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On higher Turán inequalities for the plane partitions, ellipsoidal T -designs, and j -inversion

This thesis is about combinatorics and number theory. More precisely, the content of our thesis is on *Higer Turán inequalities for plane partitions* [2], introduction of a generalization of spherical t -designs, which we call *ellipsoidal t -design* [3], and providing *inversion formulae for j -function around elliptic points* [1].

Plane partition function is a 2-dimensional analog of partition function, where you study the number of ways a number can be written (in a nice order) in an array. Here we study the roots of the doubly infinite family of Jensen polynomials $J_{\text{PL}}^{d,n}(x)$ associated to MacMahon's plane partition function $\text{PL}(n)$. Recently, Ono, Pujahari, and Rolen proved, for each $d \geq 2$, that the $J_{\text{PL}}^{d,n}(x)$ have all real roots for sufficiently large n . Here we make their result effective. Namely, if $N_{\text{PL}}(d)$ is the minimal integer such that $J_{\text{PL}}^{d,n}(x)$ has all real roots for all $n \geq N_{\text{PL}}(d)$, then we show that

$$N_{\text{PL}}(d) \leq 279928 \cdot d(d-1) \cdot \left(6d^3 \cdot (22.2)^{\frac{3(d-1)}{2}}\right)^{2d} e^{\frac{\Gamma(2d^2)}{(2\pi)^{2d+2}}}.$$

Moreover, using the ideas that led to the above inequality, we explicitly prove that $N_{\text{PL}}(3) = 26$, $N_{\text{PL}}(4) = 46$, $N_{\text{PL}}(5) = 73$, $N_{\text{PL}}(6) = 102$ and $N_{\text{PL}}(7) = 136$.

A spherical t -design is a finite set of points on a sphere such that integration of a polynomial of degree less than or equal to t is same as averaging over this set. We generalize this notion to special ellipses and call it ellipsoidal T -design (where T is a potentially infinite set), and as an example offer fixed norm shells in rings of integers of imaginary quadratic fields with class number 1.

One of the most fundamental results in the theory of elliptic functions is the inversion formulas for j -function around infinity. Recently, Hong, Mertens, Ono, and Zhang proved a conjecture of Căldăraru, He, and Huang that expresses the Taylor series of the modular j -function around the elliptic points i and $\rho = e^{\pi i/3}$ as rational functions. We extend these results and give inversion formulas for the j -function around i and ρ arising from Gauss' hypergeometric functions.

References

- [1] A. De Las Penas Castano and B. V. Pandey. Inversion formulas for the j -function around elliptic points. *Archiv der Mathematik*, 119:359–369, 2022.
- [2] B. V. Pandey. Higer Turán inequalities for plane partitions. *arXiv:2210.08617*
- [3] B. V. Pandey. Modular forms and ellipsoidal t -designs. *The Ramanujan Journal*, 58:1245–1257, 2022.

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