Topology General Exam **August 16, 2024**

Instructions: This is a four hour exam. Your solutions should be legible and clearly organized, written in complete sentences in good mathematical style on your own paper. All work should be your own—no outside sources are permitted—using methods and results from the first year topology courses. Each problem is worth the same number of points.

(1) (a) Prove that any continuous map $S^2 \to S^1$ is homotopic to a constant map.
(b) Let M_g be the compact orientable surface of genus g, i.e., the g-holed torus. Prove that for any $g \geq 1$, there exists a map $f\colon M_g \to S^1$ that is continuous and non-nullhomotopic.

(2) Let A be a path-connected subspace of a topological space X and $i\colon A \to X$ the inclusion. Show that, for any $x_0 \in A$, the induced map $i_*\colon \pi_1(A, x_0) \to \pi_1(X, x_0)$ is surjective if and only if every path in X with endpoints in A is homotopic (relative to endpoints) to a path in A. Here "relative to endpoints" means that endpoints are required to be fixed throughout a homotopy.

(3) Describe all the path-connected covering spaces of the torus T^2, up to isomorphism.

(4) Let M be a Möbius strip and $\mathbb{R}P^2$ the real projective plane.
(a) Using a CW structure or otherwise, prove that the inclusion $\partial M \to M$ induces the multiplication by 2 map on the first homology group.
(b) Explain why removing a disk from $\mathbb{R}P^2$ gives a space homeomorphic to M.
(c) Use this decomposition to compute the homology groups of $\mathbb{R}P^2$.

(5) Consider $\mathbb{R}^4$ with coordinates x, y, z, w. Consider the subset S of $\mathbb{R}^4$ defined by the equation $x^2 + y^2 = z^2 + w^2$.
(a) Prove that S is not a smooth submanifold of $\mathbb{R}^4$.
(b) Determine which points of S are manifold points. (A point $p \in S$ is a *manifold point* if there exists a neighborhood of p in S which is a manifold.)
Justify your answers.

(6) Let M, N be two smooth manifolds, where the dimension of M is m and the dimension of N is n for some $n > m > 0$. Prove that there does not exist a smooth surjective map $M \to N$.

(7) Consider the unit sphere S in $\mathbb{R}^3$, and let ω be the 2-form on $\mathbb{R}^3$ given by $\omega = x\,\mathrm{d}y \wedge \mathrm{d}z + y^2 \mathrm{d}x \wedge \mathrm{d}y$. Compute the integral $\int_S \omega$ of the restriction of ω to S.

(8) Let M be a smooth, compact n-dimensional submanifold (without boundary) of $\mathbb{R}^{n+1}$. Prove that there exists a point $p \in M$ such that the entire submanifold M lies to one side of the tangent space T_pM. (Here T_pM is thought of as a hyperplane in $\mathbb{R}^{n+1}$ which is tangent to M at p.)