REAL ANALYSIS GENERAL EXAM
AUGUST 18, 2025

- Solve the following **five** questions. The exam is 2 hours long.
- To receive full credit, you **must clearly state** which theorems you are using in your proofs, and your proofs must be **fully justified and explained**.

Question 1. Let m be the Lebesgue measure on $\mathbb{R}$ and let E be a Lebesgue measurable set such that $m(E) < \infty$. Let $f_n, f : E \to \mathbb{R}$ be measurable functions such that $f_n \to f$ almost everywhere. Assume that

$$\int_E |f_n|^3 \, dm \leq 1, \text{ for any } n \in \mathbb{N}.$$

Prove that

$$\lim_{n\to\infty} \int_E f_n \, dm = \int_E f \, dm.$$

Question 2. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f : X \to [0, \infty]$ be a non-negative $\mathcal{M}$-measurable function on X. Let

$$T = \left\{ t \in [0, \infty] : \mu\left(f^{-1}(\{t\})\right) > 0 \right\}.$$

Prove that if f is integrable, then T is a countable set.

Question 3. Let $\mathcal{M}$ denote the Lebesgue measurable sets on $\mathbb{R}$ and let m be the Lebesgue measure on $\mathcal{M}$ and τ be the counting measure on $\mathcal{M}$. Show that $m << \tau$ but $\frac{dm}{d\tau}$ doesn't exist. Explain why this doesn't contradict with the Radon-Nikodym Theorem?

Question 4. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space with $\mu(X) = \infty$.

(a) Show that there exists a disjoint sequence $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{M}$ (i.e., $E_n \cap E_m = \phi$, when $n \neq m$), such that $\cup_{n\in\mathbb{N}} E_n = X$ and $\mu(E_n) \in [1, \infty)$ for every $n \in \mathbb{N}$.

(b) Show that there exists an $\mathcal{M}$-measurable function $f : X \to [0, \infty]$ such that $f \in L^p(X, \mathcal{M}, \mu)$ for all $p \in (1, \infty]$ and $f \notin L^1(X, \mathcal{M}, \mu)$.

Question 5.

(a) Let $f \in L^1(\mathbb{R}^n, m)$. Prove that the set of points where the maximal function is finite, $\{x \in \mathbb{R}^n : Mf(x) < \infty\}$, has full measure (i.e., its complement is a set of measure zero). The Hardy-Littlewood maximal function is defined as:

$$Mf(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y) \, dm(y).$$

(b) Give the definition of the Lebesgue set and state the Lebesgue Differentiation Theorem. Give an example of a function f that is in $L^1_{loc}(\mathbb{R}, m)$ but whose Lebesgue set is not all of $\mathbb{R}$. Justify your answer.