Topology General Exam
August 24, 2012

Name: ______________________________

Instructions: This is a four hour exam and 'closed book'. There are eight problems.

1. (a) Suppose that $t \in \mathbb{R}$ is a regular value of a smooth map $f : \mathbb{R}^n \to \mathbb{R}$, and let $M = f^{-1}(t)$. Explain why M has a nowhere vanishing normal vector field.

(b) If $f(x, y, z) = x^2 + y^2 + z^2$, check that the hypothesis of part (a) holds when $t = 1$, and then draw a picture illustrating the conclusion.

2. Rigorously prove that the Möbius band is non–orientable.

3. (a) Let M and N be smooth connected closed (= compact without boundary) manifolds of the same dimension. Show that a submersion $f : M \to N$ will then be a finite sheeted covering map. (a submersion = a map whose differential is surjective at each point.)

(b) Explain why if M is a connected closed surface, and $f : M \to S^2$ is a submersion, then f must, in fact, be a diffeomorphism.

(c) Explain why if M is a connected closed surface, and $f : M \to S^1 \times S^1$ is a submersion, then M must be $S^1 \times S^1$.

4. Let $S^2 \xleftarrow{p_1} S^2 \vee S^2 \xrightarrow{p_2} S^2$ be the two 'projection maps': the other sphere is collapsed to the basepoint. Then say that a map $f : S^2 \to S^2 \vee S^2$ has type (m, n) if the degree of $p_1 \circ f$ is m and the degree of $p_2 \circ f$ is n. Let $X_f = (S^2 \vee S^2) \cup_f D^3$.

(a) Compute the homology groups of X_f if f has type $(4, 6)$, describing the homology groups as direct sums of cyclic groups, as usual.

(b) More generally, describe the homology groups of X_f if f has type (m, n).

5. Suppose that X is the union of open sets X_1 and X_2, and Y is the union of open sets Y_1 and Y_2. Let $f : X \to Y$ be a map that restricts to maps $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$, and thus also $f_{12} : X_1 \cap X_2 \to Y_1 \cap Y_2$.

Prove that, if f_1, f_2 and f_{12} all induce isomorphisms in homology, then $f_* : H_*(X) \to H_*(Y)$ will also be an isomorphism.

6. Suppose $p : \tilde{Y} \to Y$ is a double cover. If X is a space such that $H_1(X)$ is a finite group of odd order, show that any map $f : X \to Y$ lifts through p: there exists $\tilde{f} : X \to \tilde{Y}$ such that $f = p \circ \tilde{f}$. (You can assume that X is locally 'friendly'.)

7. Let $M_2(\mathbb{R})$ be the vector space of all 2×2 real matrices, and let $f : M_2(\mathbb{R}) \to \mathbb{R}$ be given by $f(A) = \det(A)$. The differential of f at $A \in M_2(\mathbb{R})$ is a linear map $d_A f : M_2(\mathbb{R}) \to \mathbb{R}$.

(a) Compute $d_A f(A)$.

(b) Show that $SL_2(\mathbb{R})$, the group of 2×2 real matrices with determinant 1, is a smooth submanifold of $M_2(\mathbb{R})$.

(c) Show that $T_I SL_2(\mathbb{R})$, the tangent space of $SL_2(\mathbb{R})$ at the identity matrix I, is the subspace of $M_2(\mathbb{R})$ consisting of matrices with trace equal to 0.

8. Recall that the Brower Fixed Point Theorem says that every continuous self map of the closed n–ball D^n has a fixed point.

(a) Prove the theorem using homology.

(b) Prove the theorem using the methods of differential topology methods. (Step 1: If a continuous f had no fixed points, a nearby smooth function would also have no fixed points.)