Session 5

In the previous example "When should I leave my house to get to the airport", we do not use directly the estimated value, because there are uncertainty and error in the estimation of how long it will take you to get to the airport. The cost of the error is not symmetric; it will cost you a lot more to be a little bit late than to be a little bit early there. Therefore, your optimal action should take that estimation error into account. You choose the action that maximizes payoff in estimation rather than the estimate of unobserved parameter and then, plugs that into the decision making formula, assuming it is correct, ignoring the error, and then make an action based on that assumption (estimated value).

Here, we need to quantify the amount of error (uncertainty) and take into account to make a rigorous optimal decision. This is why the statistical decision theory is a powerful tool because it gives you a rigorous way to deal with the error.

Actions: $a \in A$

States: $x \in X$, Parameters: $\theta \in \Theta$

$x_1, x_2, ..., x_n$ random variables on X

Suppose we have uncertainty about our parameter values and in our game directly or indirectly we are depending on the values of the unobserved parameters.

So, we suppose that we got some possibility to get better information about unobserved parameters and there can be different ways. Here, we suppose we are going to get information by using statistical sampling from the state space and the statistical inference.

EX: Who is going to win the election and it depends on the population. Therefore, you can learn by doing some statistical sampling and from that, you can get information.

When we are doing sampling, we suppose that they are independent and identically distributed (i.i.d).

f(x): Defines the probability distribution for all x_i (for now drop i)

If X is discrete, e.g. X={1,...,m}

$f(x_j) = pr\{X = x_j\}, j = 1, 2, ..., J$

More generality, $B \subseteq X$

Then, $pr\{x \subseteq B\} = \Sigma_{j \in B} f(x_j) , j = 1, 2, ..., J$

If X is continuous

$B \subseteq X$

Then, $pr\{x \subseteq B\} = \int_{x \in B} f(x) dx$

Figure 1: 1

Suppose that there are probability density depends on the values of the unobserved parameter (θ), e.g. If we believe that the values are normally distributed, then it depends on two parameters including the μ and σ of the distribution.

$f(x) = f(x|\theta)$

Ex: $x_1, x_2, ..., x_n \sim N(\mu, \sigma)$

$\theta = <\mu, \sigma^2>$

Let $\pi(\theta)$ denotes our prior believes about values of θ

We are going to use any kind of predictive model, or tool to sharpen or a probability distribution over the state variables as it was shown in the below figure. In this way, we would have more confidence when we choose an action.

Figure 2: 2

Suppose we have a loss function that depends on $\theta : L(a, \theta)$

Suppose we know exactly what θ is, then choosing optimal a^* is trivial.

$a^* = argmax L(a|\theta)$

more generally suppose our beliefs about θ represented by $\pi(\theta)$

Then, for each action $a \in A$, $L(a|\theta)$ is a random variable. When you choose an action, you are choosing in effect the lottery, and then depending on θ, you will realize some loss.

Depending on a true value of θ, get some realize

Expected loss, given beliefs $\pi(\theta)$

$E[L(a|\theta)] = \int_{\theta \in \Theta} L(a|\theta)\pi(\theta)d\theta$

EX: The airport departure decision: if you have the suppose that θ governs how long it is going to take you to get to the airport in the likelihood of the plane is late or other things; a is how early you decide to leave. Then, when you choose the action (when you are going to leave), given the beliefs you have the expected pay off that includes all of the probabilities over your beliefs about θ.

θ: The time your plane leaves. How long you will need.

$\pi(\theta)$: your beliefs about θ

Figure 3: 3

a: how long you give yourself

$$L(atheta) = \begin{cases} C(a - \theta) & \text{if } a \geq \theta \\ 1000 & \text{if } a \leq \theta \end{cases} \tag{1}$$

C: How much you value your time (\$/hr)

$E^\pi[L(a|\theta)] = \int L(a|\theta)\pi(\theta)d\theta$

If $\pi(\theta) \sim N(\mu, \sigma_\theta^2)$

Then, $E^\pi[L(a|\theta)] = \int L(a|\theta)e^{-\frac{(a-\mu_\theta)^2}{\sigma_\theta}}$

We think μ_θ is the most likelihood of θ.

We keep our example simple, and we suppose that there is just only early and late. You leave early or not, and you hit the traffic or not. So, you have two by two decisions, and if you are early you waste your time, and if you are late you miss the flight. In that case, this problem looks very much like our problem with giving the meditation to the patient.

The point is , you do not assume that $\theta = E[\theta] = \int \pi(\theta)d\theta = \mu_\theta$

and then solve $min[L(a|\mu_\theta)]$

Instead, $minE^\pi[L(a|\theta)]$

In general, "most of the time"

$E[L(a|\theta)] \neq L(a|E[\theta])$

Def: A function h() s convex if for convex B on the domain of h

Jensen's inequality implies that

If h() is a convex function, then
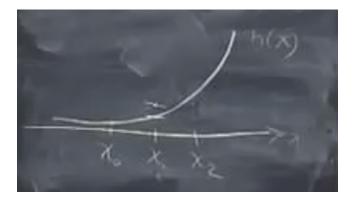
$E^{\pi}[h(x)] \geq h(E^{\pi}[x])$

Figure 4: 4

The loss of the expected value is not going to be the same as the expected value of the loss.

Therefore the action that minimizes our loss is not going to be identified by just assuming that we got the right answer for estimation of θ ($E[\theta]$) and plugging it into the equation of the loss.

Actually, we need to take into account uncertainty.

Suppose you have an opportunity to learn more about the data before you make a decision, maybe in the example by looking at the traffic forecast and update your beliefs about how long it will take you to get to the airport. The predictive tool allows you to sharpen your beliefs and allows you to make an optimization action. More generally, we got a lot of different kinds of predictive tools and what we are thinking about more is to sharpen our prior beliefs which give us a different probability over θ.

What the predictive tools do is sharping our beliefs about θ.

In general,

you get some data

$x_1, x_2, ..., x_n \longmapsto \pi(\theta \; x_1, x_2, ..., x_n)$

$\pi(\theta|x_1, x_2, ..., x_n)$: Updated or posterior beliefs

One important approach to updating the beliefs is Bayesian Analysis.