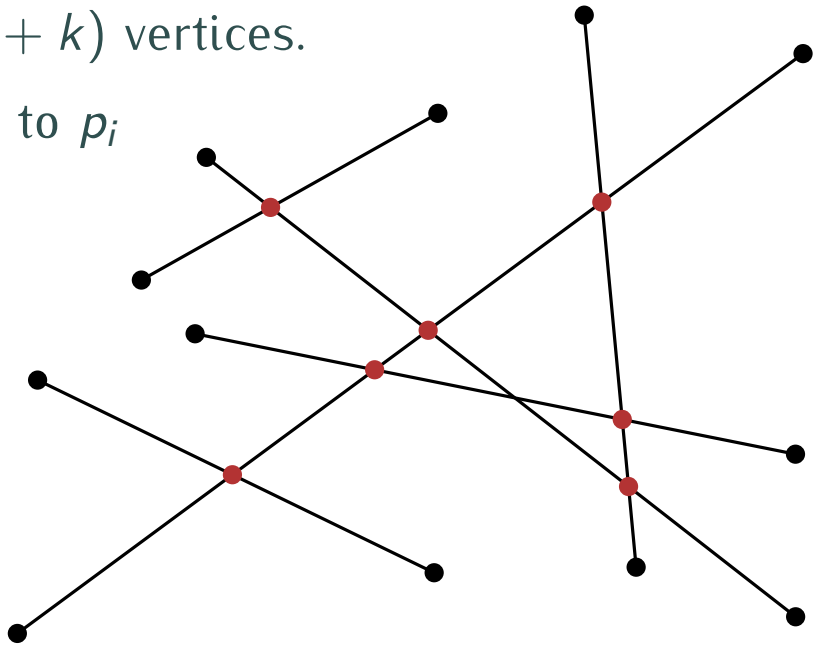

Theorem. Let S be a set of n line segments in $\mathbb{R}^2$. We can compute all k intersection points in $O((n+k)\log n)$ time.