REAL ANALYSIS GENERAL EXAM FALL 2022

Solve as many problems as you can. Full solutions on a smaller number of problems will be worth more than partial solutions on several problems.

Problem 1.

Let (X, μ) be a measure space, and for $n \in \mathbb{N}$, let $f_n \colon X \to [0, +\infty)$ be measurable and satisfy $\int f_n \, d\mu = 1$.

(a) Show that $\int \sum_{n=1}^{\infty} n^{-2} f_n \, d\mu < +\infty$.

(b) Use the previous part to show that for almost every $x \in X$, we have that

$$\lim_{n \to \infty} \frac{f_n(x)}{n^2} = 0.$$

Problem 2.

Let $f \in L^1([0,1])$.

(a) Show that $\int_0^1 \int_x^1 \frac{|f(t)|}{t} \, dt \, dx < +\infty$.

(b) Show that $\int_x^1 \frac{|f(t)|}{t} \, dt < +\infty$ for almost every $x \in [0,1]$ and that if $g(x) = \int_x^1 \frac{f(t)}{t} \, dt$, then $g \in L^1([0,1])$ and $\int_0^1 g(x) \, dx = \int_0^1 f(x) \, dx$.

Problem 3.

Let $(X, \mu), (Y, \nu)$ be σ-finite measure spaces. Suppose that

$$B \colon L^2(X, \mu) \times L^2(Y, \nu) \to \mathbb{C}$$

satisfies the following:

- $f \mapsto B(f, h)$, $g \mapsto B(k, g)$ are linear for all $h \in L^2(Y, \nu), k \in L^2(X, \mu)$.
- there is a $C \geq 0$ so that $|B(f, g)| \leq C \|f\|_2 \|g\|_2$ for all $f \in L^2(X, \mu), g \in L^2(Y, \nu)$.

Show that:

(a) for every $f \in L^2(X, \mu)$ there is a $T(f) \in L^2(Y, \nu)$ with

$$B(f, g) = \int T(f) g \, d\nu, \text{ for all } g \in L^2(Y, \nu).$$

Show that such a $T(f)$ is unique up to equality almost everywhere.

(b) the map T constructed in part (a) is linear, and

$$\|T(f)\|_2 \leq C \|f\|_2 \text{ for all } f \in L^2(X, \mu).$$

Problem 4.

Let X be a set and $\mathcal{F} \subseteq \Sigma$ be σ-algebras of subsets of X. Let $\mu \colon \Sigma \to [0,1]$ be a probability measure.

(a) Given $f \in L^1(X, \Sigma, \mu)$ show that there is $g \in L^1(X, \mathcal{F}, \mu)$ (i.e. a $g \colon X \to \mathbb{C}$ which is L^1 and $\mathcal{F}$-measurable) so that

$$\int_E f \, d\mu = \int_E g \, d\mu$$

for all $E \in \mathcal{F}$. If $\widetilde{g} \in L^1(X, \mathcal{F}, \mu)$ is another such function, show that $g = \widetilde{g}$ almost everywhere.

(b) For $f \in L^1(X, \Sigma, \mu)$, denote the g in part (a) by $\mathbb{E}_{\mathcal{F}}(f)$. Show that for all $h \in L^\infty(X, \mathcal{F}, \mu), f \in L^1(X, \Sigma, \mu)$ we have

$$\int fh\, d\mu = \int \mathbb{E}_{\mathcal{F}}(f) h\, d\mu.$$

Problem 5.

Let $E \subseteq [0,1]$ be a Borel set with $0 < m(E \cap [a,b]) < b - a$ for all $0 \leq a < b \leq 1$ (you may assume the existence of such a set for this problem). Define $f\colon [0,1] \to [0,1]$ by $f(x) = m(E \cap [0,x])$ (here m is Lebesgue measure). Show that f is continuous, strictly increasing, and that for almost every $x \in E^c$ we have that $f'(x)$ exists and is 0. (It will be helpful to use the Lebesgue differentiation theorem).