COMPLEX ANALYSIS GENERAL EXAM FALL 2023

Solve as many problems as you can. Full solutions on a smaller number of problems will be worth more than partial solutions on several problems. Don't use any of the Picard theorems.

Question 1. For $\xi \in \mathbb{R}$, compute

$$\int_{-\infty}^{\infty} \frac{\cos(\xi x)}{x^2 + 4x + 5}\, dx.$$

You must justify all estimates.

Question 2. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function whose derivative satisfies

$$|f'(z)| \leq e^{\sqrt{\log(|z|+1)}}$$

for all $z \in \mathbb{C}$. Show that there exist constants $a, b \in \mathbb{C}$ with $|a| \leq 1$ such that $f(z) = az + b$.

Question 3. Let $U := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$. Suppose that $f : \overline{U} \to \mathbb{C}$ is continuous, $\sup_{z \in \partial U} |f(z)| \leq 1$, $f|_U$ is holomorphic, and that there exist $M, \alpha > 0$ such that $|f(z)| \leq Me^{\alpha\sqrt{|z|}}$ for all $z \in U$. Show that $|f(z)| \leq 1$ for all $z \in U$. (Hint: for fixed $\varepsilon > 0$, show that the desired estimate holds for $f_\varepsilon(z) := f(z)e^{-\varepsilon z^{3/4}}$.)

Question 4. Given, $n, m \in \mathbb{N}$ with $m < 2n$ and $a, b \in \mathbb{R}$ with $b > 0$ and $a \in (-b^{1-\frac{m}{2n}}, b^{1-\frac{m}{2n}})$, show that the polynomial $P(z) := z^{2n} + az^m + b$ has exactly n zeroes (counted with multiplicity) with positive imaginary parts. (Hint: consider the case $a = 0$ first.)

Question 5. Let (f_n) be a sequence of holomorphic functions from $\mathbb{D}$ to $\mathbb{C}$ that is locally bounded, where $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ denotes the unit disc centred at the origin. Suppose that there exists a holomorphic function $f : \mathbb{D} \to \mathbb{C}$ such that the set $A = \{z \in \mathbb{D} : \lim_{n\to\infty} f_n(z) = f(z)\}$ has a limit point in $\mathbb{D}$. Show that (f_n) converges uniformly on compact sets to f.