COMPLEX ANALYSIS GENERAL EXAM SPRING 2023

Solve as many problems as you can. Full solutions on a smaller number of problems will be worth more than partial solutions on several problems. Don't use any of the Picard theorems.

Question 1. Compute
$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2 + 1}\, dx.$$
You must justify all estimates.

Question 2. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function for which there exist $M, R > 0$ such that $|f(z)| \geq M|z|^2$ for all $|z| > R$. Show that f is a polynomial of degree at least 2.

Question 3. Let $f, g : \mathbb{C} \to \mathbb{C}$ be entire functions for which there exists $\lambda \in \mathbb{R}$ such that $\operatorname{Im}(f(z)) \leq \lambda \operatorname{Im}(g(z))$ for all $z \in \mathbb{C}$. Show that there exist constants $a, b \in \mathbb{C}$ such that $f(z) = ag(z) + b$ for all $z \in \mathbb{C}$.

Question 4. Let $P(z) := z^9 + z^5 - 8z^3 + 2z + 1$. Determine the number of zeroes (counted with multiplicity) of P that lie in the annulus $A_{1,2}(0) := \{z \in \mathbb{C} : 1 < |z| < 2\}$.

Question 5. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disc, and let $\mathcal{F}$ denote the family of holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ satisfying $|f(z)| > 1$ for all $z \in \mathbb{D}$ and $f(0) = 2i$. Show that $\mathcal{F}$ is normal, so that every sequence (f_n) in $\mathcal{F}$ has a subsequence that converges uniformly on compact sets to a holomorphic function $f : \mathbb{D} \to \mathbb{C}$.