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The family index of the odd signature operator with coefficients in a flat bundle

The Atiyah-Singer index theorem provides an analytic interpretation of certain topological invariants associated to a M-bundle $f : E \to B$, M a smooth closed oriented Riemannian manifold. In particular a family of such operators $D = \{D_b\}_{b \in B}$ produces a class $index(D) \in K^*(B)$ in the complex K-theory of B.

For example, the index of the (even) signature operator D^e coincides with the signature $Sig(M)$ of the intersection form defined on the cohomology of M, $index(D^e) = Sig(M) = \langle \mathcal{L}(M), [M] \rangle \in \mathbb{Z} \cong K^0(pt.)$, where $\mathcal{L}(M) = \mathcal{L}(TM)$ is the total Hirzebruch L-class of M.

Any elliptic operator D on a manifold M may be "twisted" by a vector bundle $V \to M$, producing an operator D_V. When V is *flat* $index(D^e_V)$ again may be interpreted as the signature of an intersection form, this time defined on the *cohomology* $H^*(M; V)$ *of* M *with coefficients in* V,

$$index(D^e_V) = Sig_V(M) = \langle \mathcal{L}(M) ch([V]), [M] \rangle \in \mathbb{Z} \cong K^0(pt.).$$

Sig_V are the *higher signatures*, conjectured by S. Novikov to be invariants of the oriented homotopy type of M.

Assume M is odd dimensional. There is also an odd signature operator D^o and the family index of D^o is still related to the signature of E. It was shown by J. Ebert that the family index of D^o is always zero, $ind(D^o) = 0 \in K^1(B)$, from which it follows that $Sig(E) = 0$. Ebert's proof, essentially functional analytic in nature, is still applicable to D^o_V, so long as V is flat Hermitian. Let $V \to E$ be a flat hermitian vector bundle.

Theorem The the family index of the odd signature operator twisted by V is zero, $ind(D^o_V) = 0 \in K^1(B)$

Let $f_! : H^*(E) \to H^{*-dim(M)}(B)$ be the umkehr homomorphism (integration over the fiber), $T_vE = ker(df)$ the vertical tangent bundle of E. Applying the Chern character $ch : K \to H\mathbb{Q}$ yields the following vanishing results in rational cohomology

Theorem $f_!(\mathcal{L}(T_vE) ch([V])) = 0 \in H^*(B; \mathbb{Q})$

Theorem The higher signature $Sig_V(E) = 0$

The special case when $V = 0$ was conjectured by Atiyah and proven by J. Ebert. This result generalizes an observation made by M. Gromov (when $M = S^1$). Note the flatness of V above is essential. This can be seen in the simplest possible example where $E = S^1 \times S^1 \to B = S^1$ is the projection.

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