Improved estimators for constrained Markov chain models

Ursula U. Müller
Universität Bremen

Anton Schick*
Binghamton University

Wolfgang Wefelmeyer
Universität Siegen

Abstract

Suppose we observe an ergodic Markov chain and know that the stationary law of one or two successive observations fulfills a linear constraint. We show how to improve given estimators exploiting this knowledge, and prove that the best of these estimators is efficient.

AMS 2000 subject classifications. Primary 62M05; secondary 62G05, 62G20.

Key words and Phrases. Empirical estimator, asymptotically linear estimator, influence function, regular estimator, Markov chain model, reversible chain, symmetric chain, linear autoregression.

1 Introduction

To begin let $X_1, \dots, X_n$ be *independent* with distribution P. Let $t(P)$ be a real-valued functional, and $\hat{t}$ an estimator with influence function b in $L_2(P)$,

$$n^{1/2}(\hat{t} - t(P)) = n^{-1/2} \sum_{i=1}^{n} b(X_i) + o_P(1),$$

with $Pb = Eb(X) = 0$. If the distribution fulfills a constraint $Pv = 0$ for a known vector-valued function v with components in $L_2(P)$, we can introduce new estimators for $t(P)$,

$$\hat{t}(c) = \hat{t} - c^\top \frac{1}{n} \sum_{i=1}^{n} v(X_i)$$

*Supported in part by NSF Grant DMS0072174

with influence function $b - c^\top v$ and asymptotic variance $P[(b - c^\top v)^2]$. If $P[vv^\top]$ is invertible, then by the Schwarz inequality the asymptotic variance is minimized by $c = c_b$ with

$$c_b = (P[vv^\top])^{-1} P[vb].$$

The constant c_b depends on the unknown distribution and must be estimated, say by

$$\hat{c}_b = \Big(\sum_{i=1}^n v(X_i)v(X_i)^\top\Big)^{-1} \sum_{i=1}^n v(X_i)\hat{b}(X_i),$$

leading to the estimator $\hat{t}(\hat{c}_b)$. It is easily seen to be efficient if all we know about the distribution is that it fulfills the constraint $Pv = 0$. If $t(P)$ is linear, say $t(P) = Pf$, then estimation of $t(P)$ and c_b is particularly easy. A simple estimator of $t(P)$ is the empirical estimator $\hat{t} = \frac{1}{n}\sum_{i=1}^n f(X_i)$, with influence function $b(x) = f(x) - Pf$. Then $P[vb] = P[vf]$, and a consistent estimator of $P[vb]$ is the empirical estimator $\frac{1}{n}\sum_{i=1}^n v(X_i)f(X_i)$. We refer to Levit (1975), Haberman (1984) and the monograph of Bickel, Klaassen, Ritov and Wellner (1998, Section 3.2, Example 3).

In Section 2 we extend the results from the i.i.d. case to Markov chains $X_0, \dots, X_n$ with transition distribution Q and invariant distribution π. We consider constraints $\pi \otimes Qv = \iint \pi(dx)Q(x, dy)v(x, y) = 0$ for vector-valued functions v, now of two arguments. Our estimators can be further improved if the chain is known to be reversible. In Section 3 we illustrate our results with a simple example, estimating the variance of the invariant distribution when the mean is known to be zero. The efficient estimator simplifies for the linear autoregressive model. In Examples 3 and 4 we show how reversibility and symmetry can be described by linear constraints $\pi \otimes Qv = 0$ with infinite-dimensional v. We also construct efficient estimators for these models.

2 Results

Let $X_0, \dots, X_n$ be observations from a positive Harris recurrent and V^2-uniformly ergodic Markov chain on an arbitrary state space S with countably generated σ-field, with transition distribution Q and invariant distribution π. See e.g. Meyn and Tweedie (1993) for these concepts. We use the notation $\pi \otimes Q(dx, dy) = \pi(dx)Q(x, dy)$ and $Q_x w = \int Q(x, dy)w(x, y)$.

Let v be a k-dimensional measurable function defined on S^2 such that the constraint $\pi \otimes Qv = 0$ holds for all transition distributions Q in the model. Fix the true transition distribution Q, and let W be the set of all real-valued measurable functions w on S^2 such that $Q_x|w|/V(x)$ is bounded in x. Assume that v is in W. We refer to Schick and Wefelmeyer (2000a) for a discussion of this assumption. Set

$$H = \{h \in L_2(\pi \otimes Q) : Qh = 0\}.$$

Then $h(X_{i-1}, X_i)$ is a martingale increment.

1. Let $t(Q)$ be a real-valued functional of the transition distribution. Following the approach outlined in the Introduction for the i.i.d. case, call an estimator $\hat{t}$ *asymptotically linear* with *influence function* b if $b \in H$ and $\hat{t}$ admits the martingale approximation

$$n^{1/2}(\hat{t} - t(Q)) = n^{-1/2} \sum_{i=1}^{n} b(X_{i-1}, X_i) + o_P(1).$$

By a martingale central limit theorem, see Meyn and Tweedie (1993, Theorem 17.4.4), $\hat{t}$ is asymptotically normal with variance $\pi \otimes Qb^2$. From the constraint $\pi \otimes Qv = 0$ we obtain new estimators

$$\hat{t}(c) = \hat{t} - c^\top \frac{1}{n} \sum_{i=1}^{n} v(X_{i-1}, X_i). \tag{2.1}$$

By the martingale approximation of Gordin (1969), see Meyn and Tweedie (1993, Section 17.4), we have

$$n^{-1/2} \sum_{i=1}^{n} \Big(v(X_{i-1}, X_i) - Av(X_{i-1}, X_i) \Big) = o_P(1) \tag{2.2}$$

with

$$Av(x, y) = v(x, y) - Q_x v + \sum_{j=1}^{\infty} (Q_y^j - Q_x^{j+1}) v.$$

From (2.1) and (2.2),

$$n^{1/2}(\hat{t}(c) - t(Q)) = n^{-1/2} \sum_{i=1}^{n} \Big(b(X_{i-1}, X_i) - c^\top Av(X_{i-1}, X_i) \Big) + o_P(1).$$

By construction, $Av(X_{i-1}, X_i)$ is a martingale increment. Hence $\hat{t}(c)$ is asymptotically linear with influence function $b - c^\top Av$. Again by the martingale central limit theorem, $\hat{t}(c)$ is asymptotically normal with variance $\sigma^2 = \pi \otimes Q[(b - c^\top Av)^2]$. Assume that $\pi \otimes Q[Av \cdot Av^\top]$ is invertible. By the Schwarz inequality, the variance is minimized for $c = c_b$ with

$$c_b = (\pi \otimes Q[Av \cdot Av^\top])^{-1} \pi \otimes Q[Av \cdot b].$$

The minimal asymptotic variance is

$$\sigma_b^2 = \pi \otimes Qb^2 - \pi \otimes Q[bAv^\top](\pi \otimes Q[Av \cdot Av^\top])^{-1} \pi \otimes Q[Av \cdot b].$$

The optimal vector c_b depends on the unknown transition distribution and must be replaced by a consistent estimator $\hat{c}_b$. The estimator $\hat{t}(\hat{c}_b)$ has the same asymptotic variance as $\hat{t}(c_b)$. We arrive at the following result.

Theorem 1. *If $\hat{c}_b$ is consistent for c_b, then the estimator $\hat{t}(\hat{c}_b)$ is asymptotically linear for $t(Q)$ with influence function $b - c_b^\top Av$ and asymptotic variance σ_b^2.*

2. We show now that if $\hat{t}$ is asymptotically linear and regular, then $\hat{t}(\hat{c}_b)$ is regular and efficient in the sense of Hájek's convolution theorem. The set H introduced above consists of the functions h on S^2 for which one can construct Hellinger differentiable perturbations of Q of the form

$$Q_{nh}(x, dy) \doteq Q(x, dy)(1 + n^{-1/2}h(x, y))$$

that are again transition distributions. This means that H is the *tangent space* of the full nonparametric model. By Kartashov (1985), see also Kartashov (1996) and Greenwood and Wefelmeyer (1999), we have the perturbation expansion

$$n^{1/2}(\pi_{nh} \otimes Q_{nh}v - \pi \otimes Qv) \to \pi \otimes Q[hAv]. \tag{2.3}$$

The constraints $\pi \otimes Qv = 0$ and $\pi_{nh} \otimes Q_{nh}v = 0$ now lead to a constraint on h, namely $\pi \otimes Q[hAv] = 0$. Hence the tangent space of the constrained model consists of all functions h orthogonal to Av,

$$H_* = \{h \in H : \pi \otimes Q[hAv] = 0\}.$$

The functional $t(Q)$ is called *differentiable* at Q with *gradient* g if $g \in H$ and

$$n^{1/2}(t(Q_{nh}) - t(Q)) \to \pi \otimes Q[hg] \quad \text{for } h \in H_*. \tag{2.4}$$

The *canonical gradient* is the projection g_* of g onto H_*. The estimator $\hat{t}$ is called *regular* at Q with *limit* L if

$$n^{1/2}(\hat{t} - t(Q_{nh})) \Rightarrow L \quad \text{under } P_{nh} \text{ for } h \in H_*.$$

Here P_{nh} is the law of $X_0, \dots, X_n$ when Q_{nh} is the true transition distribution.

We recall two characterizations from the theory of efficient estimation; for appropriate versions see e.g. Wefelmeyer (1999, Sections 3 and 5). (1) *An asymptotically linear estimator is regular if and only if its influence function is a gradient.* (2) *A regular estimator is efficient if and only if it is asymptotically linear with influence function equal to the canonical gradient.*

By definition, H has the orthogonal decomposition $H = H_* \oplus [Av]$, where $[Av]$ is the linear span of Av. Hence the canonical gradient, the projection g_* of g onto H_*, can be written $g_* = g - g_v$, where g_v is the projection of g onto $[Av]$, i.e. $g_v = c_*^\top Av$ with

$$c_* = (\pi \otimes Q[Av \cdot Av^\top])^{-1}\pi \otimes Q[Av \cdot g].$$

Now let $\hat{t}$ be a regular and asymptotically linear estimator for $t(Q)$. By characterization (1), its influence function is a gradient, say g. By Theorem 1, the estimator $\hat{t}(\hat{c}_*)$ has influence function $g - c_*^\top Av = g_*$. From characterization (2) we obtain the following result.

Theorem 2. *If $\hat{t}$ is a regular and asymptotically linear estimator for $t(Q)$, and $\hat{c}_*$ is consistent for c_*, then $\hat{t}(\hat{c}_*)$ is regular and efficient for $t(Q)$ in the model constrained by $\pi \otimes Qv = 0$.*

Note that for the improvement $\hat{t}(c)$ we needed the constraint $\pi \otimes Qv = 0$ only for the true Q, while for efficiency of $\hat{t}(\hat{c}_*)$ we needed the constraint also for perturbations Q_{nh}, at least in the direction of the canonical gradient.

3. Suppose we know, in addition to $\pi \otimes Qv = 0$, that the Markov chain is *reversible*, $\pi(dx)Q(x,dy) = \pi(dy)Q(y,dx)$. By Greenwood and Wefelmeyer (1999), this puts the following additional constraint on the tangent space:

$$H_*^{\mathrm{rev}} = \{h \in H_* : Bh \text{ symmetric}\}.$$

Here B is the *adjoint* of A in the sense that for $h \in H$ and $w \in W$,

$$\pi \otimes Q[hAw] = \pi \otimes Q[Bh \cdot w].$$

Let $t(Q)$ be differentiable at Q with gradient $g \in H$ in this doubly constrained model in the sense that (2.4) holds for $h \in H_*^{\mathrm{rev}}$. As in the proof of Theorem 2 of Greenwood and Wefelmeyer (1999), the projection g_*^{rev} of g onto H_*^{rev} is obtained by symmetrizing g_*,

$$\begin{aligned}
g_*^{\mathrm{rev}}(x,y) &= \frac{1}{2}\big(g(x,y) + g(y,x)\big) - c_*^{\mathrm{rev}}\frac{1}{2}\big(v(x,y) + v(y,x)\big), \\
c_*^{\mathrm{rev}} &= \big(E[Av(X_0,X_1) \cdot Av(X_0,X_1)^\top]\big)^{-1} \\
&\quad \frac{1}{2}E\big[Av(X_0,X_1)\big(g(X_0,X_1) + g(X_1,X_0)\big)\big].
\end{aligned}$$

Here and in the following, expectations are taken with respect to the *stationary* law of the chain. Note that if $\hat{t}$ has influence function $g \in H$, then the symmetrized estimator

$$\frac{1}{2}\big(\hat{t}(X_0,\dots,X_n) + \hat{t}(X_n,\dots,X_0)\big)$$

has influence function $\frac{1}{2}\big(g(x,y) + g(y,x)\big)$. We arrive at the following result.

Theorem 3. *If $\hat{t}$ is a regular and asymptotically linear estimator for $t(Q)$, and $\hat{c}_*^{\mathrm{rev}}$ is consistent for c_*^{rev}, then*

$$\frac{1}{2}\big(\hat{t}(X_0,\dots,X_n) + \hat{t}(X_n,\dots,X_0)\big) - \hat{c}_*^{\mathrm{rev}}\frac{1}{2n}\sum_{i=1}^n \big(v(X_{i-1},X_i) + v(X_i,X_{i-1})\big)$$

is regular and efficient for $t(Q)$ in the model constrained by $\pi\otimes Qv = 0$ and reversibility.

4. In this subsection we treat the problem of estimating c_* for linear functionals $t(Q) = \pi\otimes Qf$ with f in W, and constraint $\pi\otimes Qv = Ev(X_0, X_1) = 0$. In the i.i.d. case, c_* was easy to estimate. For Markov chains, c_* involves the operator A, and estimation is less straightforward. By the martingale approximation (2.2), the empirical estimator

$$\hat{t} = \frac{1}{n}\sum_{i=1}^{n} f(X_{i-1}, X_i)$$

is asymptotically linear with influence function $b = Af$ in H. By the perturbation expansion (2.3), Af is a gradient of $\pi\otimes Qf$. Hence the empirical estimator is regular by characterization (1). If nothing is known about Q, the empirical estimator is efficient: see Penev (1991) and Bickel (1993) for functions f of one argument, and Greenwood and Wefelmeyer (1995) for functions f of two arguments; or simply note that H is the tangent space of the full nonparametric model, and hence Af is the canonical gradient of $\pi\otimes Qf$.

For $t(Q) = \pi\otimes Qf$ we have

$$c_* = c_f = (\pi\otimes Q[Av\cdot Av^\top])^{-1}\,\pi\otimes Q[Av\cdot Af] = \Sigma^{-1}F,$$

say. One checks that for vectors w and z with components in W,

$$\begin{aligned}
\pi\otimes Q[Aw\cdot Az^\top] &= E\big[\big(w(X_0,X_1) - Ew(X_0,X_1)\big)z(X_0,X_1)^\top\big]\\
&\quad +\sum_{j=1}^{\infty}\Big(E\big[\big(w(X_0,X_1) - Ew(X_0,X_1)\big)z(X_j,X_{j+1})^\top\big]\\
&\qquad +E\big[\big(w(X_j,X_{j+1}) - Ew(X_0,X_1)\big)z(X_0,X_1)^\top\big]\Big).
\end{aligned}$$

For functions of *one* argument compare Meyn and Tweedie (1993, Section 17.4.3). Now we use the constraint $Ev(X_0,X_1) = 0$ to estimate $\Sigma = \pi\otimes Q[Av\cdot Av^\top]$ by

$$\hat{\Sigma} = \frac{1}{n}\sum_{i=1}^{n} v(X_{i-1},X_i)v(X_{i-1},X_i)^\top + \sum_{j=1}^{m(n)}\frac{2}{n-j}\sum_{i=1}^{n-j} v(X_{i-1},X_i)v(X_{i+j-1},X_{i+j})^\top$$

and $F = \pi\otimes Q[Av\cdot Af]$ by

$$\begin{aligned}
\hat{F} &= \frac{1}{n}\sum_{i=1}^{n} v(X_{i-1},X_i)f(X_{i-1},X_i)\\
&+ \sum_{j=1}^{m(n)}\frac{1}{n-j}\sum_{i=1}^{n-j}\Big(v(X_{i-1},X_i)f(X_{i+j-1},X_{i+j}) + v(X_{i+j-1},X_{i+j})f(X_{i-1},X_i)\Big).
\end{aligned}$$

Since the chain is assumed V^2-uniformly ergodic, it is V^2-uniformly mixing by Meyn and Tweedie (1993, Theorem 16.1.5). To prove consistency of $\hat{F}$, set $v_K = -K \vee v \wedge K$ and write $\hat{F}_K$ for the corresponding estimator with truncated v. Since $\sum_{j=1}^{\infty} Q^j f$ converges in $L_2(\pi)$, we obtain from the Cauchy–Schwarz inequality that for each $\varepsilon > 0$ there is a K such that

$$E|\hat{F}_K - \hat{F}| \leq \varepsilon, \quad |\pi \otimes Q[Av_K \cdot Af] - \pi \otimes Q[Av \cdot Af]| \leq \varepsilon.$$

Furthermore, by straightforward calculation, for $m(n)$ tending to infinity more slowly than n,

$$E[\hat{F}_K - \pi \otimes Q[Av_K \cdot Af]]^2 \to 0.$$

Hence $\hat{F}$ is consistent. In practice $m(n)$ will be taken small. Consistency of $\hat{\Sigma}$ is proved similarly. We arrive at the following result.

Theorem 4. *If $m(n)$ tends to infinity more slowly than n, then $\hat{c}_f = \hat{\Sigma}^{-1}\hat{F}$ is consistent for c_f.*

3 Applications

Example 1. If the function v is constant in one argument, say $v(x, y) = v_1(y)$, then the constraint is $\pi \otimes Qv = \pi v_1 = 0$. In particular, for real state space $S = \mathbf{R}$ and constraint $\pi v = 0$ with $v(x, y) = y$, the chain has mean zero. A natural estimator for the *variance* $t(Q) = E(X - EX)^2$ of the invariant distribution is the empirical estimator $\frac{1}{n}\sum_{i=1}^n X_i^2 - (\frac{1}{n}\sum_{i=1}^n X_i)^2$. Since $EX = 0$, we have $E(X - EX)^2 = EX^2$, and an asymptotically equivalent estimator is the empirical second moment $\frac{1}{n}\sum_{i=1}^n X_i^2$. By Theorem 2, a better estimator is

$$\hat{t}(\hat{c}_f) = \frac{1}{n}\sum_{i=1}^n X_i^2 - \hat{c}_f \frac{1}{n}\sum_{i=1}^n X_i,$$

with $\hat{c}_f$ a consistent estimator of $(\pi \otimes Q[(Av)^2])^{-1}\pi \otimes Q[Av \cdot Af]$ for $v(x, y) = y$ and $f(x, y) = y^2$.

Example 2. Consider the linear autoregressive model of order one, $X_i = \rho X_{i-1} + \varepsilon_i$, where the innovations ε_i are i.i.d. with mean zero, finite variance σ^2, finite fourth moment and $|\rho| < 1$. Then the invariant distribution π has mean zero. This is a submodel of Example 1. For this submodel, the operator A and the estimator for c_f simplify. Let us again consider the problem of estimating the variance $t(Q) = E(X - EX)^2 = EX^2$ of the invariant distribution. For $w \in L_2(\pi)$,

$$Q_y^j w = Ew\Big(\sum_{k=0}^{j-1} \rho^k \varepsilon_{i-k} + \rho^j y\Big).$$

In particular, for $v(x,y) = y$ and $f(x,y) = y^2$,

$$Av(x,y) = \frac{1}{1-\rho}(y - \rho x), \quad Af(x,y) = \frac{1}{1-\rho^2}(y^2 - \rho^2 x^2 - \sigma^2).$$

Hence

$$\pi \otimes Q[(Av)^2] = \frac{\sigma^2}{(1-\rho)^2}, \quad \pi \otimes Q[Av \cdot Af] = \frac{\alpha_3}{(1-\rho)(1-\rho^2)},$$

where $\alpha_3 = E\varepsilon^3$ is the third moment of the innovation distribution.

Estimate the autoregression coefficient ρ by the least squares estimator

$$\hat{\rho} = \sum_{i=1}^n X_{i-1} X_i \Big/ \sum_{i=1}^n X_{i-1}^2,$$

the innovations ε_i by $\hat{\varepsilon}_i = X_i - \hat{\rho} X_{i-1}$, and σ^2 and α_3 by the empirical moments based on the estimated innovations,

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n \hat{\varepsilon}_i^2, \quad \hat{\alpha}_3 = \frac{1}{n} \sum_{i=1}^n \hat{\varepsilon}_i^3.$$

We obtain

$$\hat{t}(\hat{c}_f) = \frac{1}{n} \sum_{i=1}^n X_i^2 - \frac{\hat{\alpha}_3}{(1+\hat{\rho})\hat{\sigma}^2} \frac{1}{n} \sum_{i=1}^n X_i.$$

We note that for $\rho = 0$ the observations are $X_i = \varepsilon_i$ and i.i.d., and the estimator $\hat{t}(\hat{c}_f)$ is asymptotically equivalent to the estimator obtained in the i.i.d. case.

To estimate c_f, we have used the information that the Markov chain is an AR(1) model. This information simplifies $\hat{c}_f$ but does not improve the estimator $\hat{t}(\hat{c}_f)$ asymptotically. We refer to Schick and Wefelmeyer (2000b, Section 6) for better estimators of EX^2, and to Schick and Wefelmeyer (2000c) for efficient estimators of general linear functionals of invariant laws of linear time series.

Remark 1. Constraints $\pi \otimes Qv = 0$ for functions $v(x,y) = u(x)w(y) - u(y)w(x)$ describe symmetries of the joint law of two successive observations with respect to time reversal. If such constraints hold for a sufficiently large class of functions, e.g. — in the case of real state space — for all indicators $u(x) = 1_{(-\infty,a]}(x)$ and $w(y) = 1_{(-\infty,b]}(y)$ with $a, b \in \mathbf{R}$, then the chain is reversible. Let $t(Q)$ be differentiable, and let $\hat{t} = \hat{t}(X_0, \dots, X_n)$ be an asymptotically linear estimator for $t(Q)$. By the arguments in Subsection 3 of Section 2, the symmetrized estimator

$$\frac{1}{2}\big(\hat{t}(X_0, \dots, X_n) + \hat{t}(X_n, \dots, X_0)\big)$$

is efficient for $t(Q)$ if the chain is known to be reversible.

Remark 2. For real state space, constraints $\pi \otimes Qv = 0$ for functions $v(x, y) = z(x, y) - z(-x, -y)$ describe symmetries of the joint law of two successive observations with respect to reflection at zero. If such constraints hold for a sufficiently large class of functions, e.g. for all functions $z(x, y) = 1_{(-\infty,a]}(x)1_{(-\infty,b]}(y)$ with $a, b \in \mathbf{R}$, then

$$\pi(dx)Q(x, dy) = \pi(-dx)Q(-x, -dy)$$

and therefore $\pi(dx) = \pi(-dx)$ and $Q(x, dy) = Q(-x, -dy)$. In this case, we do not need the results of Section 2. Note also that the condition $Q(x, dy) = Q(-x, -dy)$ implies

$$\int \pi(-dx)Q(x, dy) = \int \pi(-dx)Q(-x, -dy) = \pi(-dy),$$

and hence $\pi(dx) = \pi(-dx)$ holds automatically. The tangent space of the model constrained by symmetry of the transition distribution, $Q(x, dy) = Q(-x, -dy)$, is

$$H_* = \{h \in H : h(x, y) = h(-x, -y)\}.$$

Write $f^-(x, y) = f(-x, -y)$. It is straightforward to check that $Af^- = (Af)^-$. For $h \in H_*$ we have $h = h^-$ and

$$\pi \otimes Q[hAf] = \pi \otimes Q[h^-(Af)^-] = \frac{1}{2}\pi \otimes Q[h(Af + (Af)^-)] = \frac{1}{2}\pi \otimes Q[hA(f + f^-)].$$

Hence the projection of Af onto H_* is $\frac{1}{2}A(f + f^-)$. By the martingale approximation (2.2), this is the influence function of the symmetrized empirical estimator

$$\hat{t}_* = \frac{1}{2n}\sum_{i=1}^{n}\big(f(X_{i-1}, X_i) + f(-X_{i-1}, -X_i)\big),$$

which is therefore efficient for $\pi \otimes Qf$ under the constraint $Q(x, dy) = Q(-x, -dy)$.

Similarly as in Remark 1, the result generalizes to arbitrary differentiable functionals $t(Q)$ with gradient $g \in H$. Let $\hat{t}$ be an asymptotically linear estimator for $t(Q)$ with influence function g. Then the symmetrized estimator

$$\hat{t}_* = \frac{1}{2}\big(\hat{t}(X_0, \ldots, X_n) + \hat{t}(-X_0, \ldots, -X_n)\big)$$

is efficient for $t(Q)$ if the chain is known to be symmetric.

References

Bickel, P. J. (1993). Estimation in semiparametric models. In: *Multivariate Analysis: Future Directions* (C. R. Rao, ed.), 55–73, North-Holland, Amsterdam.

Bickel, P. J., Klaassen, C. A. J., Ritov, Y. and Wellner, J. A. (1998). *Efficient and Adaptive Estimation for Semiparametric Models.* Springer, New York.

Gordin, M. I. (1969). The central limit theorem for stationary processes. *Soviet Math. Dokl.* **10**, 1174–1176.

Greenwood, P. E. and Wefelmeyer, W. (1995). Efficiency of empirical estimators for Markov chains. *Ann. Statist.* **23**, 132–143.

Greenwood, P. E. and Wefelmeyer, W. (1999). Reversible Markov chains and optimality of symmetrized empirical estimators. *Bernoulli* **5**, 109-123.

Haberman, S. J. (1984). Adjustment by minimum discriminant information. *Ann. Statist.* **12**, 971–988.

Kartashov, N. V. (1985). Criteria for uniform ergodicity and strong stability of Markov chains with a common phase space. *Theory Probab. Math. Statist.* **30**, 71–89.

Kartashov, N. V. (1996). *Strong Stable Markov Chains*, VSP, Utrecht.

Levit, B. Y. (1975). Conditional estimation of linear functionals. *Problems Inform. Transmission* **11**, 39–54.

Meyn, S. P. and Tweedie, R. L. (1993). *Markov Chains and Stochastic Stability*, Springer, London.

Penev, S. (1991). Efficient estimation of the stationary distribution for exponentially ergodic Markov chains. *J. Statist. Plann. Inference* **27**, 105–123.

Schick, A. and Wefelmeyer, W. (2000a). Estimating joint distributions of Markov chains. To appear in: *Stat. Inference Stoch. Process.* **3**.

Schick, A. and Wefelmeyer, W. (2000b). Efficient estimation in invertible linear processes. Technical Report, Department of Mathematical Sciences, Binghamton University.

Schick, A. and Wefelmeyer, W. (2000c). Estimating invariant laws of linear processes by U-statistics. Technical Report, Department of Mathematical Sciences, Binghamton University.

Wefelmeyer, W. (1999). Efficient estimation in Markov chain models: an introduction. In: *Asymptotics, Nonparametrics, and Time Series* (S. Ghosh, ed.), 427-459, Statistics: Textbooks and Monographs 158, Dekker, New York.