

CS4102 Algorithms

Spring 2021 – Floryan and Horton

Module 3, Day 6: Last DP Algorithms and
(1) Gerrymandering (not in the textbook)

Gerrymandering

- Manipulating electoral district boundaries to favor one political party over others
- Coined in an 1812 Political cartoon
- Governor **Elbridge Gerry** signed a bill that redistricted Massachusetts to benefit his Democratic-Republican Party



The Gerrymander

SUPREME COURT OF THE UNITED STATES

Syllabus

VIRGINIA HOUSE OF DELEGATES ET AL. v. BIRMINGHAM

APPEAL FROM

No. 18-2

After the 2010 Census, State's Senate districts suddenly gerrymandered. Equal Protection (collectively) brought suit, arguing that the bench took into account where a third party's unconstitutional interests for the General Attorney General to this Court. Held: The House of Representatives or in its

SUPREME COURT OF THE UNITED STATES

Syllabus

NBC NEWS

North Carolina judges slam GOP gerrymandering in stinging ruling, reject district maps

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POLITICS NEWS

North Carolina gerrymanders district maps

The ruling found state constitution.

LAW

Supreme Court Rules Partisan Gerrymandering Is Beyond The Reach Of Federal Courts

June 27, 2019 · 10:17 AM ET

Heard on [All Things Considered](#)



Sept. 3, 2019, 7:36 PM EDT / Updated
By Jane C. Timm

A panel of three judges in North Carolina rejected legislative district maps on an extreme partisan gerrymandering claim. The constitution.

4-Minute Listen

[+ PLAYLIST](#) [d](#) [r](#) [e](#) [m](#)



According to the Supreme Court

- Gerrymandering cannot be used to:
 - Disadvantage racial/ethnic/religious groups
- It can be used to:
 - Disadvantage political parties

SUPREME COURT OF THE UNITED STATES

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VIRGINIA HOUSE OF DELEGATES ET AL. v.
BETHUNE-HILL ET AL.

APPEAL FROM THE UNITED STATES DISTRICT COURT FOR THE
EASTERN DISTRICT OF VIRGINIA

No. 18-281. Argued March 18, 2019—Decided June 17, 2019

After the 2010 census, Virginia redrew legislative districts for the State's Senate and House of Delegates. Voters in 12 impacted House districts sued two state agencies and four election officials (collectively, State Defendants), charging that the redrawn districts were racially gerrymandered in violation of the Fourteenth Amendment's Equal Protection Clause. The House of Delegates and its Speaker (collectively, the House) intervened as defendants, participating in the bench trial, on appeal to this Court, and at a second bench trial, where a three-judge District Court held that 11 of the districts were unconstitutionally drawn, enjoined Virginia from conducting elections for those districts before adoption of a new plan, and gave the General Assembly several months to adopt that plan. Virginia's Attorney General announced that the State would not pursue an appeal to this Court. The House, however, did file an appeal.

Held: The House lacks standing, either to represent the State's interests or in its own right. Pp. 3–12.

SUPREME COURT OF THE UNITED STATES

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RUCHO ET AL. v. COMMON CAUSE ET AL.

APPEAL FROM THE UNITED STATES DISTRICT COURT FOR THE
MIDDLE DISTRICT OF NORTH CAROLINA

No. 18-422. Argued March 26, 2019—Decided June 27, 2019*

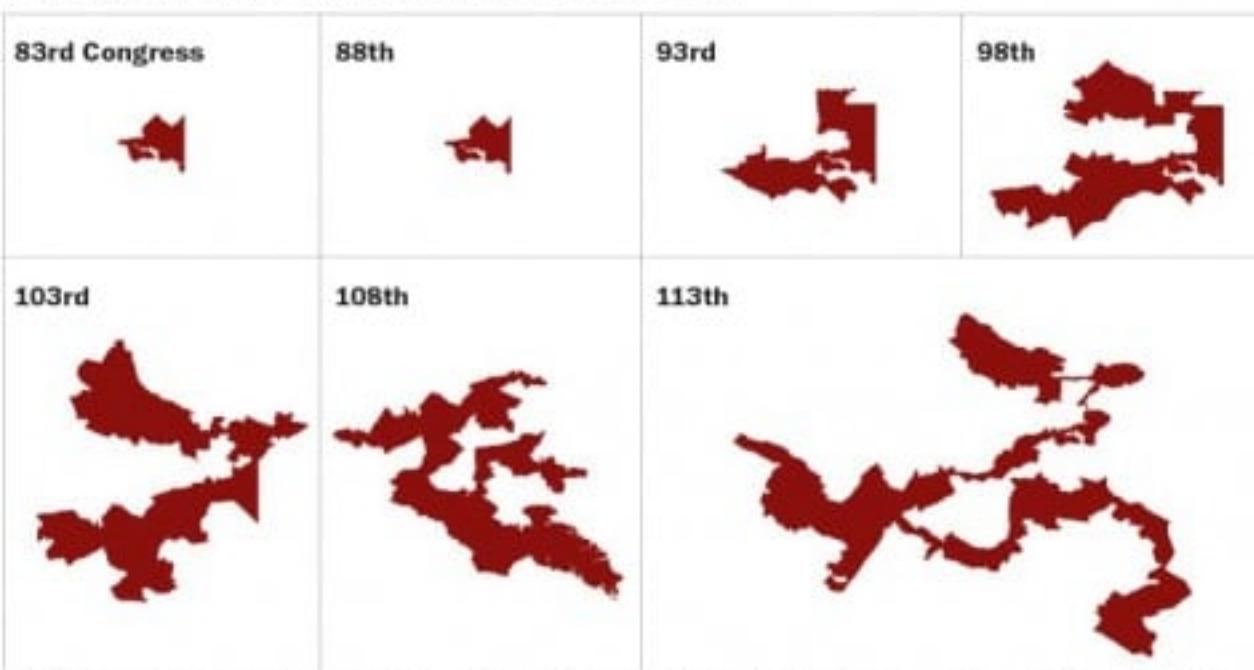
Voters and other plaintiffs in North Carolina and Maryland filed suits challenging their States' congressional districting maps as unconstitutional partisan gerrymanders. The North Carolina plaintiffs claimed that the State's districting plan discriminated against Democrats, while the Maryland plaintiffs claimed that their State's plan discriminated against Republicans. The plaintiffs alleged violations of the First Amendment, the Equal Protection Clause of the Fourteenth Amendment, the Elections Clause, and Article I, §2. The District Courts in both cases ruled in favor of the plaintiffs, and the defendants appealed directly to this Court.

Held: Partisan gerrymandering claims present political questions beyond the reach of the federal courts. Pp. 6–34.

(a) In these cases, the Court is asked to decide an important question of constitutional law. Before it does so, the Court "must find that the question is presented in a 'case' or 'controversy' that is . . . 'of a Judiciary Nature.'" *DaimlerChrysler Corp. v. Cuno*, 547 U. S. 332, 342. While it is "the province and duty of the judicial department to

Gerrymandering Today

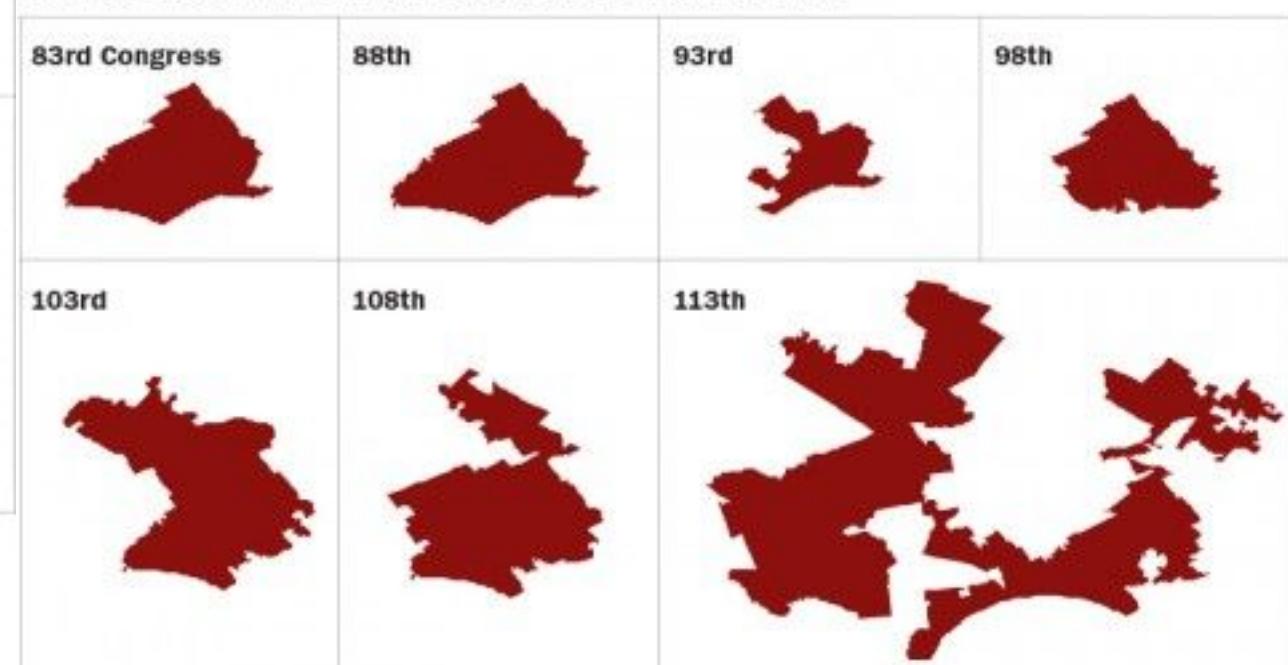
THE EVOLUTION OF MARYLAND'S THIRD DISTRICT



SOURCE: Shapefiles maintained by Jeffrey B. Lewis, Brandon DeVine, Lincoln Pitcher and Kenneth C. Martis, UCLA.
Drawn to scale.

GRAPHIC: The Washington Post. Published May 20, 2014

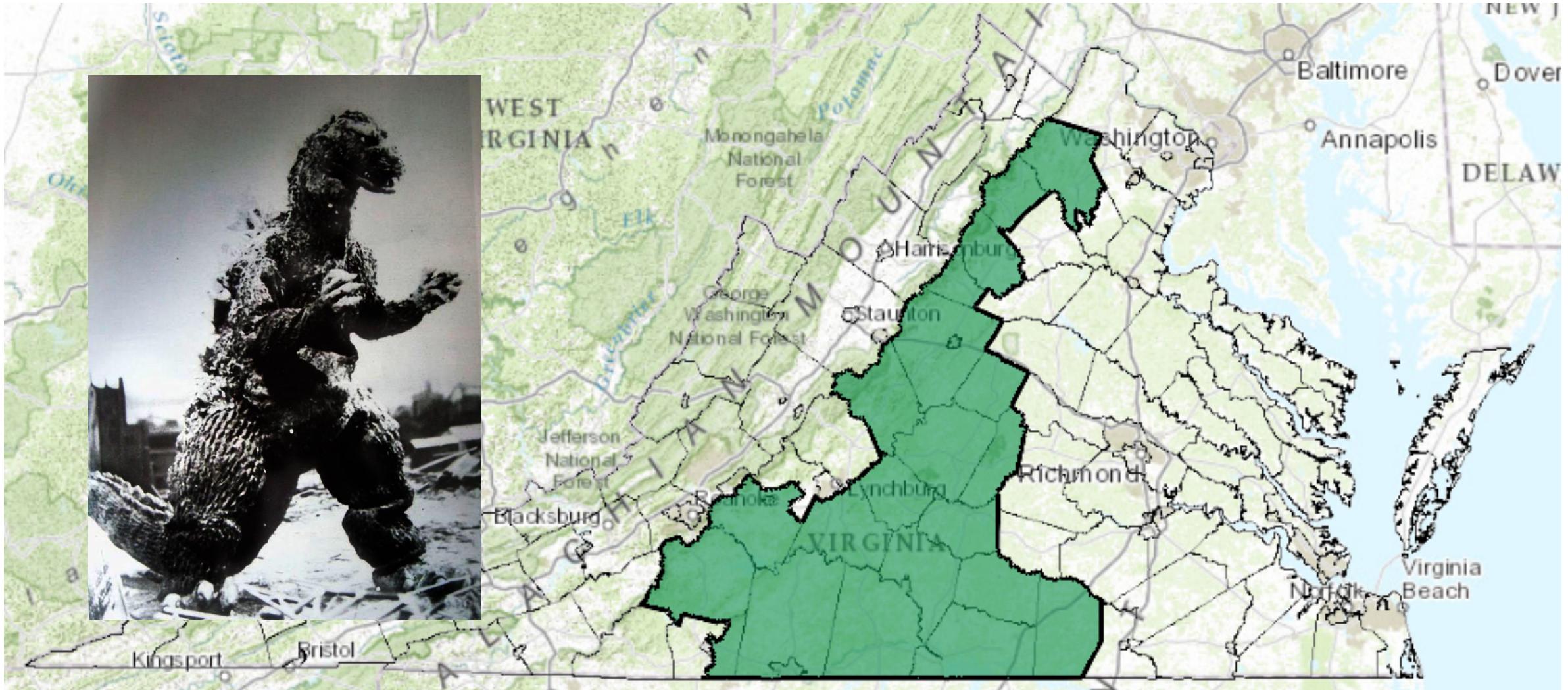
THE EVOLUTION OF PENNSYLVANIA'S SEVENTH DISTRICT



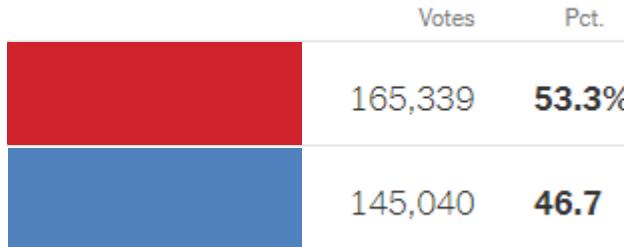
SOURCE: Shapefiles maintained by Jeffrey B. Lewis, Brandon DeVine, Lincoln Pitcher and Kenneth C. Martis, UCLA.
Drawn to scale.

GRAPHIC: The Washington Post. Published May 20, 2014

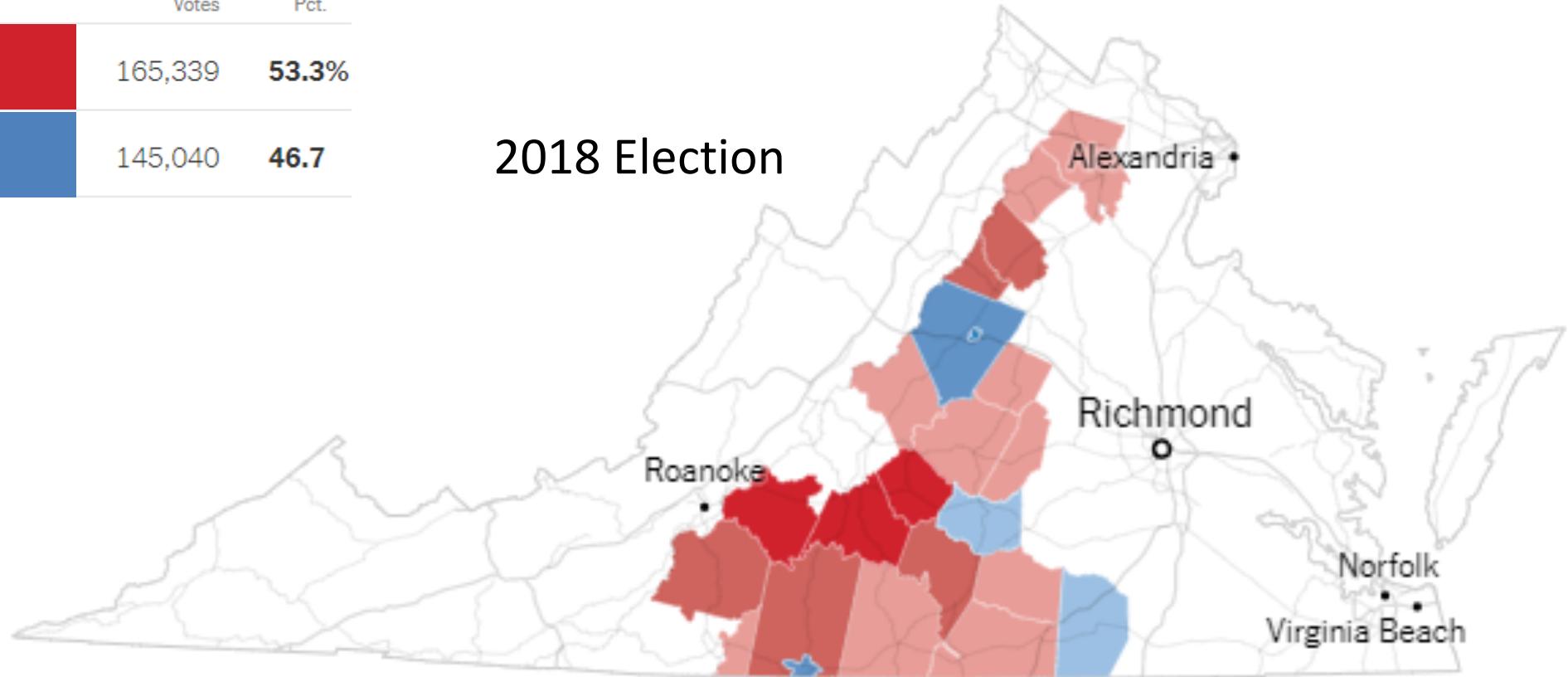
VA 5th Congressional District (2020)



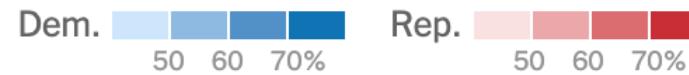
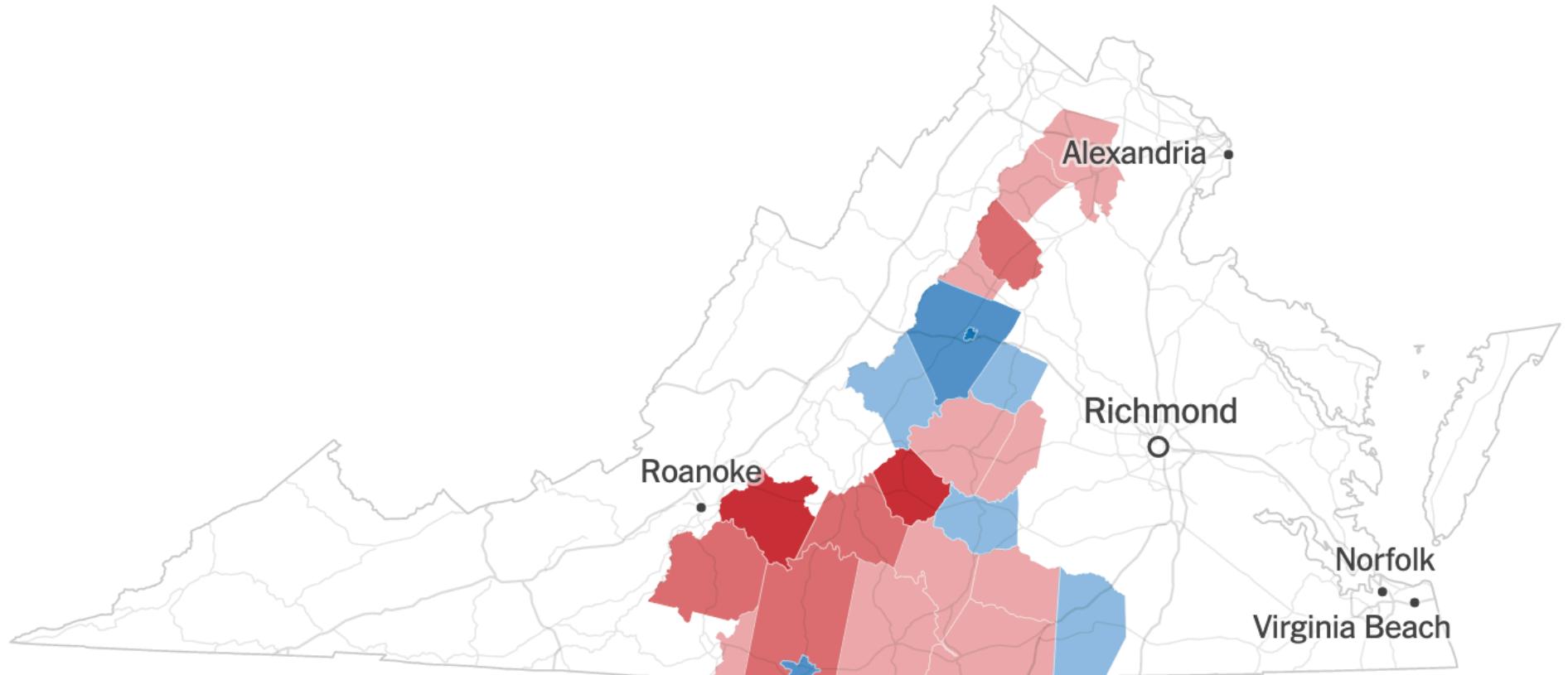
VA 5th District



2018 Election



5th District 2020 HR Results



Some Thinking Before 2012 VA Congressional Redistricting

Redistricting commission proposal

The bipartisan panel created by the governor cited four measures as guides for its recommendations: Voting Rights Act considerations, equal population, compactness, and municipal and county boundary lines. This is one of three options it endorsed for new boundaries on Virginia's 11 congressional districts.



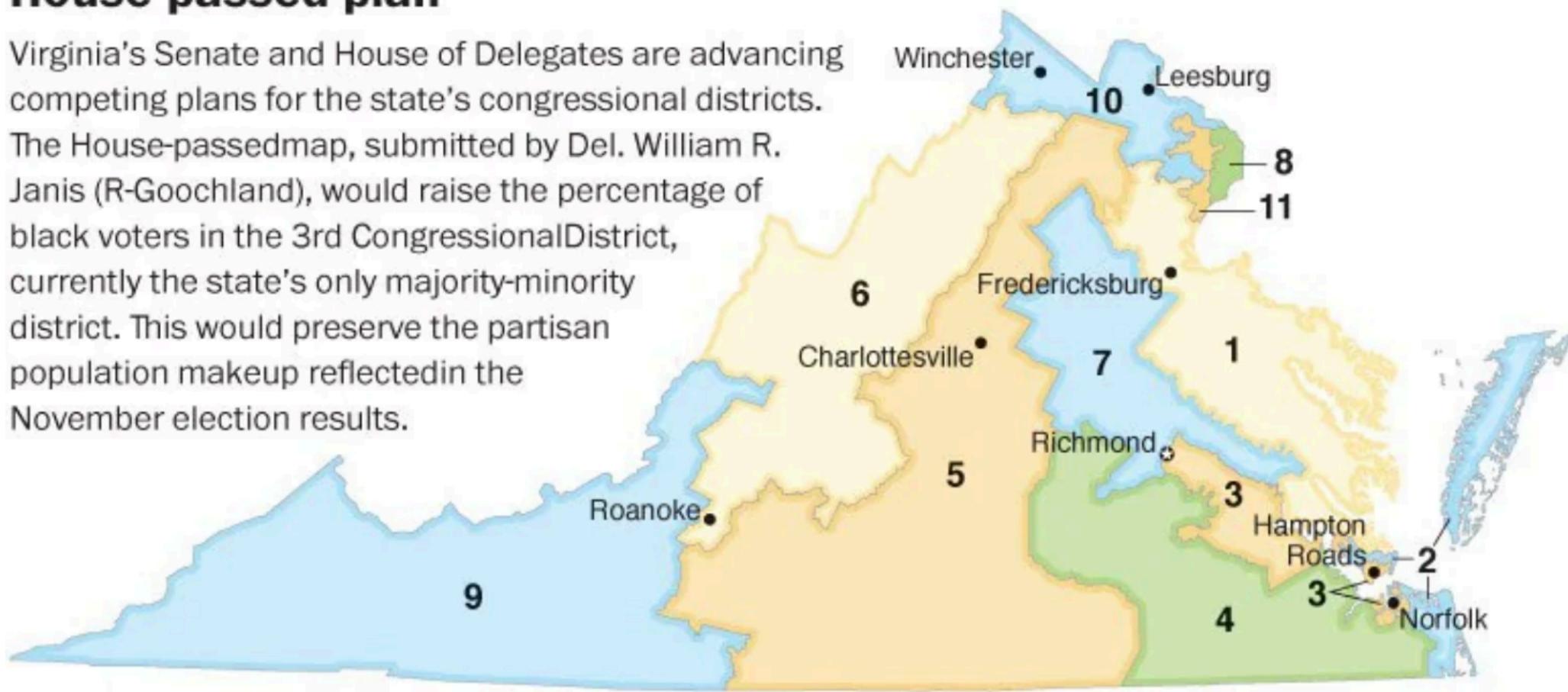
From: "Incumbents, not voters, shaping Virginia's congressional districts." Washington Post, 2011

Political Reality and 2012 VA Congressional Redistricting

House-passed plan

Virginia's Senate and House of Delegates are advancing competing plans for the state's congressional districts.

The House-passed map, submitted by Del. William R. Janis (R-Goochland), would raise the percentage of black voters in the 3rd Congressional District, currently the state's only majority-minority district. This would preserve the partisan population makeup reflected in the November election results.

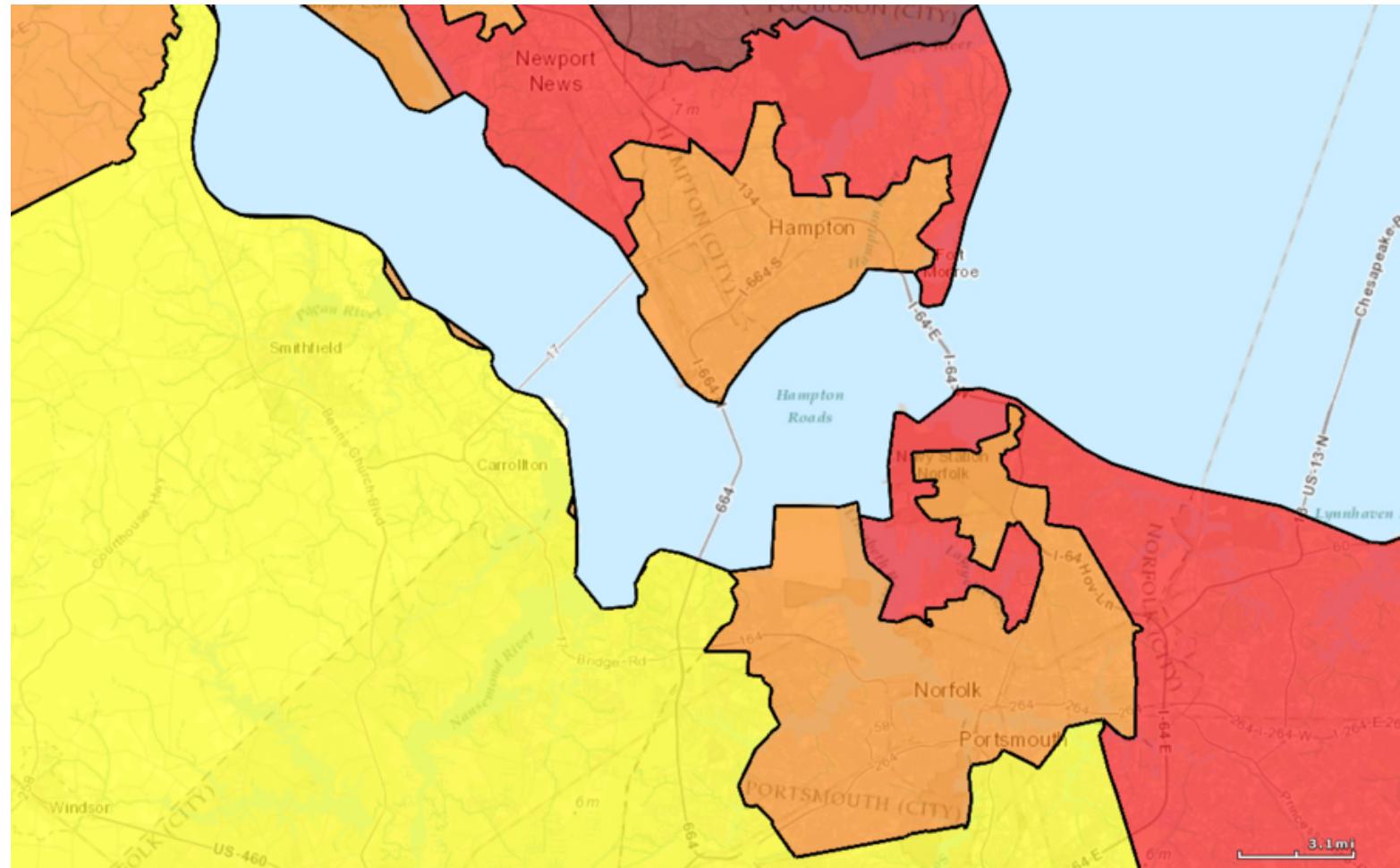


Seriously?

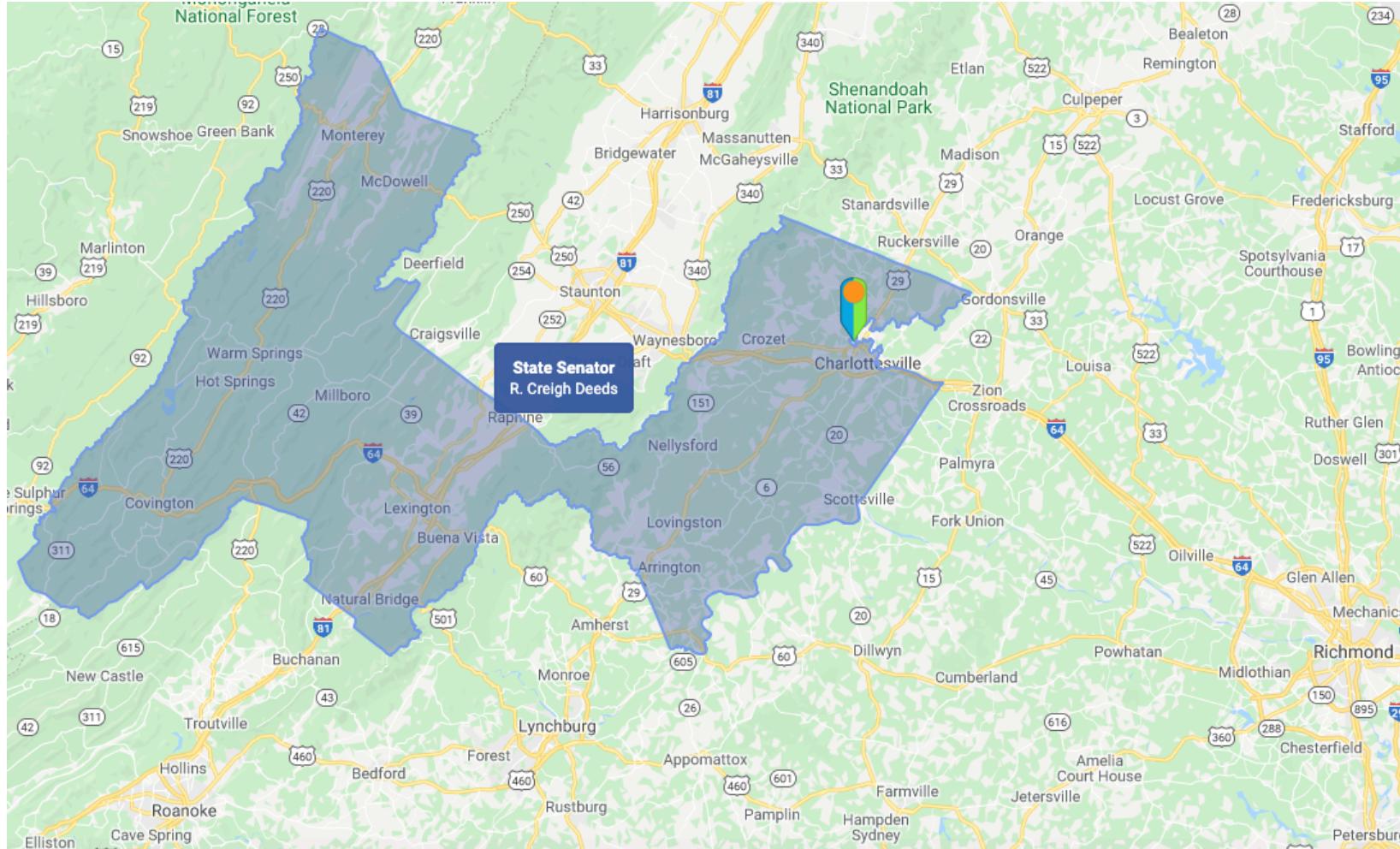
- Computers make it really effective!
 - Close-up on part of 3rd District
 - This was 2013-2017; court ordered it changed
- Virginia will do redistricting soon under a new system

Learn More about VA redistricting:

<https://www.vpap.org/redistricting/>
<https://redistricting.dls.virginia.gov/>

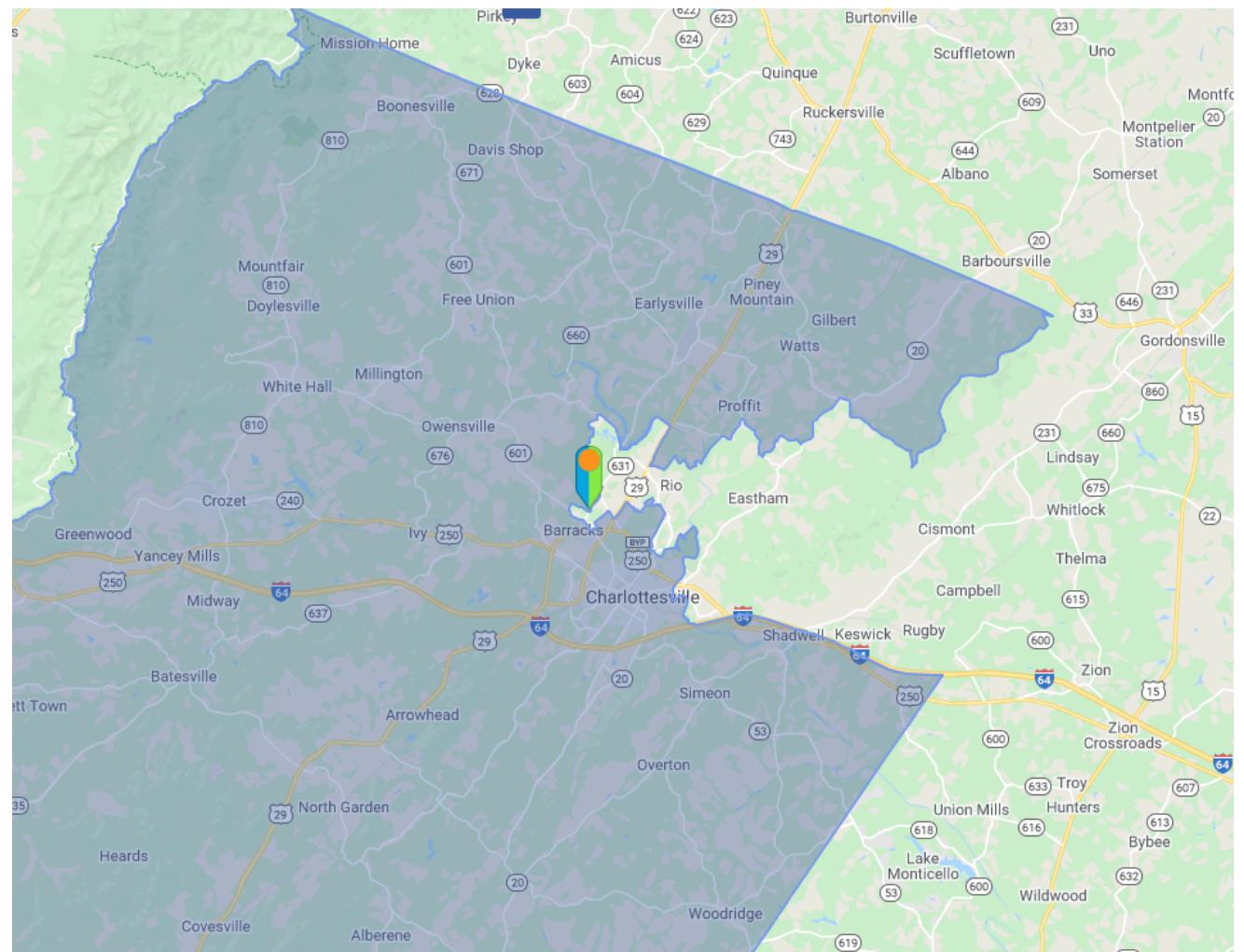
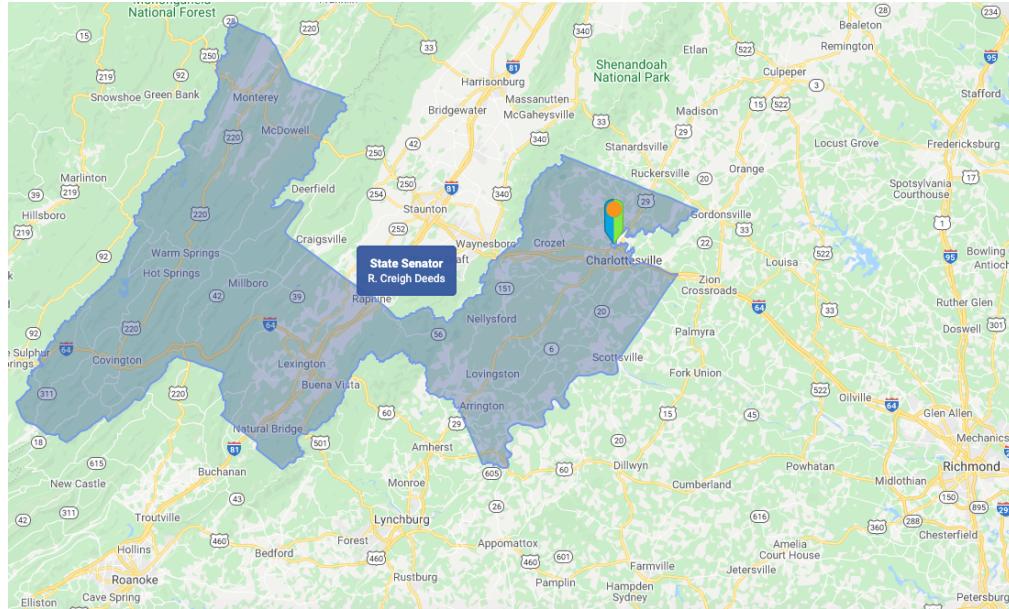


VA State Senate District 25 (2020)

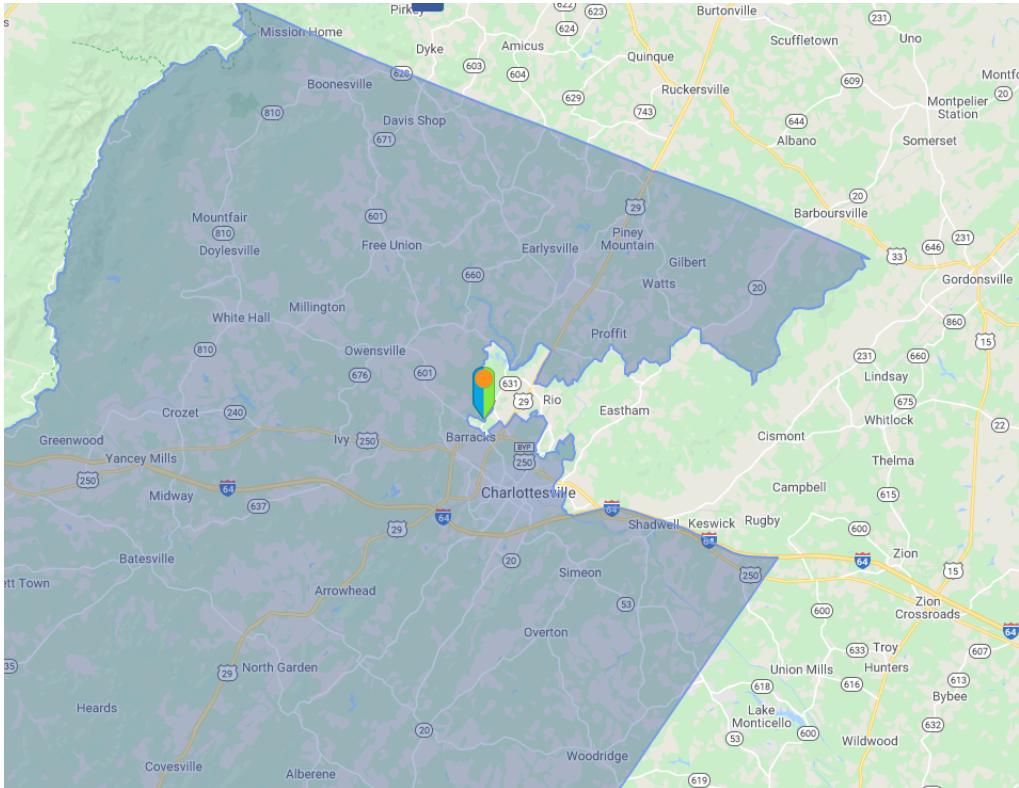


My house marked!

VA State Senate District 25 (2020)



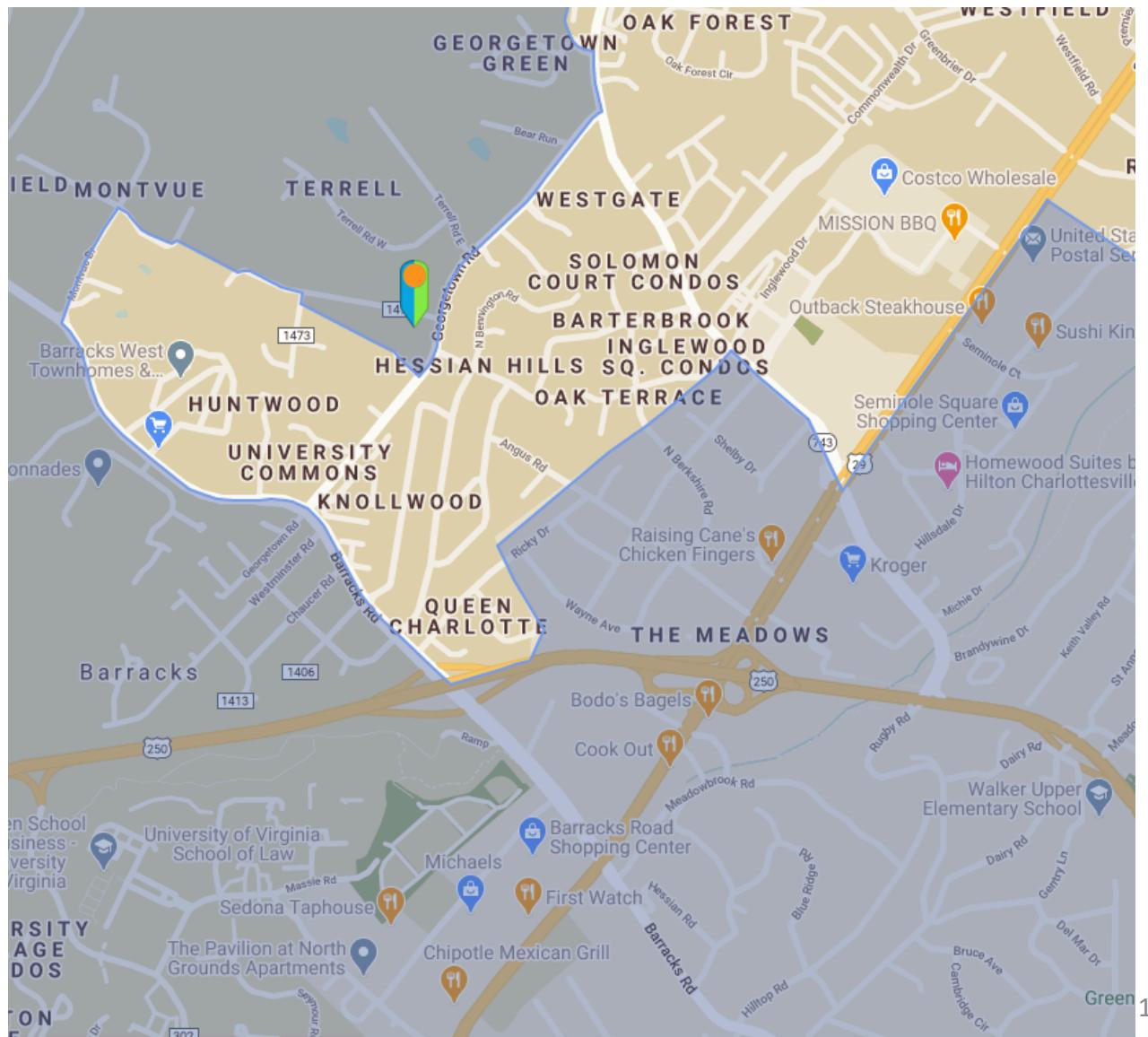
VA State Senate District 25 (2020)



Learn More about VA redistricting:

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<https://redistricting.dls.virginia.gov/>



An Algorithm to Gerrymander

- States are broken into precincts
- All precincts have the same size
- We know the votes for 2 parties in each precinct
- Group precincts into districts to maximize the number of districts won by my party

It's really a bit more complicated than this...

Overall: R:217 D:183



The “Regular” Party



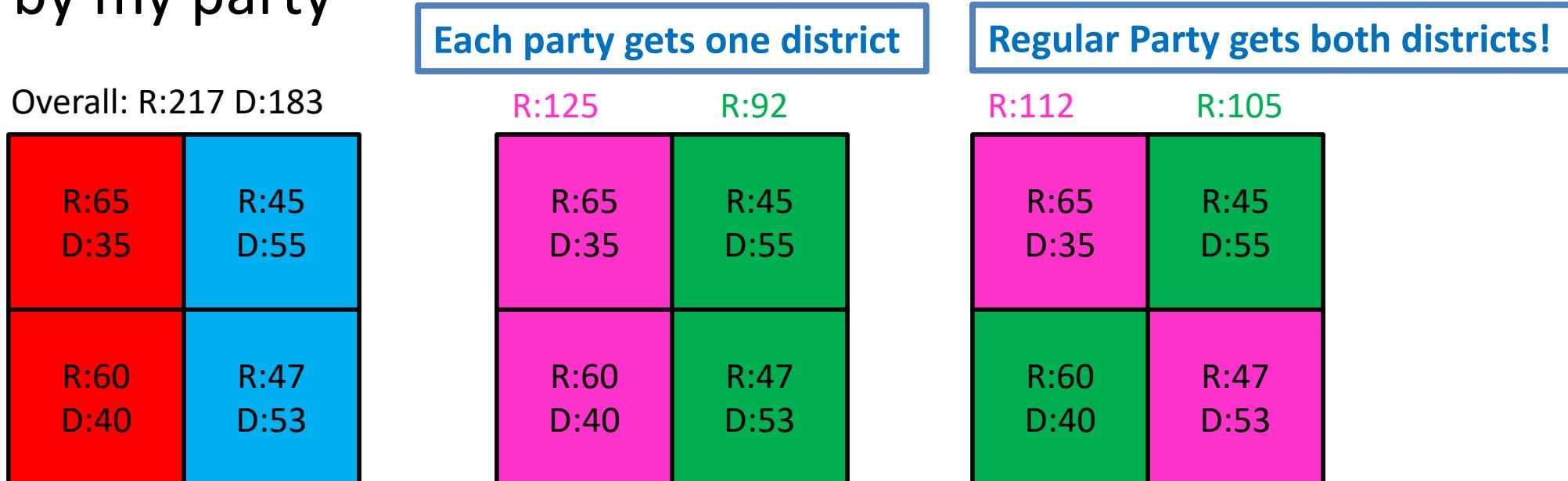
The “Diet” Party

VS



How does it work?

- States are broken into precincts
- All precincts have the same size
- We know the votes for 2 parties in each precinct
- Group precincts into districts to maximize the number of districts won by my party



Gerrymandering Problem Statement

- Given:
 - A list of precincts p_1, p_2, \dots, p_n and $R(p_i)$, number of votes for “Regular Party”
 - Each precinct has exactly m voters (So mn total voters)
- Output:
 - Two districts $D_1, D_2 \subset \{p_1, p_2, \dots, p_n\}$
 - Where $|D_1| = |D_2|$
 - So exactly $\frac{mn}{2}$ votes per district
 - $R(D_1) > \frac{mn}{4}$ and $R(D_2) > \frac{mn}{4}$
 - $R(D_i)$ gives number of “Regular Party” voters in D_i
 - $R(D_i) > \frac{mn}{4}$ means D_i is majority “Regular Party”
 - “failure” if no such solution is possible

Valid Gerrymandering:
Both districts go to
Regular Party!

More than 50% of the $\frac{mn}{2}$ votes

Dynamic Programming

- Requires **Optimal Substructure**
 - Solution to larger problem contains the solutions to smaller ones
- Idea:
 1. Identify the recursive structure of the problem
 - What is the “last thing” done?
 2. Save the solution to each subproblem in memory
 3. Select a good order for solving subproblems
 - “Top Down”: Solve each recursively
 - “Bottom Up”: Iteratively solve smallest to largest

Dynamic Programming

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Consider the last precinct

After assigning the first $n - 1$ precincts

$$p_1, p_2, \dots, p_{n-1}$$

D_1
 k precincts
 x voters for R

D_2
 $n - k - 1$ precincts
 y voters for R

If we assign
 p_n to D_1

p_n

If we assign
 p_n to D_2

D_1
 $k + 1$ precincts
 $x + R(p_n)$ voters for R

Valid gerrymandering if:
 $k + 1 = \frac{n}{2}$
both $x + R(p_n), y > \frac{mn}{4}$

D_2
 $n - k$ precincts
 $y + R(p_n)$ voters for R

Valid gerrymandering if:
 $n - k = \frac{n}{2}$
both $x, y + R(p_n) > \frac{mn}{4}$

D_1
 $k + 1$ precincts
 $x + R(p_n)$ voters for R

D_2
 $n - k - 1$ precincts
 y voters for R

World One

D_1
 k precincts
 x voters for R

D_2
 $n - k$ precincts
 $y + R(p_n)$ voters for R

World Two

Define Recursive Structure

$S(j, k, x, y) = \text{True}$

$n \times n \times mn \times mn$

if from among the first j precincts:
 k are assigned to D_1
exactly x vote for R in D_1
exactly y vote for R in D_2

4D Dynamic Programming!!!

Two ways to satisfy $S(j, k, x, y)$:

D_1

$k - 1$ precincts

$x - R(p_j)$ voters for R

D_2

$j - k$ precincts

y voters for R

D_1

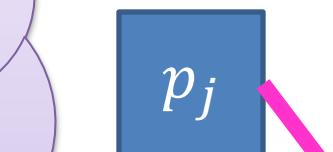
k precincts

x voters for R

D_2

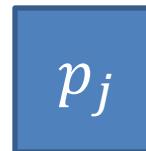
$j - 1 - k$ precincts

$y - R(p_j)$ voters for R



Then assign
 p_j to D_1

OR



Then assign
 p_j to D_2

$S(j, k, x, y) = \text{True if:}$

from among the first j precincts

k are assigned to D_1

exactly x vote for R in D_1

exactly y vote for R in D_2

$$S(j, k, x, y) = S(j - 1, k - 1, x - R(p_j), y) \vee S(j - 1, k, x, y - R(p_j))$$

Final Algorithm

$$S(j, k, x, y) = S(j - 1, k - 1, x - R(p_j), y) \vee S(j - 1, k, x, y - R(p_j))$$

Initialize $S(0,0,0,0)$ = True

for $j = 1, \dots, n$:

 for $k = 1, \dots, \min(j, \frac{n}{2})$:

 for $x = 0, \dots, jm$:

 for $y = 0, \dots, jm$:

$S(j, k, x, y) =$

$$S(j - 1, k - 1, x - R(p_j), y) \vee S(j - 1, k, x, y - R(p_j))$$

Search for True entry at $S(n, \frac{n}{2}, > \frac{mn}{4}, > \frac{mn}{4})$

$S(j, k, x, y)$ = True if:

from among the first j precincts

k are assigned to D_1

exactly x vote for R in D_1

exactly y vote for R in D_2

Run-Time

$$S(j, k, x, y) = S(j - 1, k - 1, x - R(p_j), y) \vee S(j - 1, k, x, y - R(p_j))$$

Initialize $S(0,0,0,0) = \text{True}$

n for $j = 1, \dots, n$:

$\frac{n}{2}$ for $k = 1, \dots, \min(j, \frac{n}{2})$:

$\Theta(n^4m^2)$

nm for $x = 0, \dots, jm$:

nm for $y = 0, \dots, jm$:

$S(j, k, x, y) =$

$$S(j - 1, k - 1, x - R(p_j), y) \vee S(j - 1, k, x, y - R(p_j))$$

Search for True entry at $S(n, \frac{n}{2}, > \frac{mn}{4}, > \frac{mn}{4})$

Can We Visualize this 4D “Table”?

$S(j, k, x, y) = \text{True if:}$

- from among the first j precincts n
- k are assigned to D_1 $n/2$
- exactly x vote for R in D_1 nm
- exactly y vote for R in D_2 nm

To get a solution: search for True entry at $S(n, \frac{n}{2}, > \frac{mn}{4}, > \frac{mn}{4})$

$$\Theta(n^4m^2)$$

- This looks big! Yes, and it's interesting too! ☺
- Inputs:
 - List (size n) of precincts and counts of voters for Regular Party, $R(p_i)$
 - Number of voters (integer m)
- n is a **size** of one of the inputs
 - If n doubles, twice as many items in the list that's our input
- But m is an input **value** (not a size)
 - If m doubles, it's still one integer, one input item
 - But the amount of work grows
 - The complexity depends on the size of this single integer

Size of a Numeric Input-Value

Question: How do we measure the size of an integer?

Answer: the number of bits to represent it.

Example:

The value 4 (decimal) in binary is 100, so the size of “value 4” is 3.

If the size grows by 1, that’s 4 bits. With 4 bits, the value could be 1000 or 8 decimal.

Wait, what? Size of input grows by 1, and the value doubles (4 to 8). That sounds like exponential! 2^n vs. 2^{n+1}

Pseudo-Polynomial Time

Yes, the *inputSize* (in bits) of value m is $\log_2 m$

$$\text{inputSize} = \log_2 m$$

$$m = 2^{\text{inputSize}}$$

$$\text{So } m^2 = (2^{\text{inputSize}})^2 = 2^{2 \cdot \text{inputSize}}$$

Gerrymandering's run-time is exponential because of **size of input m**

- Because run-time $\Theta(n^4 m^2)$ written in terms of the *value* of m , not the *size* of m
- Input size is really $n + |m| = n + \log m$

This is called **pseudo-polynomial time** (https://en.wikipedia.org/wiki/Pseudo-polynomial_time)

We've seen others like this! Knapsack DP $\Theta(n \cdot C)$ and Coin-changing DP $\Theta(n \cdot A)$