Solve as many problems as you can. Full solutions on a smaller number of problems will be worth more than partial solutions on several problems. Throughout $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Don't use any of the Picard theorems.

Problem 1.

Compute, for $b, \xi > 0$ and $a \in \mathbb{R}$

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{xe^{-ix\xi}}{(x-a)^2+b}\,dx.$$

Show all estimates.

Problem 2.

Define $g\colon \mathbb{C}\setminus\{1\} \to \mathbb{C}$ by $g(z) = \frac{z+1}{z-1}$, and let $f\colon \mathbb{C}\setminus\{1\} \to \mathbb{C}$ be given by $f(z) = e^{g(z)}$.

(i) Give an explicit description of $g(\mathbb{D})$.

(ii) Show that f is bounded on $\mathbb{D}$.

Problem 3.

Fix an integer $k \in \mathbb{N}$. Suppose that $a_0, \cdots, a_k \in \mathbb{C}$, set $p(z) = \sum_{j=0}^{k} a_j z^j$ and assume that $p(z)$ has no zeroes on $\partial\mathbb{D}$. Prove that there is an $\varepsilon > 0$ so that if $b_0, \cdots, b_k \in \mathbb{C}$ with $\left(\sum_{j=0}^{k} |b_j - a_j|^2\right)^{1/2} < \varepsilon$, then setting $q(z) = \sum_{j=0}^{k} b_j z^j$ we have that p, q have the same number of zeroes in $\mathbb{D}$, counted with multiplicity.

Problem 4.

Let $A = \{z \in \mathbb{C} : 0 < |z| < 1\}$ and $B = \{z \in \mathbb{C} : 1 < |z| < 2\}$,

(i) Suppose that $\Omega \subseteq \mathbb{C}$ is a bounded open set and that $\phi\colon A \to \Omega$ is a holomorphic bijection. Show that ϕ extends to a holomorphic function $\widetilde{\phi}\colon \mathbb{D} \to \overline{\Omega}$ and that $\widetilde{\phi}(0) \in \partial\Omega$. (Suggestion: it may be helfpul to prove that if z_n is a sequence in A and $z_n \to 0$ then for every compact $K \subseteq \Omega$ we have that $\{n : \phi(z_n) \in K\}$ is finite).

(ii) Show that there is no holomorphic bijection from A to B.

Problem 5.

For $p \in \mathbb{C}$ and $s > 0$, we use $B_s(p) = \{z \in \mathbb{C} : |z - p| < s\}$.

(i) Let $U \subseteq \mathbb{C}$ be open, $p \in U$ and $s > 0$ with $\overline{B_s(p)} \subseteq U$. Show that if $f\colon U \to \mathbb{C}$ is holomoprhic, then

$$f(p) = \frac{1}{\pi s^2} \int\int_{B_s(p)} f(x+iy)\,dx\,dy.$$

(Hint: compute the integral in polar coordinates).

(ii) Let $\mathcal{F}$ be the family of all holomorphic functions $f\colon \mathbb{D} \to \mathbb{C}$ with

$$\sup_{0<s<1} \int\int_{B_s(0)} |f(x+iy)|\,dx\,dy \le 1.$$

Show that $\mathcal{F}$ is *normal* meaning that if $(f_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{F}$, then there is a subsequence $(f_{n_k})_k$ in $\mathcal{F}$ which converges uniformly on compact subsets of $\mathbb{D}$.