Mathematical Modeling of a Three Axis Robotic Arm

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Abstract

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This presentation models and simulates the spatial motion of a three-degree-of-freedom robotic arm using Python. Joint coordinates are calculated based on link lengths (L_1, L_2) and joint angles (θ, ϕ_1, ϕ_2) through trigonometric expressions. The geometric and trigonometric conditions for the arm's endpoint to reach a target surface (e.g., a whiteboard) are analyzed, including valid joint angle ranges. Vector analysis, trigonometric transformations, and the law of cosines are employed, and the results are visualized and tested in Python. The project offers a theoretical and practical approach to solving inverse kinematics problems in robotics.

Outline of the Talk

1. **Introduction and Applications**
   - Real-world applications of 3 variable robotic arms (e.g., assembly lines, surgery, 3D printing, education)
   - Motivation: Need for precise control in spatial environments
2. **Problem Definition**
   - What is spatial positioning?
   - Why modeling robotic arm kinematics is challenging and important
3. **Mathematical Representation of the Robotic Arm**
   - Definition of link lengths (L_1, L_2) and joint angles (θ, ϕ_1, ϕ_2)
   - Description of coordinate system and base point (x_0, y_0, z_0)

Outline of the Talk

4. **Forward and Inverse Kinematics**
   - Computing joint positions (x_1, y_1, z_1) and end-effector (x_2, y_2, z_2)
   - Deriving inverse kinematic expressions for θ, ϕ_1, ϕ_2
5. **Geometric and Trigonometric Constraints**
   - Applying the law of cosines and triangle relationships
   - Constraints for reaching a target (e.g., whiteboard surface)
6. **Python-Based Simulation and Visualization**
   - Implementation of kinematic model in Python
   - 3D graphical representation and animation of the robotic arm

Applications of Three Axis Robotic Arm

- **Assembly Lines:** 3D positioning and precise assembly of components.
- **Welding and Painting Applications:** Performing tasks at complex angles.
- **Surgical Robots:** Medical operations requiring precise, multi-axis movements.
- **3D Printing and CNC Machining:** Executing complex toolpaths in manufacturing.
- **Research and Education:** Used in advanced robotics control and kinematic studies.

Applications of Two-Arm Robotic Systems

Excavation and Industrial Robotics

Excavator system

Industrial robot arm

Problem Statement

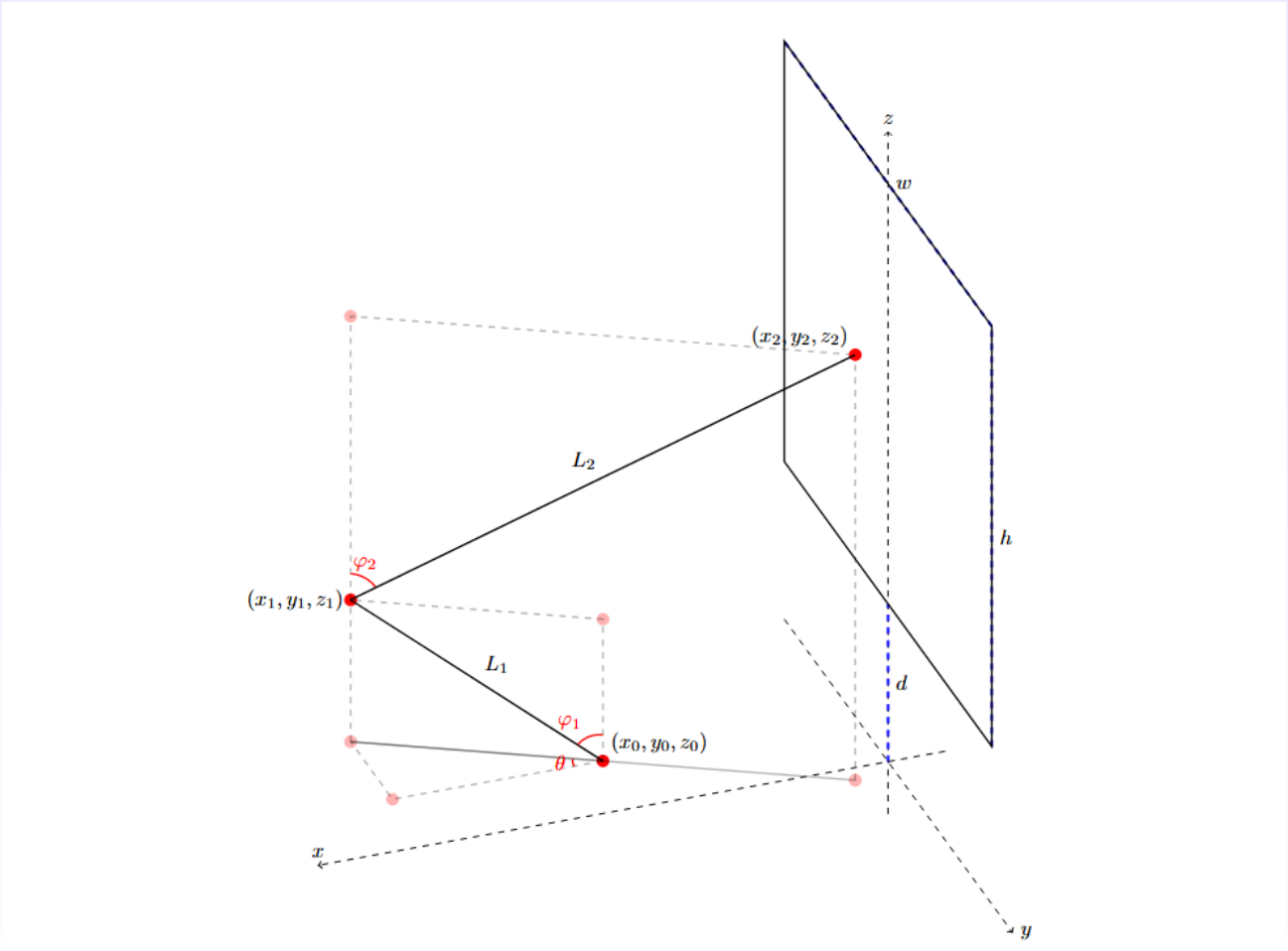
Kinematic Modeling of a 3 Variable Robotic Arm

The precise control and spatial positioning of robotic arms with multiple variables present significant challenges in robotics.

Modeling the kinematics of a three-axis robotic arm mathematically is essential for understanding its movement, enabling accurate targeting and manipulation within a 3D workspace.

This project aims to develop a comprehensive mathematical model and simulation framework to solve the forward and inverse kinematics problems for such robotic arms, ensuring efficient and reliable operation in real-world applications.

Representation of Robot Arm Angles and Coordinates



Mathematical Representation of the Robotic Arm

Link Lengths and Joint Angles

Let (x_0, y_0, z_0) denote the base point of the robotic arm. Let L_1 and L_2 be the lengths of the first and second links, respectively.

- ϕ_1 is the angle between link L_1 and the z-axis.
- ϕ_2 is the angle between link L_2 and the z-axis.
- θ is the angle between the x-axis and the xy-projection of L_1.

Mathematical Representation of the Robotic Arm

Forward Kinematic Equations

Using trigonometric relationships, the coordinates of the joints are:

Intermediate point (joint 1):

$$x_1 = x_0 + L_1 \sin(\phi_1) \cos(\theta)$$
$$y_1 = y_0 - L_1 \sin(\phi_1) \sin(\theta)$$
$$z_1 = z_0 + L_1 \cos(\phi_1)$$

End-effector (joint 2):

$$x_2 = x_1 - L_2 \sin(\phi_2) \cos(\theta)$$
$$y_2 = y_1 + L_2 \sin(\phi_2) \sin(\theta)$$
$$z_2 = z_1 + L_2 \cos(\phi_2)$$

Mathematical Representation of the Robotic Arm

Angle Formulas for θ, ϕ_1, ϕ_2

Angle θ (rotation in xy-plane):

$$\theta = -\arctan\left(\frac{y_2 - y_0}{x_2 - x_0}\right)$$

Angle ϕ_1 (inclination of first link):

$$\phi_1 = \arctan\left(\frac{x_1 - x_0}{z_1 - z_0 \cdot \sec(\theta)}\right)$$

Angle ϕ_2 (inclination of second link):

$$\phi_2 = \arctan\left(\frac{y_2 - y_1}{z_2 - z_1 \cdot \csc(\theta)}\right)$$

These expressions allow us to compute the required joint angles based on spatial coordinates of the arm.

Animation Program Code

```
import numpy as np
import matplotlib.pyplot as plt
from matplotlib.animation import FuncAnimation

# Calculates the planar angle (theta) between two points in the XY plane
def tf(x, y, u, v):
    return np.pi - np.arctan2(v - y, u - x)
# Computes the 3D Euclidean distance between two points (used as l3)
def l3f(x, y, z, u, v, w):
    return np.linalg.norm([u - x, v - y, w - z])
# Calculates the angle (in radians) using the Law of Cosines
def af(l, k, h):
    return np.arccos((k**2 + h**2 - l**2) / (2 * k * h))
# Alternative version of af() with different argument order
def bf(l, k, h):
    return af(k, l, h)
```

Animation Program Code

```
# Calculates the vertical angle based on the Z-axis height difference
def df(z, w, h):
    return np.arccos(np.abs(w - z) / h)
# Calculates the first joint angle phi1
def f1f(z, w, l, k, h):
    return bf(l, k, h) - df(z, w, h)
# Calculates the second joint angle phi2
def f2f(z, w, l, k, h):
    return af(l, k, h) + df(z, w, h)

# parameters
w, h, d = 10, 8, 5
l1, l2 = 5, 10
x0, y0, z0 = 5, 0, 1
```

Animation Program Code

```
# The logo is parameterized over the interval t E [0, 14),
def batmanlogo_scalar(t):
    pi = np.pi
    # Transforms and scales the shape to fit the drawing area
    def transform(x, y):
        return [x / 2, y / 2 + 8]
    # Piecewise construction of the Batman logo using
    trigonometric and polynomial segments
    if 0 <= t < 1:
        return transform(0.6 * t, 4)
    elif 1 <= t < 2:
        return transform(0.6 + 0.6 * (t - 1), 4 + 0.94 * (t - 1))
    elif 2 <= t < 3:
        return transform(1.2, 4.9434 - 1.4434 * (t - 2))
    elif 3 <= t < 4:
        s = (0.576 + pi) * (t - 3) - pi
        return transform(1.52 * np.cos(s) + 2.72, 1.52 * np.sin(s
    ) + 3.5)
```

Animation Program Code

```
    elif 4 <= t < 5:
        return transform(8 * np.cos(1.83 * (5 - t) - 0.785), 5 *
    np.sin(1.83 * (5 - t) - 0.785))
    elif 5 <= t < 6:
        s = 4.4 + 2.26 * (6 - t)
        return transform(-12.43 + 5.212 * s - 0.375 * s**2,
    -21.52 + 7.03 * s - 0.65 * s**2)
    elif 6 <= t < 7:
        s = 3.25 * (7 - t)
        return transform(s, -2 - 0.75 * (s - 2)**2)
    elif 7 <= t < 8:
        s = 3.25 * (t - 7)
        return transform(-s, -2 - 0.75 * (s - 2)**2)
    elif 8 <= t < 9:
        s = 4.4 + 2.26 * (t - 8)
        return transform(12.43 - 5.212 * s + 0.375 * s**2, -21.52
     + 7.03 * s - 0.65 * s**2)
    elif 9 <= t < 10:
        s = 1.83 * (t - 9) - 0.785
        return transform(-8 * np.cos(s), 5 * np.sin(s))
```

Animation Program Code

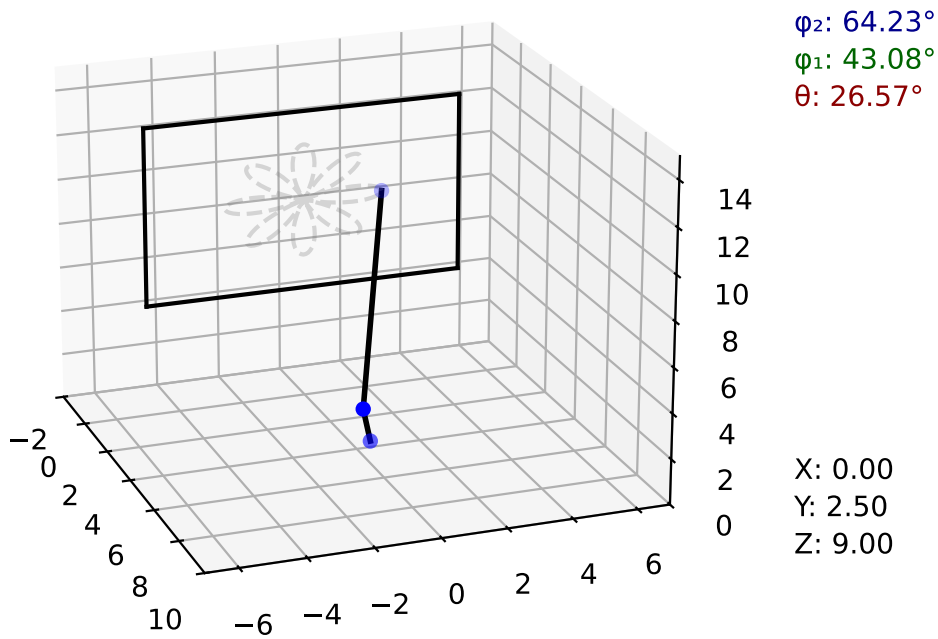
```
    elif 10 <= t < 11:
        s = (0.576 + pi) * (11 - t) - pi
        return transform(-1.52 * np.cos(s) - 2.72, 1.52 * np.sin(
    s) + 3.5)
    elif 11 <= t < 12:
        return transform(-1.2, 4.9434 - 1.4434 * (12 - t))
    elif 12 <= t < 13:
        return transform(-0.6 - 0.6 * (13 - t), 4 + 0.94 * (13 -
    t))
    elif 13 <= t < 14:
        return transform(-0.6 * (14 - t), 4)
    else:
        return [0, 0]
```

Animation Program Code

```
# Returns x and y coordinates for the given time(s) t by evaluating the Batman logo path.
def batmanlogo(t):
    t = np.asarray(t)
    if t.ndim == 0:
        return batmanlogo_scalar(t)
    else:
        result = np.array([batmanlogo_scalar(ti) for ti in t])
        return result[:, 0], result[:, 1]
# Converts the 2D Batman logo path into 3D by assigning x = 0.
def batmanlogo3d(t):
    x, y = batmanlogo(t)
    return [0 * np.array(x), x, y]
# Helper functions to extract the X, Y, and Z coordinates from the 3D path.
def x_t(t): return batmanlogo3d(t)[0]
def y_t(t): return batmanlogo3d(t)[1]
def z_t(t): return batmanlogo3d(t)[2]
```

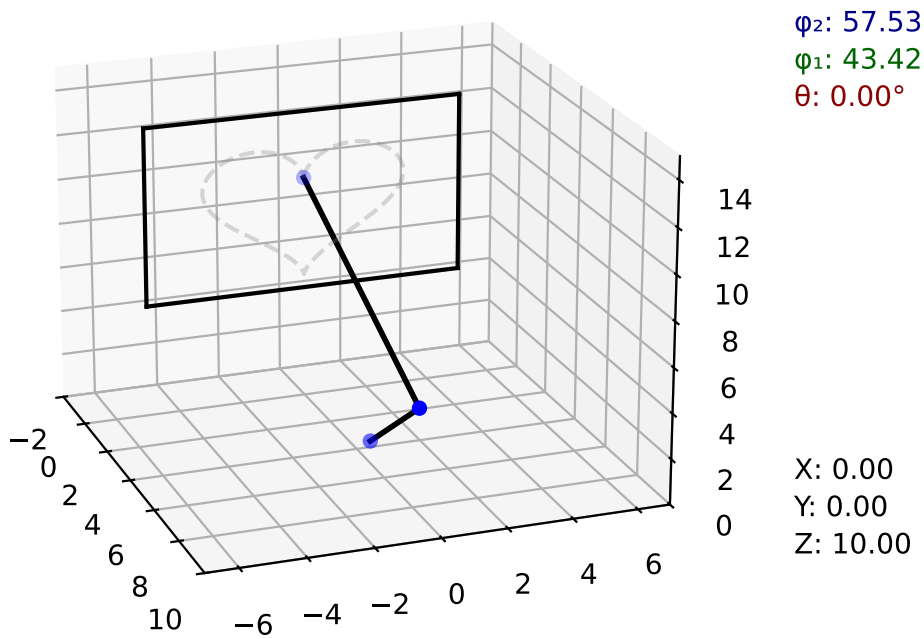
Python Simulation Codes and Graphical Representation

Python Animations



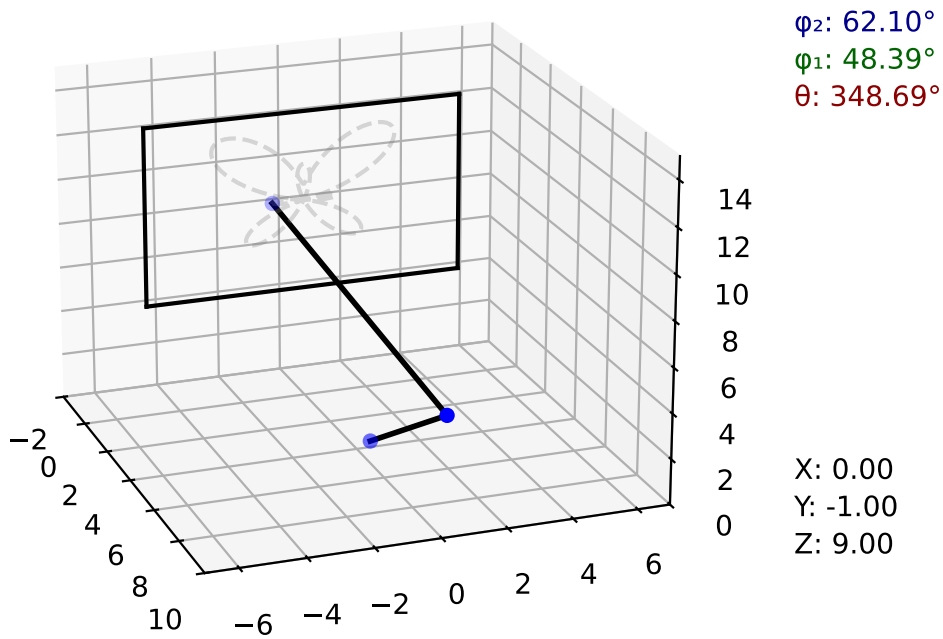
Python Simulation Codes and Graphical Representation

Python Animations



Python Simulation Codes and Graphical Representation

Python Animations



Python Simulation Codes and Graphical Representation

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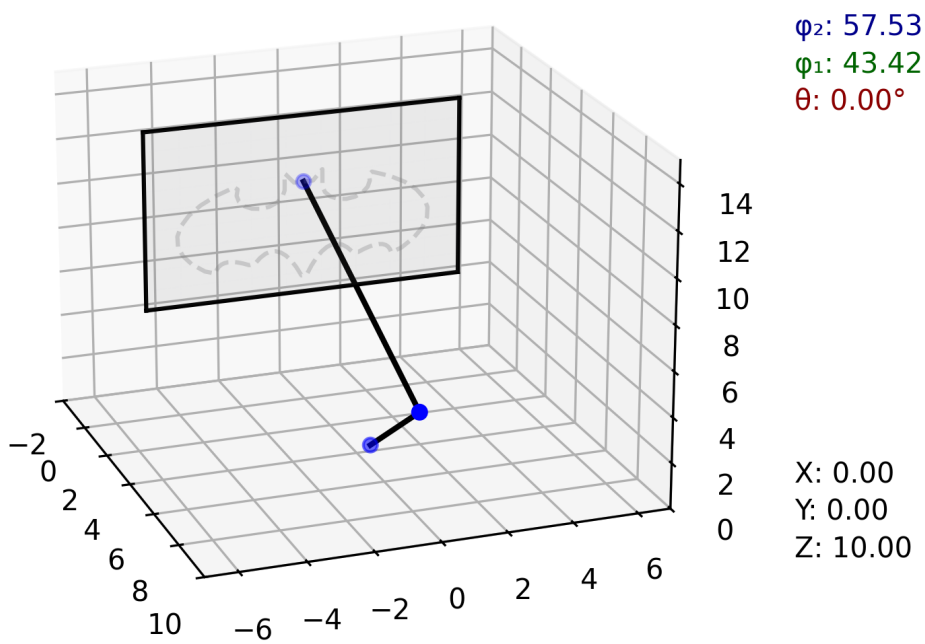
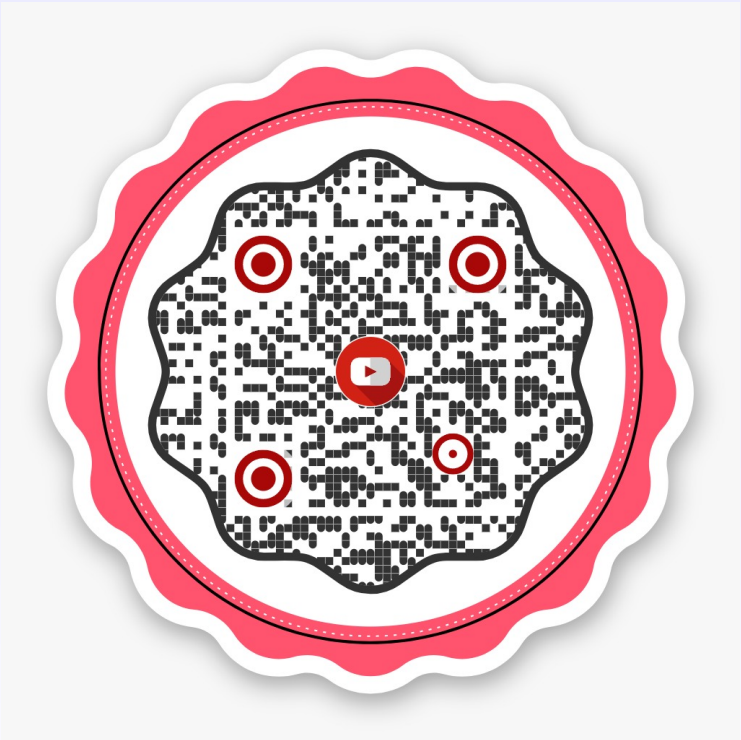


Thank You



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