COMPLEX ANALYSIS GENERAL EXAM JANUARY 2025

Solve as many problems as you can. Full solutions on a smaller number of problems will be worth more than partial solutions on several problems. Don't use any of the Picard theorems. Throughout, $\mathbb{D}$ denotes the unit disc $\{z \in \mathbb{C} : |z| < 1\}$.

Question 1. Let $f : \overline{\mathbb{D}} \to \mathbb{C}$ be a continuous function for which $f|_{\mathbb{D}}$ is holomorphic and $|f(z)| = 1$ for all $z \in \partial\mathbb{D}$. Show that that there exists a finite collection of points $z_1, \ldots, z_n \in \mathbb{D}$ and some $\theta \in \mathbb{R}$ such that

$$f(z) = e^{i\theta} \prod_{j=1}^{n} \frac{z_j - z}{1 - \overline{z_j} z}.$$

Question 2. Let $f : \mathbb{C} \setminus \{2, 3\} \to \mathbb{C}$ be the holomorphic function

$$f(z) := \frac{z}{(z-2)(z-3)}.$$

For each of the following regions, determine the Laurent series of f that converges on the given region.

(a) $\{z \in \mathbb{C} : |z| < 2\}$
(b) $\{z \in \mathbb{C} : 2 < |z| < 3\}$
(c) $\{z \in \mathbb{C} : |z| > 3\}$

Question 3. Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a harmonic function, and suppose that $u(x, y) \leq x$ for all $(x, y) \in \mathbb{R}^2$. Show that there exists a constant $c \in (-\infty, 0]$ such that $u(x, y) = x + c$ for all $(x, y) \in \mathbb{R}^2$.

Question 4. Using Cauchy's residue theorem, show that for $a, b \in \mathbb{R}$ with $0 < b < a$,

$$\int_0^{2\pi} \frac{1}{(a + b\sin\theta)^2} \, d\theta = \frac{2\pi a}{(a^2 - b^2)^{3/2}}.$$

Question 5.

(a) Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function satisfying $f(0) = \frac{2}{3}$. Show that $|f'(0)| \leq \frac{5}{9}$.
(b) Determine all holomorphic functions $f : \mathbb{D} \to \mathbb{D}$ that simultaneously satisfy both $f(0) = \frac{2}{3}$ and $|f'(0)| = \frac{5}{9}$.