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Sistem Persamaan Linear

LINEAR EQUATION

- A **linear equation** in the variables $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \ ,$$

where *b* and the coefficients $a_1, \ldots, a_n$ are real or complex numbers that are usually known in advance.

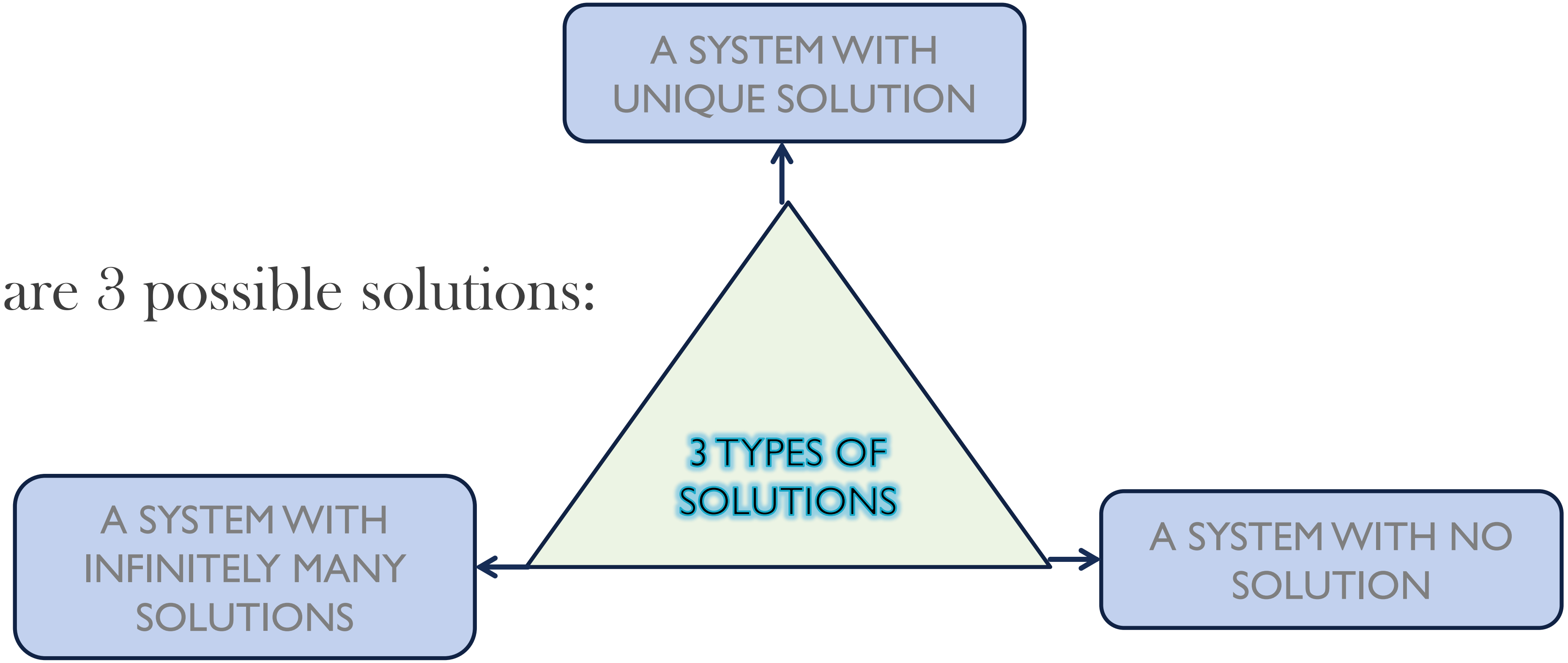
- A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same variables — say, $x_1, \ldots, x_n$.

LINEAR EQUATION

- A **solution** of the system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, \ldots, s_n$ are substituted for $x_1, \ldots, x_n$, respectively.
- The set of all possible solutions is called the **solution set** of the linear system.
- Two linear systems are called **equivalent** if they have the same solution set.

TYPES OF SOLUTIONS TO SYSTEMS OF LINEAR EQUATIONS

There are 3 possible solutions:



A SYSTEMS WITH UNIQUE SOLUTION

Consider the system:

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ 0x_1 + x_2 &= 2 \end{aligned}$$

Augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & \boxed{1} & 2 \end{array}\right]$$

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ x_1 + 2.2 &= 3 \\ x_1 + 4 &= 3 \\ x_1 &= 3 - 4 \\ &= -1 \end{aligned}$$

The system has unique solution:

$$\boxed{\underset{S2}{x_2 = 2}, \quad \underset{S1}{x_1 = -1}}$$

A SYSTEMS WITH INFINITELY MANY SOLUTION

Consider the system:

$$x_1 + 2x_2 = 3$$
$$0x_1 + 0x_2 = 0$$

$x_1 + 2s = 3$

$x_1 = 3 - 2s$

$x_2 = s$

Augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

The system has many solutions: let $x_2 = s$, where s is called a free variable.

Then,

$$x_1 = 3 - 2s,$$

A SYSTEMS WITH NO SOLUTION

- Consider the system: $$\begin{aligned} x_1 + 2x_2 &= 3 \\ 0x_1 + 0x_2 &= 2 \end{aligned}$$
- Augmented matrix: $$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 2 \end{array}\right]$$
- The system has no solution, since coefficient of x_2 is '0'.

$$\left(0x_2 \neq 2\right)$$

SOLVING SYSTEMS OF EQUATIONS

Systems of linear equations :

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

$$
\left[\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & & \vdots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{array}\right]
$$

SOLVING SYSTEMS OF EQUATIONS

4 methods used to solve systems of equations.

1) The Inverse of the Coefficient Matrix
2) Gauss Elimination
3) Gauss-Jordan Elimination
4) Cramer’s Rule

THE INVERSE OF THE COEFFICIENT MATRIX

SOLVING SYSTEMS OF EQUATIONS

Matrix Form: $\boldsymbol{AX = B}$

$A^{-1} A . X = A^{-1} . B$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} . x_n &= b_1 \\ a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} . x_n &= b_2 \\ &\vdots \\ a_{m1} . x_1 + a_{m2} . x_2 + \dots + a_{mn} . x_n &= b_n \end{aligned}$$

To find $\boldsymbol{X}$: $\boxed{\boldsymbol{X = A^{-1} B}}$

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THE INVERSE OF THE COEFFICIENT MATRIX

Method : $\boxed{X = A^{-1} B}$

Example:

Solve the system by using A^{-1}, the inverse of the coefficient matrix:

$$\begin{aligned} x + y + z &= 0 \\ 2x - y + z &= -1 \\ -x + 3y - z &= -8 \end{aligned}$$

$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ -1 & 3 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ -1 \\ -8 \end{bmatrix}$

A^{-1} ?

THE INVERSE OF THE COEFFICIENT MATRIX

Solution:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ -1 & 3 & -1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ -1 \\ -8 \end{bmatrix}$$

$$AX = B:$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ -1 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -8 \end{bmatrix}$$

THE INVERSE OF THE COEFFICIENT MATRIX

Find A^{-1} : $\boxed{A^{-1} = \frac{1}{|A|} adj[A]}$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 4 & 3 & -1 \end{bmatrix}$$

$|A| = a_{11} . A_{11} + a_{12} A_{12} + a_{13} . A_{13}$

$= 1.(-2) + 1.1 + 1.5$

$= -2 + 1 + 5 = 4$

Cofactor of A : $\begin{bmatrix} -2 & 1 & 5 \\ 4 & 0 & -4 \\ 2 & 1 & -3 \end{bmatrix}$

$A_{11} = (-1)^{1+1} \begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} = (-1)^2 ((-1).(-1) - (3.1))$

$= 1(1-3) = -2$

$A_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} = (-1)^3 (1.(-1) - (3.1))$

$= (-1)(-1-3) = (-1)(-4)$

$= 4$

Therefore:

$$adj[A] = \begin{bmatrix} -2 & 4 & 2 \\ 1 & 0 & 1 \\ 5 & -4 & -3 \end{bmatrix}, \quad |A| = 4$$

THE INVERSE OF THE COEFFICIENT MATRIX

Find X: $\boxed{X = A^{-1}B}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -2 & 4 & 2 \\ 1 & 0 & 1 \\ 5 & -4 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \\ -8 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -2.0 + 4.(-1) + 2(-8) \\ 1.0 + 0.(-1) + 1.(-8) \\ 5.0 + (-4)(-1) + (-3).(-8) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -20 \\ -8 \\ 28 \end{bmatrix}$$

$$= \begin{bmatrix} -5 \\ -2 \\ 7 \end{bmatrix}$$

$$\therefore x = -5, y = -2, z = 7$$

THE INVERSE OF THE COEFFICIENT MATRIX

Example 2:

Solve the system by using A^{-1}, the inverse of the coefficient matrix:

$$x - 3y + z = 1$$
$$2x - y + 2z = 2$$
$$x + 2y - 3z = -1$$

Answer : $x = \frac{1}{2},\ y = 0,\ z = \frac{1}{2}$

GAUSS ELIMINATION

GAUSS ELIMINATION

- Consider the systems of linear eq:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

GAUSS ELIMINATION

Write in augmented form : $[A|B]$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{array}\right]$$

Using ERO (ELEMENTARY ROW OPERATION), such that A may be reduce in REF (ROW ECHELON FORM)/Upper Triangular

GAUSS ELIMINATION

Example:

Solve the system by using Gauss Elimination method:

$$x + y + z = 0$$
$$2x - y + z = -1$$
$$-x + 3y - z = -8$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & -1 & 1 & -1 \\ -1 & 3 & -1 & -8 \end{array}\right]$$

GAUSS ELIMINATION

Solution:

Write in augmented form:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & -1 & 1 & -1 \\ -1 & 3 & -1 & -8 \end{array}\right]$$

Reduce to REF : (Diagonal = 1)

$2+(-2).1 \quad (-1)+(-2).1 \quad 1+(-2).1 \quad (-1)+(-2).0$

$4+(-4) \quad 0+(-1)\frac{4}{3}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & -1 & 1 & -1 \\ -1 & 3 & -1 & -8 \end{array}\right] \underset{H_{31(1)}}{\overset{H_{21(-2)}}{\sim}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & -1 & -1 \\ 0 & 4 & 0 & -8 \end{array}\right] \overset{H_{32(\frac{4}{3})}}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & -1 & -1 \\ 0 & 0 & -\frac{4}{3} & -\frac{28}{3} \end{array}\right] \underset{H_{3\left(-\frac{3}{4}\right)}}{\overset{H_{2(-\frac{1}{3})}}{\sim}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 7 \end{array}\right]$$

$-8+(-1)(\frac{4}{3})$

$\therefore \boxed{z = 7}$

$\boxed{y + \frac{1}{3} z = \frac{1}{3}}$

$y + \frac{1}{3}.7 = \frac{1}{3}$

$y = \frac{1}{3} - \frac{7}{3}$

$y = -2$

$\boxed{x + y + z = 0}$

$x - 2 + 7 = 0$

$x + 5 = 0$

$x = -5$ ✓

GAUSS JORDAN ELIMINATION

GAUSS JORDAN ELIMINATION

Written in augmented form : $[A|B]$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{array}\right]$$

- Using ERO (ELEMENTARY ROW OPERATION), such that A may be reduce in RREF (REDUCED ROW ECHELON FORM)/IDENTITY (***DIAGONAL = 1, OTHER ENTRIES = 0***)

Reduce to RREF : (Diagonal = 1, Other entries = 0)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & -1 & 1 & -1 \\ -1 & 3 & -1 & -8 \end{array}\right] \underset{\mathrm{H}_{31(1)}}{\overset{\mathrm{H}_{21(-2)}}{\sim}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & -1 & -1 \\ 0 & 4 & 0 & -8 \end{array}\right] \overset{H_{32(\frac{4}{3})}}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & -1 & -1 \\ 0 & 0 & -\frac{4}{3} & -\frac{28}{3} \end{array}\right] \underset{H_{3\left(-\frac{3}{4}\right)}}{\overset{H_{2(-\frac{1}{3})}}{\sim}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 7 \end{array}\right]$$

$\frac{1}{3} + (-\frac{1}{3}).1$ $\quad \frac{1}{3} + (-\frac{1}{3}) \cdot 7$

$$\underset{\mathrm{H}_{13(-1)}}{\overset{\mathrm{H}_{23(-\frac{1}{3})}}{\sim}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & -7 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 7 \end{array}\right] \overset{H_{12(-1)}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 7 \end{array}\right]$$

$1 + (-1).1 = 0$

$-7 + (-1)(-2) = -5$

$1 + (-1).1$ $\quad 0 + (-1).7$

$z = 7$ $\quad y = -2$ $\quad x = -5$

Example 2:

Solve the system by Gauss elimination and Gauss Jordan Elimination.

$$
\begin{aligned}
x-3y+z&=1\\
2x-y+2z&=2\\
x+2y-3z&=-1
\end{aligned}
$$

Answer : $x=\frac{1}{2},\ y=0,\ z=\frac{1}{2}$

GAUSS JORDAN ELIMINATION

Example 4:

Solve the system by Gauss Jordan elimination.

$$x + y + z = 4$$
$$x - y - z = 0$$
$$x - y + z = 0$$

Answer : $x = 2,\ y = 2,\ z = 0$

CRAMMER RULES

CRAMER'S RULE

Theorem 5

Cramer's Rule for 3x3 system

Given the system:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_1 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_2 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\}$$

with :

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$$

CRAMER'S RULE

If :

$$D_{x_1} = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} \quad D_{x_2} = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix} \quad D_{x_3} = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

Then :

$$x_1 = x = \frac{D_{x_1}}{D} \quad x_2 = y = \frac{D_{x_2}}{D} \quad x_3 = z = \frac{D_{x_3}}{D}$$

CRAMER'S RULE

$$D_{x_1} = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} \quad D_{x_2} = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix} \quad D_{x_3} = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

Step to solve n linear inhomogeneous equations in n variables proceed as follows:

i. Compute $|D|$, the determinant of the coefficient matrix, and, if $|D| \neq 0$, proceed to the next step.

ii. Compute the modified coefficient determinants $|D_i|, i = 1, 2, ..., n$ where D_i is derived from D by replacing the i-th column of D by the inhomogeneous vector B;

iii. The solutions $x_1, x_2, ..., x_n$ are given by $x_i = \dfrac{|D_i|}{|D|}$ for $i = 1, 2, ..., n$.

iv. If $|D| = 0$ the Cramer's rule cannot be applied. In such a case, either a unique solution to the system does not exist or there is no solution.

CRAMER'S RULE

Example 5:

Solve the system by using the Cramer's Rule.

$$
\begin{aligned}
x + y + z &= 4 \\
x - y - z &= 0 \\
x - y + z &= 0
\end{aligned}
$$

CRAMER'S RULE

Solution

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$|D| = a_{11} \cdot A_{11} + a_{12} \cdot A_{12} + a_{13} \cdot A_{13}$

$(-1)^{1+1} \begin{vmatrix} -1 & -1 \\ -1 & 1 \end{vmatrix}$

$(-1)^{1+2} \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$

$(-1)^{1+3} \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix}$

Determinant of A :

$$|A| = D = 1\begin{vmatrix} -1 & -1 \\ -1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} = -4$$

$-1-(+1) = -2 \qquad -(1-(-1)) = -2 \qquad -1-(-1) = 0$

CRAMER'S RULE

$$D_x = \begin{vmatrix} 4 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & 1 \end{vmatrix} = 4\begin{vmatrix} -1 & -1 \\ -1 & 1 \end{vmatrix} = -8 \quad \rightarrow \quad x = \frac{-8}{-4} = 2$$

$$D_y = \begin{vmatrix} 1 & 4 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{vmatrix} = -4\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = -8 \quad \rightarrow \quad y = \frac{-8}{-4} = 2$$

$$D_x = \begin{vmatrix} 1 & 1 & 4 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \end{vmatrix} = 4\begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} = 0 \quad \rightarrow \quad z = \frac{0}{-4} = 0$$

CRAMER'S RULE

Example 6:

Solve the system by using Cramer's Rule.

$$x + 2y = z - 1$$
$$x = 4 + y - z$$
$$x + y - 3z = -2$$

Answer :

$$x = 2,\ y = -1,\ z = 1$$