DIFERENSIAL UNTUK FUNGSI MULTI VARIABEL

Untuk diingat :
Derivatif digunakan sebagai analisis perubahan suatu fenomena

Motivasi (1)

Heat index sebagai fungsi temperature dan humidity

Relative humidity (%)

Actual temperature (°F) T \ H	50	55	60	65	70	75	80	85	90
90	96	98	100	103	106	109	112	115	119
92	100	103	105	108	112	115	119	123	128
94	104	107	111	114	118	122	127	132	137
96	109	113	116	121	125	130	135	141	146
98	114	118	123	127	133	138	144	150	157
100	119	124	129	135	141	147	154	161	168

Bagaimana kenaikan heat index terhadap kenaikan suhu , jika humidity konstan di 70% ?

Motivasi (1)

Misalkan heat indeks $H = H(T, H)$ sebagai fungsi dua variabel yakni temperature (=T) dan humadity (=H).

Bagaimana kenaikan heat index terhadap kenaikan suhu , jika humidity konstan di 70% ?

Solusi. Sebagai derivatif $\boldsymbol{H = H(T, H)}$ terhadap vairabel T dengan H = 70, kenaikannya

$$= \lim_{x \to 0} \frac{H(T + x, 70) - H(T, 70)}{x}$$

Hal ini yang dinamakan derivatif partial terhadap variabel T.

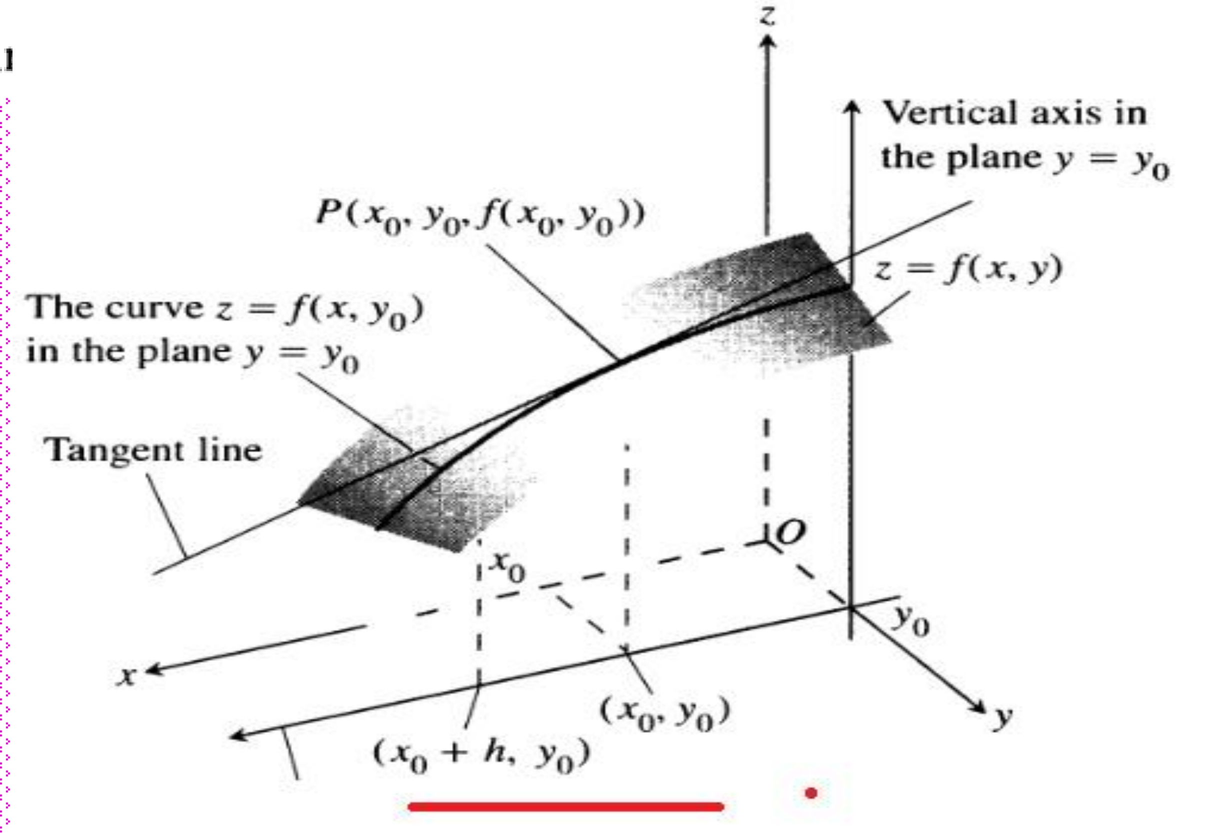
Motivasi 2 : Tafsiran geometris

Definition

The **partial derivative of $f(x, y)$ with respect to x** at the point (x_0, y_0) is

$$\boxed{\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = \left.\frac{d}{dx} f(x, y_0)\right|_{x=x_0} = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}} \qquad (1)$$

provided the li



DEFINISI DERIVATIF PARSIAL

The **partial derivative** of $z = f(x, y)$ with respect to x at the point (x, y) is

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

provided this limit exists.

The **partial derivative** of $z = f(x, y)$ with respect to y at the point (x, y) is

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

provided this limit exists.

DERIVATIF PARSIAL

EXAMPLE 1 Find the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(4, -5)$ if

$$f(x, y) = x^2 + 3xy + y - 1.$$

Solution

To find $\partial f/\partial x$, we regard y as a constant and differentiate with respect to x:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy + y - 1) = 2x + 3 \cdot 1 \cdot y + 0 - 0 = 2x + 3y.$$

The value of $\partial f/\partial x$ at $(4, -5)$ is $2(4) + 3(-5) = -7$.

To find $\partial f/\partial y$, we regard x as a constant and differentiate with respect to y

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + 3xy + y - 1) = 0 + 3 \cdot x \cdot 1 + 1 - 0 = 3x + 1.$$

The value of $\partial f/\partial y$ at $(4, -5)$ is $3(4) + 1 = 13$.

DERIVATIF PARSIAL

EXAMPLE 2 Find $\partial f/\partial y$ if $f(x, y) = y \sin xy$.

Solution

We treat x as a constant and f as a product of y and $\sin xy$:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y \sin xy) = y\frac{\partial}{\partial y} \sin xy + (\sin xy)\frac{\partial}{\partial y}(y)$$

$$= (y \cos xy)\frac{\partial}{\partial y}(xy) + \sin xy = xy \cos xy + \sin xy.$$

DERIVATIF PARSIAL

EXAMPLE 3 $\boxed{\text{Find } f_x}$ if $f(x, y) = \dfrac{2y}{y + \cos x}$.

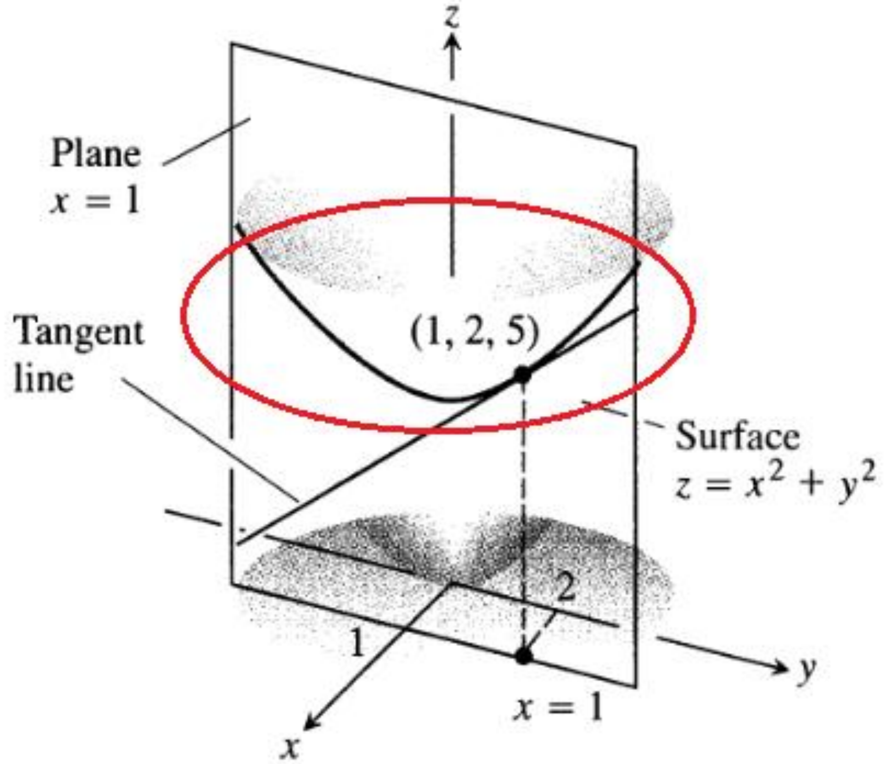
Solution

We treat f as a quotient. With y held constant, we get

$$f_x = \frac{\partial}{\partial x}\left(\frac{2y}{y + \cos x}\right) = \frac{(y + \cos x)\dfrac{\partial}{\partial x}(2y) - 2y\dfrac{\partial}{\partial x}(y + \cos x)}{(y + \cos x)^2}$$

$$= \frac{(y + \cos x)(0) - 2y(-\sin x)}{(y + \cos x)^2} = \boxed{\frac{2y \sin x}{(y + \cos x)^2}}.$$

EXAMPLE 4 The plane $x = 1$ intersects the paraboloid $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at (1, 2, 5) (Fig. 12.16).



DERIVATIF PARSIAL

Solution The slope is the value of the partial derivative $\partial z/\partial y$ at (1, 2):

$$\left.\frac{\partial z}{\partial y}\right|_{(1,2)} = \left.\frac{\partial}{\partial y}(x^2 + y^2)\right|_{(1,2)} = \left.2y\right|_{(1,2)} = 2(2) = 4.$$

As a check, we can treat the parabola as the graph of the single-variable function $z = (1)^2 + y^2 = 1 + y^2$ in the plane $x = 1$ and ask for the slope at $y = 2$. The slope, calculated now as an ordinary derivative, is

$$\left.\frac{dz}{dy}\right|_{y=2} = \left.\frac{d}{dy}(1 + y^2)\right|_{y=2} = \left.2y\right|_{y=2} \boxed{= 4.}$$

❑

DERIVATIF PARSIAL

EXAMPLE 5 Find $\partial z/\partial x$ if the equation

$$yz - \ln z = x + y$$

defines z as a function of the two independent variables x and y and the partial derivative exists.

Solution treating z as a differentiable function of x:

$$\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x}\ln z = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x}$$

$$y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1 + 0$$

$$\left(y - \frac{1}{z}\right)\frac{\partial z}{\partial x} = 1$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{z}{yz - 1}.}$$

❑

DERIVATIF PARSIAL
Order Tinggi

DERIVATIF PARSIAL ORDER DUA

When we differentiate a function $f(x, y)$ twice, we produce its second order derivatives. These derivatives are usually denoted by

$$\frac{\partial^2 f}{\partial x^2} \quad \text{or} \quad f_{xx}$$

$$\frac{\partial^2 f}{\partial y^2} \qquad f_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} \qquad f_{yx}$$

$$\frac{\partial^2 f}{\partial y \partial x} \qquad f_{xy}$$

The defining equations are

$$\boxed{\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)}, \qquad \boxed{\frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),}$$

$\frac{\partial^2 f}{\partial x \partial y}$ Means the same thing. $f_{yx} = (f_y)_x$

EXAMPLE If $f(x, y) = x\cos y + ye^x$, then

$$\boxed{\frac{\partial f}{\partial x} = \cos y + ye^x}$$

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = ye^x.$$

Also,

$$\boxed{\frac{\partial f}{\partial y} = -x\sin y + e^x}$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -x\cos y.$$

Theorem 2

Euler's Theorem (The Mixed Derivative Theorem)

If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

In Exercises 47–50, verify that $w_{xy} = w_{yx}$.

47. $w = \ln(2x + 3y)$

48. $w = e^x + x \ln y + y \ln x$

49. $w = xy^2 + x^2y^3 + x^3y^4$

50. $w = x \sin y + y \sin x + xy$

Derivatif parsial :

1. Aturan Rantai
2. Derivatif implisit

The Chain Rule for Functions of Three Variables

Chain Rule for Functions of Three Independent Variables

If $w = f(x, y, z)$ is differentiable and x, y, and z are differentiable functions of t, then w is a differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}. \tag{4}$$

Find dw/dt if

$$w = xy + z, \qquad x = \cos t, \qquad y = \sin t, \qquad z = t$$

Solution

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \\
&= (y)(-\sin t) + (x)(\cos t) + (1)(1) \\
&= (\sin t)(-\sin t) + (\cos t)(\cos t) + 1 \\
&= -\sin^2 t + \cos^2 t + 1 = 1 + \cos 2t
\end{aligned}$$

Diferensiasi Implisit

We suppose that is given implicitly as a function by $z = f(x, y)$ an equation of the $F(x, y, z) = 0$ form . This means $F(x, y, f(x, y))$ that for all $f(x, y)$ in the domain of $f(.,)$.

$$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}}$$

Diferensiasi Implisit : Contoh

Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

Let $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1$. Then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2 + 6xz}{3z^2 + 6xy} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

PROBLEM OPTIMASI : MAKS / MIN dengan DERIVATIF PARSIAL

APLIKASI MAX/MIN

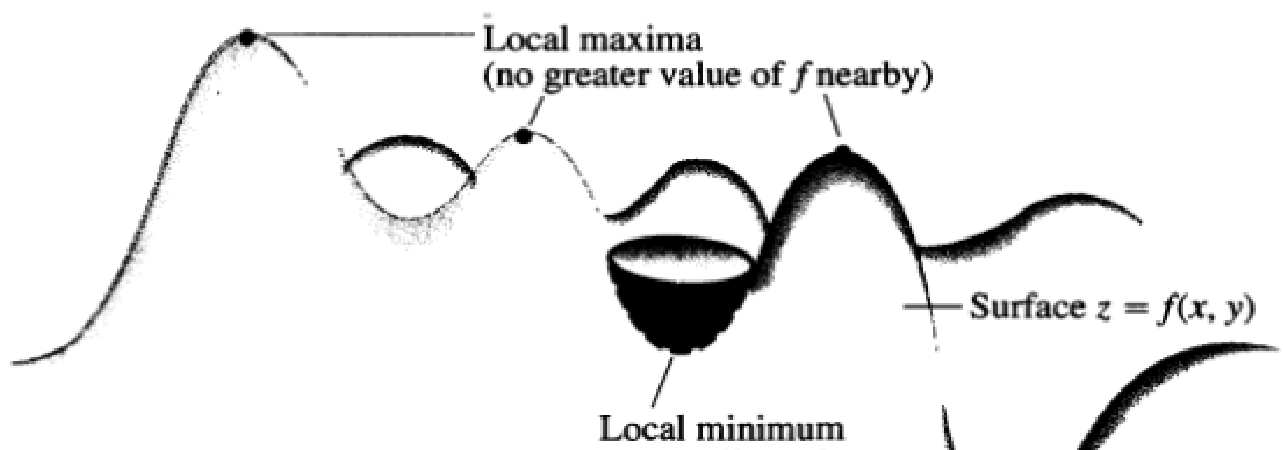
Definitions

Let $f(x, y)$ be defined on a region R containing the point (a, b). Then

1. $f(a, b)$ is a **local maximum** value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).
2. $f(a, b)$ is a **local minimum** value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).

First Derivative Test for Local Extreme Values

If $f(x, y)$ has a local maximum or minimum value at an interior point (a, b) of its domain, and if the first partial derivatives exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.



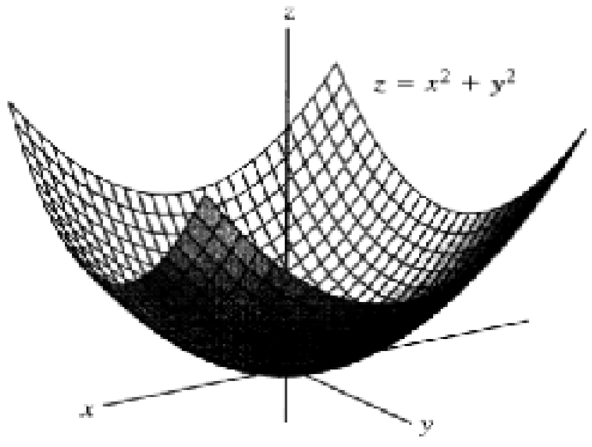
APLIKASI MAX/MIN

EXAMPLE Find the local extreme values of $f(x, y) = x^2 + y^2$.

Solution The domain of f is the entire plane (so there are no boundary points) and the partial derivatives $f_x = 2x$ and $f_y = 2y$ exist everywhere. Therefore, local extreme values can occur only where

$$f_x = 2x = 0 \qquad \text{and} \qquad f_y = 2y = 0.$$

The only possibility is the origin, where the value of f is zero. Since f is never negative, we see that the origin gives a local minimum (Fig. 12.49).



APLIKASI MAX/MIN

If $f(x, y)$ has continuous second partial derivatives in a neighborhood of a critical point (x_0, y_0), and if a number D (called the discriminant) is defined by

$$\boxed{D = f_{xx}(x_0, y_0)\, f_{yy}(x_0, y_0) - [\, f_{xy}(x_0, y_0)]^2,}$$

then (x_0, y_0) is

(i) *a maximum point if $D > 0$ and $f_{xx}(x_0, y_0) < 0$;*
(ii) *a minimum point if $D > 0$ and $f_{xx}(x_0, y_0) > 0$;*
(iii) *a saddle point if $D < 0$.*

Further, if $D = 0$, then no conclusion can be drawn, and any of the behaviors described in (i) *to* (iii) *can occur.*

APLIKASI MAX/MIN

Example Find the critical points of the function

$$z = 3x^2 + 2xy + y^2 + 10x + 2y + 1,$$

and use the second derivative test to classify them.

Solution Here we have

$$\frac{\partial z}{\partial x} = 6x + 2y + 10 = 0 \qquad \text{and} \qquad \frac{\partial z}{\partial y} = 2x + 2y + 2 = 0,$$

so the system of equations we must solve is

$$3x + y = -5,$$
$$x + y = -1.$$

By simple manipulations we easily see that $x = -2$ and $y = 1$, so there is a single critical point $(-2, 1)$. At this point we have

$$D = z_{xx}z_{yy} - {z_{xy}}^2 = 6 \cdot 2 - 2^2 = 8 > 0,$$

and since $z_{xx} = 6 > 0$, the critical point is a minimum point.

Contoh 2 :Larson and Edward

A rectangular box is resting on the *xy*-plane with one vertex at the origin. The opposite vertex lies in the plane $6x + 4y + 3z = 24$, as shown in Figure 13.73. **Find the maximum volume of the box**.

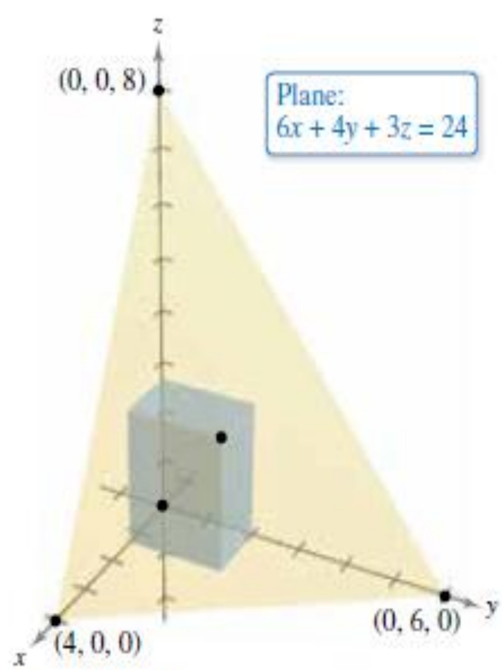


Figure 13.73

Misalkan x, y, dan z merepresentasikan Panjang, lebar, dan tinggi kotak. Volume kotak

$$V(x, y, z) = xyz$$

Karena $6x + 4y + 3z = 24$ maka

$$z = \frac{1}{3}(-6x - 4y + 24)$$

Sehingga

$$V(x, y, z) = xy\left(\frac{1}{3}(-6x - 4y + 24)\right)$$

$$V(x, y) = (24xy - 6x^2y - 4xy^2)/3$$

Contoh 2 :Larson and Edward (Lanjutan....)

Karena

$$V(x,y) = \frac{1}{3}(24xy - 6x^2y - 4xy^2)$$

maka

$$V_x(x,y) = \frac{1}{3}(24y - 12xy - 4y^2)$$

$$V_y(x,y) = \frac{1}{3}(24x - 6x^2 - 8xy)$$

Titik stasionernya adalah

$$(0,0), (4,0), (0,6), \left(\frac{4}{3}, 2\right)$$

Ujilah nilai D dan $V_{xx}(x,y) = -4y$ untuk titik stasioner $\left(\frac{4}{3}, 2\right)$.

Didapat $D = \frac{64}{3} > 0$ dan $V_{xx}\left(\frac{4}{3}, 2\right) = -8 < 0$. Volumenya

$$Max\ V = V\left(\frac{4}{3}, 2\right) = \frac{64}{9}$$

PROBLEM OPTIMASI : MAKS / MIN dengan PELIPAT LAGRANGE

PELIPAT LAGRANGE

we learned how to calculate maximum and minimum values of a function $z = f(x, y)$ *constraint* in the form of an equation

$$g(x, y) = 0.$$

The *method of Lagrange multipliers* is simply the following handy device for obtaining equations Define a function $L(x, y, \lambda)$ of the three variables x, y, and λ by

$$L(x, y, \lambda) = f(x, y) - \lambda g(x, y),$$

and observe that equations (5) are equivalent, in the same order, to the equations

$$\frac{\partial L}{\partial x} = 0, \qquad \frac{\partial L}{\partial y} = 0, \qquad \frac{\partial L}{\partial \lambda} = 0.$$

The variable λ is called the *Lagrange multiplier*.

Example Find the point on the plane $x + 2y + 3z = 6$ that is closest to the origin.

Solution We want to minimize the distance $\sqrt{x^2 + y^2 + z^2}$ subject to the constraint $x + 2y + 3z - 6 = 0$. If the distance is a minimum, its square is a mini-

Let

$$\boxed{L = x^2 + y^2 + z^2 - \lambda(x + 2y + 3z - 6).}$$

Then the equations we must solve are

$$\frac{\partial L}{\partial x} = 2x - \lambda = 0,$$

$$\frac{\partial L}{\partial y} = 2y - 2\lambda = 0,$$

$$\frac{\partial L}{\partial z} = 2z - 3\lambda = 0,$$

$$\frac{\partial L}{\partial \lambda} = -(x + 2y + 3z - 6) = 0.$$

Substituting the values for x, y, and z from the first three equations into the fourth gives

$$\tfrac{1}{2}\lambda + 2\lambda + \tfrac{9}{2}\lambda = 6 \qquad \text{or} \qquad \tfrac{14}{2}\lambda = 6 \qquad \text{or} \qquad \lambda = \tfrac{6}{7}.$$

It now follows that $x = \frac{3}{7}$, $y = \frac{6}{7}$, and $z = \frac{9}{7}$, so the desired point is $(\frac{3}{7}, \frac{6}{7}, \frac{9}{7})$.

Contoh 2 :Larson and Edward

A rectangular box is resting on the *xy*-plane with one vertex at the origin. The opposite vertex lies in the plane $6x + 4y + 3z = 24$, as shown in Figure 13.73. **Find the maximum volume of the box using Lagrange's Method**

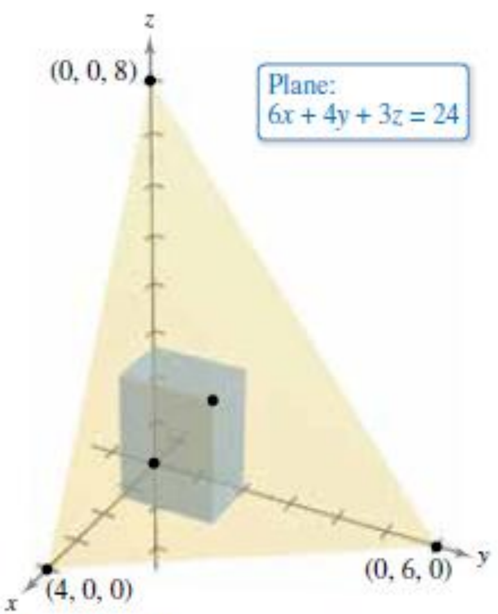


Figure 13.73