Topology General Exam **August 15, 2025**

Instructions: This is a four hour exam. Your solutions should be legible and clearly organized, written in complete sentences in good mathematical style on your own paper. All work should be your own—no outside sources are permitted—using methods and results from the first year topology courses. Each problem is worth the same number of points.

(1) Let $X = S^1 \times \mathbb{R}P^2$.
   (a) Compute $\pi_1(X)$.
   (b) Describe the universal cover of X and explicitly describe each element in the group of deck transformations of the universal cover.

(2) Let $x_1, x_2, \ldots, x_{10}$ be 10 distinct points on the 2-torus T^2. Let X be the quotient of T^2 obtained by identifying all 10 points.
   (a) Compute $\pi_1(X)$.
   (b) Compute $H_n(X)$ for all $n \geq 0$.

(3) (a) Let $f : S^1 \to \mathbb{R}$ be a continuous map. Show that there exists some $x \in S^1$ such that $f(x) = f(-x)$.
   (b) Show that any continuous map $g : \mathbb{R}P^2 \times \mathbb{R}P^2 \to T^2$ is nullhomotopic.

(4) Let (C, d) and (C', d') be chain complexes of abelian groups. The *mapping cone* of a chain map $f : (C, d) \to (C', d')$, denoted $(cone(f), d'')$, is defined by

$$cone(f)_n := C_{n-1} \oplus C'_n, \quad d''(c, c') = (-d(c), d'(c') - f(c)).$$

Prove that $(cone(f), d'')$ is a chain complex.

(5) Consider the family of smooth maps $f_c\colon \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$f_c(x, y) = (x, y, cx + y^2), \quad c \in \mathbb{R}.$$

Let $S \subset \mathbb{R}^3$ be the surface defined by

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 + y^2\}.$$

   (a) For which values of $c \in \mathbb{R}$ is the map f_c transverse to the surface S?
   (b) For what values of $c \in \mathbb{R}$ is the intersection of the image of f_c with S a manifold?

(6) Consider the 3-form

$$\theta = e^w \left(x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy\right)$$

on $\mathbb{R}^4$ where the coordinates are x, y, z, w. Let H be the hemisphere $x^2 + y^2 + z^2 + w^2 = 1, w \geq 0$, with the orientation induced as being part of the boundary of the 4-ball with the standard orientation. Compute

$$\int_S \mathrm{d}\, \theta.$$

(7) Let $p\colon M \to N$ be a smooth covering map between closed manifolds of some dimension n. Suppose N is oriented. Show that M is orientable. Pick an orientation on M and compute the degree of p in terms of the number of sheets of the covering.