Topology General Exam **January 8, 2025**

Student ID: ________________________

Instructions: This is a four hour exam. Your solutions should be legible and clearly organized, written in complete sentences in good mathematical style. All work should be your own—no outside sources are permitted—using methods and results from the first year topology course topics. There are 8 problems; each problem is worth the same number of points.

1. Show that any map $\mathbb{R}P^2 \vee \mathbb{R}P^2 \to S^1$ is nullhomotopic.

2. Consider the space $X = T \vee S^2$, where $T = S^1 \times S^1$ is the torus and S^2 is the 2-sphere.

(a) Compute $\pi_1(X)$.

(b) Compute the homology groups of X.

(c) Describe the isomorphism classes of covering spaces of X.

3. Consider a commutative diagram of abelian groups:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r, "i"] \arrow[d] & A_2 \arrow[r, "j"] \arrow[d] & A_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B_1 \arrow[r] \arrow[d] & B_2 \arrow[r] \arrow[d] & B_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_1 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

Suppose that all columns are exact, and that the two bottom rows are exact. Prove that the top row is exact at A_2, i.e. that $\mathrm{im}(i) = \ker(j)$.

4. Let $f\colon S^n \to S^n$ be a smooth map that has no fixed points, where S^n is the n-sphere. Find the degree of f.

5. Consider $\mathrm{SL}(2,\mathbb{R})$, the set of 2×2 matrices of determinant 1.

(a) Prove that $\mathrm{SL}(2,\mathbb{R})$ is a smooth manifold.

(b) Prove that the tangent space $\mathrm{T}_I\,\mathrm{SL}(2,\mathbb{R})$ at the identity matrix is the set of 2×2 traceless matrices (i.e. matrices with trace equal to zero).

(c) Prove that for an arbitrary matrix $A\in\mathrm{SL}(2,\mathbb{R})$, the tangent space $\mathrm{T}_A\,\mathrm{SL}(2,\mathbb{R})$ consists of matrices of the form $A\cdot M$ where the trace of M is zero.

6. Consider S^2, the unit sphere in $\mathbb{R}^3$, and let $f\colon S^2 \to \mathbb{R}^2$ be given by

$$f(x, y, z) = (x^2 + y^2, z^2 + 1).$$

Consider the submanifold L of $\mathbb{R}^2$ consisting of all points $\{(t, t) \mid t \in \mathbb{R}\}$.

Is f transverse to L? (That is, is it true that $\mathrm{d}_p f(\mathrm{T}_p S^2) + \mathrm{T}_{f(p)} L = \mathrm{T}_{f(p)} \mathbb{R}^2$ for all $p \in f^{-1}(L)$?)

7. Consider the 1-form $\omega = \sum_{i=1}^{n} x_i \, \mathrm{d}x_i$ on $\mathbb{R}^n$. Show that there exists an $(n-1)$-form $\alpha \in \Omega^{n-1}(\mathbb{R}^n \smallsetminus \{0\})$ such that

$$\omega \wedge \alpha = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n.$$