



UNIVERSITY of VIRGINIA

DEPARTMENT of MATHEMATICS

On the coefficients of q -series and modular forms

This thesis studies the coefficients of q -series, particularly in connection with partitions and modular forms. Our main tools are the circle method, which we use to prove asymptotics and inequalities for partition functions, and newforms, which we use to give an effective solution to a variation of Lehmer's conjecture.

Wright's circle method is a very general and useful technique for producing asymptotic expansions for a sequence $a(n)$ given certain asymptotic properties of its generating function, $F(q) = \sum_{n \geq 0} a(n)q^n$. We apply effective versions of Wright's technique to prove inequalities between partition functions. For example, if $D_{r,t}(n)$ denotes the number of parts equivalent to r modulo t among all partitions of n into distinct parts, we use Wright's method to prove that

$$D_{r,t}(n) = \frac{3^{\frac{1}{4}} e^{\pi\sqrt{\frac{n}{3}}}}{2\pi t n^{\frac{1}{4}}} \left(\log(2) + \left(\frac{\sqrt{3}\log(2)}{8\pi} - \frac{\pi}{4\sqrt{3}} \left(r - \frac{t}{2} \right) \right) n^{-\frac{1}{2}} + O_t(n^{-1}) \right)$$

and in particular that when $1 \leq r < s \leq t$ and $2 \leq t \leq 10$, $D_{r,t}(n) > D_{s,t}(n)$ for all $n > 8$. Using a similar technique, we resolve a conjecture of Coll, Mayers and Mayers by showing that the infinite product

$$\sum_{n \geq 0} a(n)q^n = \prod_{n \geq 1} \frac{1}{1 + (-1)^{2n-1}q^{2n-1}}$$

has only non-negative coefficients; we do this by proving an effective asymptotic formula for $a(n)$.

Hook numbers of partitions play many important roles in number theory and representation theory. In this thesis, we consider the function $p_t(a, b; n)$ that counts the number of partitions of n whose number of t -hooks, that is hook numbers which are multiples of t , are congruent to a modulo b . In joint work with Pun and Bringmann, Males and Ono, we give exact formulas for $p_t(a, b; n)$ using Rademacher's circle method, and we use this formula to show that equidistribution modulo primes b holds if and only if $b|t$.

We also consider an important application partition asymptotics to certain kinds of partition inequalities; in particular the Turán inequalities. These inequalities arise as a criterion for proving hyperbolicity of the Jensen polynomials. Using important work of Griffin, Ono, Rolen and Zagier along with asymptotics, we prove that the k -regular partition function $p_k(n)$ satisfies all the Turán inequalities for $n \gg 0$.

Lehmer has conjectured that the coefficients $\tau(n)$ of Ramanujan's delta function $\Delta = q \prod_{n \geq 1} (1 - q^n)^{24}$ are never zero. Although we cannot resolve this question, in joint work with Balakrishnan, Ono and Tsai, we give an algorithm for explicitly solving equations of the form $\tau(n) = \pm \ell^m$, where ℓ is any odd prime. Our method relies crucially on the deep results of Bilu, Hanrot, and Voutier on primitive prime divisors of Lucas sequences and on properties of Atkin-Lehner newforms.

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