COMPLEX ANALYSIS GENERAL EXAM AUGUST 2025

Solve as many problems as you can. Full solutions on a smaller number of problems will be worth more than partial solutions on several problems. Don't use any of the Picard theorems. Throughout, $\mathbb{D}$ denotes the unit disc $\{z \in \mathbb{C} : |z| < 1\}$.

Question 1. Using Cauchy's residue theorem, compute the definite integral

$$\int_0^{2\pi} \frac{1}{5 + 4\cos\theta}\, d\theta.$$

Question 2. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function satisfying $|f(2z)| \leq 2|f(z)|$ for all $z \in \mathbb{C}$. Show that either $f(z) = f(0)$ for all $z \in \mathbb{C}$ or $f(z) = f'(0)z$ for all $z \in \mathbb{C}$.

Question 3. Show that the only solution inside the unit disc $\mathbb{D}$ to the equation $e^z = 2z+1$ is $z = 0$. (You may use without proof the fact that $e \approx 2.718 < 3$.)

Question 4. Let $U := \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$. Let $f : \overline{U} \to \mathbb{C}$ be a continuous bounded function for which $f|_U$ is holomorphic. Suppose that there exist constants $M_1, M_2 \geq 0$ such that

$$\sup_{\mathrm{Re}(z)=0} |f(z)| \leq M_0, \qquad \sup_{\mathrm{Re}(z)=1} |f(z)| \leq M_1.$$

Show that for $r \in [0, 1]$,

$$\sup_{\mathrm{Re}(z)=r} |f(z)| \leq M_0^{1-r} M_1^r.$$

(Hint: for fixed $\varepsilon > 0$, show that $f_\varepsilon(z) := f(z) M_0^{z-1} M_1^{-z} e^{\varepsilon(z^2-1)}$ satisfies $\sup_{z\in\overline{U}} |f_\varepsilon(z)| \leq 1$.)

Question 5. Let $\mathcal{F}$ denote the set of holomorphic functions f on the disc $B_2(0) = \{z \in \mathbb{C} : |z| < 2\}$ that satisfy

$$\int_0^{2\pi} |f(e^{i\theta})|\, d\theta \leq 1.$$

Let

$$\mathcal{G} := \{g : \mathbb{D} \to \mathbb{C} : g = f|_{\mathbb{D}} \text{ for some } f \in \mathcal{F}\}.$$

Show that $\mathcal{G}$ is a normal family.