REAL ANALYSIS GENERAL EXAM JANUARY 2025

Solve as many problems as you can. Full solutions on a smaller number of problems will be worth more than partial solutions on several problems.

Question 1. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $(f_n), (g_n)$ be sequences of functions in $L^1(X, \mathcal{M}, \mu)$ that converge pointwise a.e. to functions $f, g \in L^1(X, \mathcal{M}, \mu)$ respectively. Suppose that $|f_n| \leq g_n$ a.e. and that

$$\lim_{n\to\infty} \int_X g_n(x)\, d\mu(x) = \int_X g(x)\, d\mu(x).$$

Show that

$$\lim_{n\to\infty} \int_X f_n(x)\, d\mu(x) = \int_X f(x)\, d\mu(x).$$

Question 2.

(a) Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Show that if $p, p' \in [1, \infty]$ with $p < p'$, then $L^p(X, \mathcal{M}, \mu) \supseteq L^{p'}(X, \mathcal{M}, \mu)$.

(b) Show that if $p, p' \in [1, \infty]$ with $p < p'$, the spaces $L^p(\mathbb{R}) \setminus L^{p'}(\mathbb{R})$ and $L^{p'}(\mathbb{R}) \setminus L^p(\mathbb{R})$ are both nonempty. (Note that the Lebesgue measure on $\mathbb{R}$ is *not* a finite measure, but merely a σ-finite measure.)

Question 3. Let $\mathcal{H}$ be a separable Hilbert space. A sequence (v_m) in $\mathcal{H}$ *converges weakly* to $v \in \mathcal{H}$ if

$$\lim_{m\to\infty} \langle v_m, w\rangle = \langle v, w\rangle$$

for every $w \in \mathcal{H}$. Show that for any sequence (v_m) in $\mathcal{H}$ for which $\sup_{m\in\mathbb{N}} \|v_m\|$ is finite, there exists a subsequence (v_{m_k}) that converges weakly to some $v \in \mathcal{H}$.

Question 4. Define the Dirac delta measure δ_0 on the Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$ of $\mathbb{R}$ by

$$\delta_0(A) := \begin{cases} 1 & \text{if } A \ni 0, \\ 0 & \text{otherwise.} \end{cases}$$

For each $r > 0$, let ν_r be the measure on the Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$ of $\mathbb{R}$ given by

$$\nu_r(A) := \frac{1}{2r} m(A \cap [-r, r]),$$

where m denotes the Lebesgue measure on $\mathbb{R}$. Show that for every continuous function $f : \mathbb{R} \to \mathbb{C}$, we have that

$$\lim_{r\searrow 0} \int_{\mathbb{R}} f(x)\, d\nu_r(x) = \int_{\mathbb{R}} f(x)\, d\delta_0(x).$$

Question 5.

(a) State the Riemann–Lebesgue lemma for the Fourier transform on $\mathbb{R}^n$.

(b) Show that there does not exist a function $g \in L^1(\mathbb{R}^n)$ that satisfies $f * g = f$ for all $f \in L^1(\mathbb{R}^n)$.