REAL ANALYSIS GENERAL EXAM AUGUST 2024

Solve as many problems as you can. Full solutions on a smaller number of problems will be worth more than partial solutions on several problems.

Question 1. Let $A \subset \mathbb{R}$ be a Lebesgue measurable set of finite measure. For $r \in \mathbb{R} \setminus \{0\}$, let $rA := \{x \in \mathbb{R} : r^{-1}x \in A\}$ and let $A \triangle rA := (A \setminus rA) \cup (rA \setminus A)$. Show that

$$\lim_{r \to 1} m(A \triangle rA) = 0.$$

Question 2. Let μ be a finite Borel measure on $\mathbb{R}$. For $\xi \in \mathbb{R}$, define

$$\widehat{\mu}(\xi) := \int_{\mathbb{R}} e^{-2\pi i x \xi}\, d\mu(x).$$

Suppose that

$$\lim_{\xi \to 0} \frac{\widehat{\mu}(\xi) - \widehat{\mu}(0)}{\xi^2} = 0.$$

(a) Show that

$$\int_{\mathbb{R}} x^2\, d\mu(x) = 0.$$

(b) Deduce that for any open interval $(a, b) \subseteq \mathbb{R}$, we have that

$$\mu((a,b)) = \begin{cases} \mu(\mathbb{R}) & \text{if } 0 \in (a,b), \\ 0 & \text{if } 0 \notin (a,b). \end{cases}$$

(The measure μ is a scalar multiple of the Dirac delta mass, though you do not need to prove this fact.)

Question 3. Fix $\alpha \in (0, 1]$. The space $C^{0,\alpha}([0, 1])$ of *Hölder continuous functions* consists of functions $f : [0, 1] \to \mathbb{C}$ for which

$$\|f\| := |f(0)| + \sup_{\substack{x,y \in [0,1] \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

is finite. This is a vector space (though you do not need to prove this fact).

(a) Show that $\| \cdot \|$ is a norm on $C^{0,\alpha}([0, 1])$.

(b) Show that $(C^{0,\alpha}([0, 1]), \| \cdot \|)$ is a Banach space.

Question 4. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Let $p \in [1, \infty)$ and $f \in L^p(X, \mathcal{M}, \mu)$. In what follows, you may freely use without proof the fact that the sets $\{x \in X : |f(x)| > \alpha\}$ and $\{(x, \alpha) \in X \times (0, \infty) : |f(x)| > \alpha\}$ are measurable.

(a) Show that

$$\int_X |f(x)|^p \, d\mu(x) = \int_0^\infty \alpha^{p-1} \mu(\{x \in X : |f(x)| > \alpha\}) \, d\alpha.$$

(b) Show that for all $\alpha > 0$,

$$\mu(\{x \in X : |f(x)| > \alpha\}) \le \frac{\|f\|_p^p}{\alpha^p}.$$

Question 5. Let $\phi \in \mathscr{S}(\mathbb{R}^n)$ be a fixed Schwartz function. The Fourier multiplier M acts on functions $f \in L^2(\mathbb{R}^n)$ according to the formula

$$\widehat{Mf}(\xi) := \phi(\xi)\widehat{f}(\xi).$$

That is, M is the operation of taking the Fourier transform of f, multiplying by the function $\phi(\xi)$, and then taking the inverse Fourier transform. Show that M is a bounded linear map from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.