Asymptotics of the Chebyshev Polynomials of General Sets

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Joint Work with Jacob Christiansen and Maxim Zinchenko

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The *Chebyshev polynomial of degree* n is the monic polynomial, T_n, with

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The minimizer is unique (as we'll see below in the case that $\mathfrak{e} \subset \mathbb{R}$), so it is appropriate to speak of *the* Chebyshev polynomial rather than *a* Chebyshev polynomial.

Chebyshev Polynomials

Chebyshev invented his explicit polynomials which obey $Q_n(\cos(\theta)) = \cos(n\theta)$ not because of their functional relation but because they are the best approximation on $[-1,1]$ to x^n by polynomials of degree $n-1$. In this regard, Sodin and Yuditskii unearthed the following quote from a 1926 report by Lebesgue on the work of S. N. Bernstein.

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This quote is a little bizarre given that, as we'll see, Borel (who was Lebesgue's thesis advisor) made important contributions to the subject in 1905!

The Alternation Theorem

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We say that P_n, a degree n polynomial, has an *alternating set* in $\mathfrak{e} \subset \mathbb{R}$ if there exists $\{x_j\}_{j=0}^n \subset \mathfrak{e}$ with

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While the basic idea of the following theorem goes back to Chebyshev, the result itself is due to Borel and Markov, independently, around 1905.

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Conversely, let P_n be a degree n monic polynomial with an alternating set and suppose that $\|T_n\|_{\mathfrak{e}} < \|P_n\|_{\mathfrak{e}}$.

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At the alternating points for Q, we must have $T_n = S_n$, so they must be equal polynomials since there are $n+1$ points and their difference has degree at most $n-1$.

Alternation and Zeros

If T_n is the Chebyshev polynomial for $\mathfrak{e} \subset \mathbb{R}$ and $x_0 < x_1 < \ldots < x_n$ is an alternating set for T_n,

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Fact 1 All the zeros of the Chebyshev polynomials of a set $\mathfrak{e} \subset \mathbb{R}$ lie in $\mathbb{R}$ and all are simple and lie in $\mathrm{cvh}(\mathfrak{e})$.

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Here, $\mathrm{cvh}(\mathfrak{e})$ is the convex hull of $\mathfrak{e}$ and that result follows from $x_0, x_n \in \mathfrak{e}$.

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Above the top zero (resp. below the bottom zero) of T_n, $|T_n(x)|$ is monotone increasing (resp. decreasing). It follows that $x_n = \sup_{y \in \mathfrak{e}} y$ (resp $x_0 = \inf_{y \in \mathfrak{e}} y$) so

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Coulomb Energies and All That

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$$C(\mathfrak{e}) = \exp(-R(\mathfrak{e})) \qquad R(\mathfrak{e}) = \log(1/C(\mathfrak{e}))$$

Equilibrium Measures and All That

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The function $\Phi_{\mathfrak{e}}(z) = \int_{\mathfrak{e}} d\rho_{\mathfrak{e}}(x) \log |x - z|^{-1}$ is called the *equilibrium potential*.

Green's Function

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If $\mathfrak{e}$ is a period–n set, one can prove that $\mathfrak{e}_n = \mathfrak{e}$ so that all the period–n sets are precisely the possible $\mathfrak{e}_n$'s.

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Theorem (*Faber–Fekete–Szegő Theorem*) For any compact subset $\mathfrak{e} \subset \mathbb{C}$, we have that

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As in the work of Faber, it is natural to look for an analytic function, $B_{\mathfrak{e}}(z)$, with $|B_{\mathfrak{e}}(z)| = \exp(-G_{\mathfrak{e}}(z))$ on Ω, the unbounded component of $(\mathbb{C} \cup \{\infty\}) \setminus \mathfrak{e}$.

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Put differently, $B_{\mathfrak{e}}(z)$ can be continued along any curve in Ω and there is a map from the fundamental group of Ω to $\partial\mathbb{D}$, which is a character (i.e. group homomorphism), so that after continuation around a closed curve, $B_{\mathfrak{e}}(z)$ is multiplied by the character applied to that curve.

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If $T_n(z)B_{\mathfrak{e}}(z)^n C(\mathfrak{e})^{-n}$ had a limit, that limit cannot be n independent since the character is n dependent. Widom had the idea that there should be functions $F_\chi(z)$ defined for each χ in the character group and continuous in χ so the limit is the F_χ, call it F_n, associated to the character of $B_{\mathfrak{e}}(z)^n$. As a function of n, the limit will be almost periodic!

Widom's Minimizers

He even found a candidate for the functions! Let $F_\chi(z)$ be that function among all character automorphic functions, $A(z)$, on Ω with character χ and with $A(\infty) = 1$, that minimizes $\sup_{z\in\Omega}\{|A(z)|\}$.

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Widom proved uniqueness of the minimizer and found a formula for it (in terms of some theta functions and solutions of some implicit equations). He also proved that $\|F_\chi\|_\Omega$ is continuous in χ. Because of the uniqueness, one can prove that the functions, $F_\chi(z)$, defined for $z \in \Omega$, are continuous in χ on the compact set of characters, uniformly locally in z (but as functions on the covering space not uniformly in all z).

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Since $|B_{\mathfrak{e}}(z)| \to 1$ and $\|F_n\|_\Omega$ is taken as $z \to \mathfrak{e}$, the z asymptotics and norm limit fit together.

Theorem (*Widom*) Let $\mathfrak{e}$ be a finite gap subset of $\mathbb{R}$. Let $F_n(z)$ be as above. Then

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Note: This is Widom's conjecture for $\mathfrak{e} \subset \mathbb{R}$; he made the conjecture for more general cases of $\mathfrak{e} \subset \mathbb{C}$.

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It was this example that led Widom to his conjecture.

Totik–Widom bounds

Theorem (Totik's $1/n$ bound) If $\mathfrak{e}$ is a finite gap set, the period n sets $\tilde{\mathfrak{e}}_n \supset \mathfrak{e}$ can be chosen so that $C(\tilde{\mathfrak{e}}_n) \leq C(\mathfrak{e})\left(1 + \frac{E}{n}\right)$ for some constant E.

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This complements the $2C(\mathfrak{e})^n$ lower bound. Because of his asymptotic result, Widom already had this bound in 1969 but Totik's proof was much simpler. Neither proof has very explicit estimates for D. Even though they only had the result for finite gap sets, we will say that a general set $\mathfrak{e}$ has *Totik–Widom bounds*, if there is an upper bound of the above form.

Canonical Generators

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It seems to us likely that, in some sense, this condition holds generically. It follows from results of Totik that among the 2ν dimensional set of unions of exactly ν disjoint unions, the set where the condition fails is a countable union of varieties of dimension $\nu + 1$ so the set where it fails is both of 2ν Lebesgue measure zero and a nowhere dense F_σ.

PW and DCT sets

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Let $\chi \in \pi_1^*$ and let $H^\infty(\Omega, \chi)$ be the family of bounded character automorphic analytic functions on Ω. We say that a compact set, $\mathfrak{e} \in \mathbb{C}$, has the *PW property* if and only if for every $\chi \in \pi_1^*$, we have that $H^\infty(\Omega, \chi)$ contains non–constant functions.

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where $\mathcal{C}$ is the set of critical points of $G_{\mathfrak{e}}$ (i.e. points in the unbounded component of Ω where $G'_{\mathfrak{e}}(w) = 0$). In particular, any connected set has PW.

PW and DCT sets

In the case of $\mathfrak{e} \subset \mathbb{R}$ which are regular (i.e. $G_{\mathfrak{e}}$ continuous on $\mathfrak{e}$), there is one critical point in each gap and the value of $G_{\mathfrak{e}}$ there is the maximum value in the gap.

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$$Q_\chi = F_\chi / \|F_\chi\|_\infty, \quad F_\chi = Q_\chi / Q_\chi(\infty),$$

$$Q_\chi(\infty) = 1/\|F_\chi\|_\infty$$

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Totik Widom Bounds

Recall that we say that $\mathfrak{e} \subset \mathbb{R}$ obeys a Totik-Widom bound if there is a D with $\|T_n\|_{\mathfrak{e}} \leq DC(\mathfrak{e})^n$ and that this was only known for finite gap sets.

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Homogeneous sets are regular and obey a Parreau Widom condition (a theorem of Jones and Marshall). This explicit constant is interesting even for the finite gap case. We also proved a weak converse:

Theorem If $\mathfrak{e} \subset \mathbb{C}$ is a regular set for which a TW bound holds and $\mathfrak{e}$ has a canonical generator, then $\mathfrak{e}$ is a PW set.

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For a time I suspected the answer was yes, then no, and now I'm unsure, but I lean towards yes. A key example is the solid Koch snowflake. Since it is simply connected, it is a PW set. On the other hand the fact that its boundary has dimension greater than 1 makes it a candidate for failure of TW bounds if the theorem does not extend to the general complex case so much so I suspect it was false. But, recently, Andrievskii proved a TW bound holds for a class of sets that includes the Koch snowflake, so I've oscillated back.

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The proof of this in Part 2 is simpler than the proof of Part 1 (and doesn't require Widom's result on the norm apriori but rather proves it).

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Since almost periodicity of the limit is part of Szegő–Widom asymptotics, this is a kind of converse to

DCT⇒Szegő–Widom asymptotics.

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since regularity of $\mathfrak{e}$ implies $G_{\mathfrak{e}}$ vanishes at the ends of each gap so the maximum is taken a critical point w_j.

Exponentiating and using $\|T_n\|_{\mathfrak{e}} \le 2C(\mathfrak{e}_n)^n$ we get the result. ∎

Proof of the Lemma

Because the integrated equilibrium measure of $\mathfrak{e}_n$ is $\frac{1}{\pi n} \arccos\left(\frac{T_n(x)}{\|T_n\|_\mathfrak{e}}\right)$, each band of $\mathfrak{e}_n$ has $\rho_{\mathfrak{e}_n}$ measure $\frac{1}{n}$ and the part of a band from a zero of T_n to a nearby band edge has $\rho_{\mathfrak{e}_n}$ measure $\frac{1}{2n}$.

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Case 2 (T_n *has a zero in* K) By the Alternation Theorem, one of the two extreme points immediately below the zero must lie in $\mathfrak{e}$, so there is at most a half band below the zero. Similarly, at most a half band above, so no more than a full band. ■

Size of $\mathfrak{e}_n \setminus \mathfrak{e}$

In fact, one can prove if there is a zero not too close to a gap edge and n is large, then there is exactly a full exponentially small (in Lebesgue measure) band of $\mathfrak{e}_n$ totally inside K.

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Similarly if, for j large T_{n_j} has a zero, x_j, in K and $x_j \to x_\infty \in K$, then only x_∞ is asymptotically in $\mathfrak{e}_{n_j} \cap K$ in the sense that $\bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} (K \cap \mathfrak{e}_{n_j}) = \{x_\infty\}$.

Overall Strategy

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For any gap set, S, we define the associated Blaschke product

$$B_S(z) = \prod_{K_k \in \mathcal{G}_0} B_{\mathfrak{e}}(z, x_k)$$

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The proof has two steps: first, prove that the limit is the Blaschke product of this gap set and, secondly, prove that any such product is a dual Widom maximizer. The proof of the second half follows, in part, ideas of Volberg–Yuditskii, who considered a related problem and uses some deep 1997 results of Sodin–Yuditskii on the Abel map in this setting.

Limits of M_n are Gap Blaschke Products

Rather than control M_n globally, it suffices, by a little complex analysis, to prove convergence of the absolute values and only for z in a small neighborhood of ∞. Taking into account the formula we had for $|M_n(z)|$ and taking logs, it suffices to prove that, for z near ∞, we have that

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Part 1, we proved Totik–Widom bounds for PW sets, $\mathfrak{e} \subset \mathbb{R}$ by using that when $z = \infty$, we have that

$$h_n(\infty) = \int_{\bigcup_{K_j \in \mathcal{G}} K_j} G_{\mathfrak{e}}(x) d\rho_n(x)$$

A Poisson Formula

We proved this by thinking of $d\rho_n$ as harmonic measure at ∞, i.e. if H is harmonic on $(\mathbb{C} \cup \{\infty\}) \setminus \mathfrak{e}_n$ with boundary values $H(x)$ on $\mathfrak{e}_n$, then $H(\infty) = \int_{\mathfrak{e}_n} H(x) d\rho_n(x)$.

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By using the PW bound and the fact that $\int G_{\mathfrak{e}}(x,z)\delta(x-x_k) = G_{\mathfrak{e}}(x_k,z)$, it suffices to prove that, for all $K \in \mathcal{G}$, we have that

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If the zeros in a gap have a limit, x_k, in the gap, there is a single narrow band of ρ_n–weight $1/n$ near the point so the first case is handled. If there is a zero that approaches an edge, the limit is 0 since regularity implies the Green's function $G_{\mathfrak{e}}(x,z) = G_{\mathfrak{e}}(z,x)$ vanishes as x approaches $\mathfrak{e}$. $\blacksquare$

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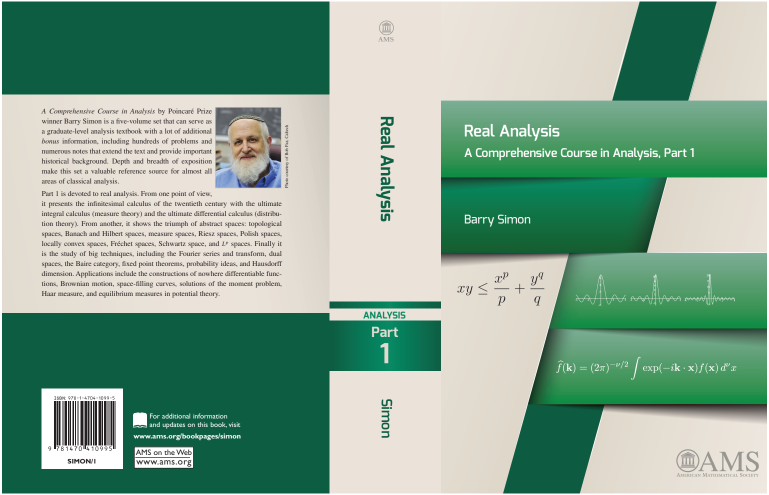

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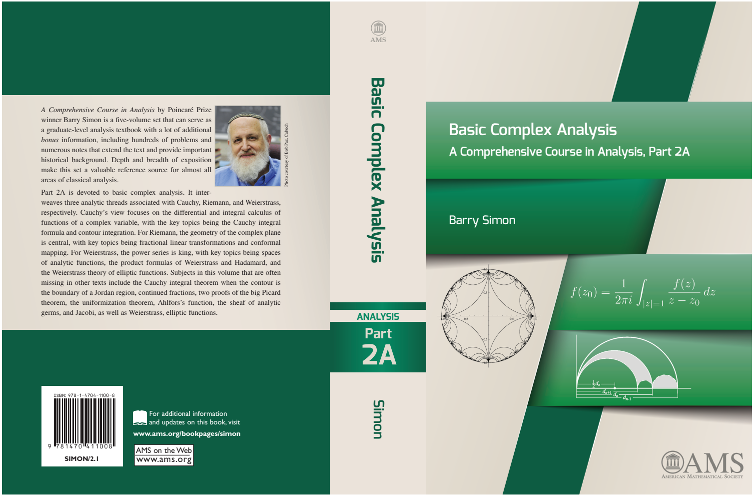

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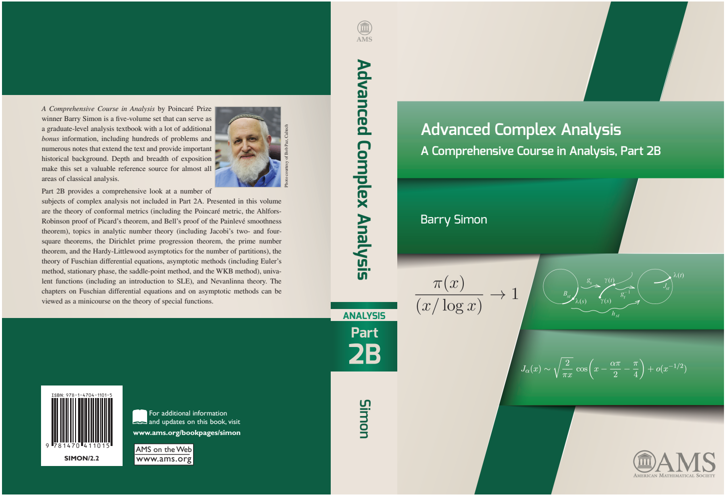

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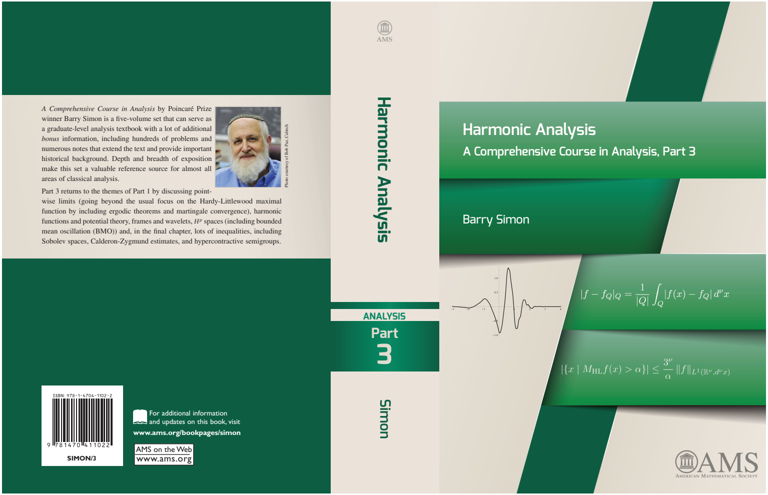

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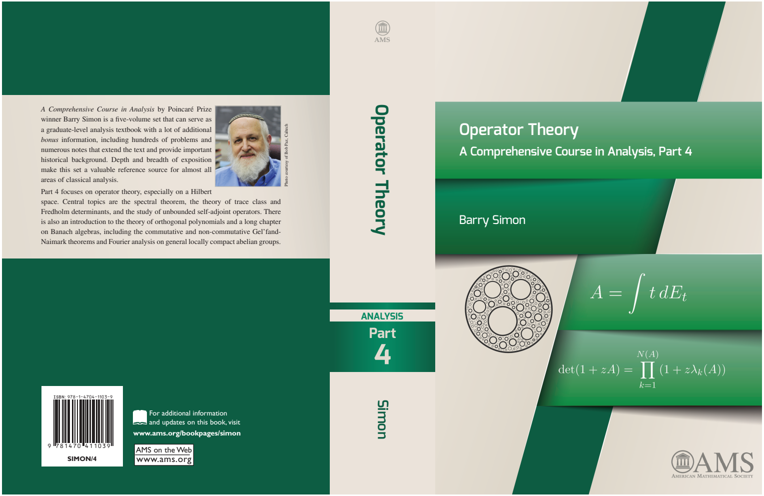

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