Topology General Exam **January 11, 2023**

Student ID: ________________________

Instructions: This is a four hour exam. Your solutions should be legible and clearly organized, written in complete sentences in good mathematical style. All work should be your own—no outside sources are permitted—using methods and results from the first year topology course topics. There are 8 problems; each problem is worth the same number of points.

1. Let $f, g\colon \mathbb{R}^1 \longrightarrow \mathbb{R}^2$ be two smooth maps. Prove that the subset of $\mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{R}^2$ consisting of all points $(x, y, z) \in \mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{R}^2$ satisfying the equation $f(x) - g(x) = z$ is a submanifold. What is its dimension?

2. Consider two topological spaces X, Y obtained from a disk by removing two smaller disks from its interior, and then identifying all three boundary curves to a single curve, respecting the orientations as shown in the figure. Compute the homology groups of both X and Y with $\mathbb{Z}$ coefficients.

3. Show that any smooth map $f\colon \mathbb{R}P^n \to \mathbb{R}P^n$ has a fixed point, when n is even.

4. Prove that 1-forms $\theta_1, \ldots, \theta_n$ on a smooth n-manifold M are linearly independent (thought of as sections of the cotangent bundle) if and only if $\theta_1 \wedge \ldots \wedge \theta_n$ is non-vanishing.

5. Suppose the following diagram of abelian groups and homomorphisms commutes:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & B_1 \arrow[r, "\beta"] \arrow[d] & B_2 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d, "\alpha"'] & C \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_2 \arrow[r] & E \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Assume that the rows and columns form exact sequences. Prove that there are isomorphisms $\ker\alpha \cong \ker\beta$ and $\operatorname{coker}\alpha \cong \operatorname{coker}\beta$.

6. Let $M = \{x^2+y^2+z^2 = 1\} \subset \mathbb{R}^3$, and for a real number $c \neq 0$ let $N = \{x^2-y^2+z^2 = c\}$. For which values of c is the intersection between M and N transverse? For which values of c is this intersection a submanifold of $\mathbb{R}^3$?

7. Let $\exp\colon \mathbb{C} \to \mathbb{C} \smallsetminus \{0\}$ be the complex exponential map $\exp(z) = e^z$ (recall that exp is a universal covering map). If X is a connected, locally path-connected space and $f\colon X \to \mathbb{C} \smallsetminus \{0\}$ a continuous map, a *logarithm* of f is a continuous map $g\colon X \to \mathbb{C}$ such that $\exp \circ g = f$. Prove that if such an X has finite fundamental group, then any continuous map $f\colon X \to \mathbb{C} \smallsetminus \{0\}$ has a logarithm.

8. The connected sum of the 2-torus T and the projective plane $\mathbb{R}P^2$ is a smooth closed 2-manifold M whose fundamental group has a presentation $\pi_1(M) = \langle a, b, c | aba^{-1}b^{-1}c^2 = 1\rangle$. Determine the number of connected, regular 3-fold covering spaces of M up to equivalence.

Hint: First show that this number is the same as the number of surjections $\pi_1(M) \to \mathbb{Z}/3\mathbb{Z}$ up to equivalence, where two surjections f, g are equivalent if and only if $g = \phi \circ f$ for some automorphism ϕ of $\mathbb{Z}/3\mathbb{Z}$.