Solve as many problems as you can. Full solutions on a smaller number of problems will be worth more than partial solutions on several problems. Throughout $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Don't use any of the Picard theorems.

Problem 1.

Compute, for $0 < \alpha < 1$,

$$\int_0^\infty \frac{x^\alpha}{x^2+4}\,dx.$$

You must show all estimates.

Problem 2.

Suppose that $U \subseteq \mathbb{C}$ is open and $f\colon U \to \mathbb{C} \setminus \{0\}$ is holomorphic. Prove that $u(z) = \log|f(z)|$ is harmonic.

Problem 3.

Let $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. Suppose that $f\colon \overline{\mathbb{H}} \to \mathbb{C}$ is continuous and that $f\big|_{\mathbb{H}}$ is holomorphic. Suppose that $f(\mathbb{R}) \subseteq \mathbb{R}$ and that $f(\mathbb{H}) \subseteq \{z \in \mathbb{C} : \mathrm{Im}(z) \geq 0, \mathrm{Re}(z) \geq 0\}$. Show that f is constant.

Problem 4.

Suppose that f and g are analytic in an open subset U of $\mathbb{C}$ containing $\{z \in \mathbb{C} : |z| \leq 1\}$. Suppose that f has a zero of order 1 at $z = 0$, and no other zero in U.

(i) For $\zeta \in \mathbb{C}$, set $f_\zeta = f(z) + \zeta g(z)$. Show that if $|\zeta|$ is small enough (depending upon g), then f_ζ has no zeroes on $\partial\mathbb{D}$, has a unique zero in $\mathbb{D}$, and that the order of this zero is 1.

(ii) Show that for all sufficiently small $|\zeta|$, if z_ζ is the unique zero of f_ζ in $\mathbb{D}$, then

$$z_\zeta = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} w\frac{f_\zeta'(w)}{f_\zeta(w)}\,dw.$$

Hint: evaluate the integral by the residue theorem.

Problem 5.

Let U be a connected open subset of $\mathbb{C}$. Suppose that $(f_n)_{n=1}^\infty$ is a sequence of holomorphic functions on U, and that there is a continuous $g\colon U \to [0, +\infty)$ with $|f_n(z)| \leq g(z)$ for all $n \in \mathbb{N}$ and all $z \in U$. Assume that $p \in U$ and that $r > 0$ has $B_r(p) \subseteq U$ and that $\lim_{n\to\infty} f_n(z) = 0$ for all $z \in B_r(p)$.

(i) Suppose that $h\colon U \to \mathbb{C}$, and that there is a subsequence $(f_{n_k})_k$ of $(f_n)_n$ with $f_{n_k} \to_{k\to\infty} h$ uniformly on compact sets. Show that necessarily $h = 0$.

(ii) Show that $f_n \to 0$ uniformly on compact subsets of U.