Class 12: Review

Schedule

Problem Set 5 is due **Friday at 6:29pm**.

See Class 11 Notes for information and preparation advice for Exam 1, which will be in class next Thursday, 5 October.

Strong Induction Principle

Let P be a predicated on $\mathbb{N}$. If

– $P(0)$ is true, and
– $(\forall m \in \mathbb{N}, m \leq n.P(n)) \implies P(n+1)$ for all $n \in \mathbb{N}$,

then

– $P(m)$ is true for all $m \in \mathbb{N}$.

As an inference rule:

$$\frac{P(0), \forall n \in \mathbb{N}.(P(0) \vee P(1) \wedge \cdots \wedge P(n)) \implies P(n+1)}{\forall m \in \mathbb{N}.P(m)}$$

With arbitrary basis, $b \in \mathbb{N}$:

$$\frac{P(b), \forall n \in \mathbb{N}.(P(b) \vee P(b+1) \wedge \cdots \wedge P(n)) \implies P(n+1)}{\forall m \in \{b, b+1, b+2, \ldots\}.P(m)}$$

Show that *strong* induction is not actually stronger than regular induction. (Hint: if the predicate for strong induction is $P(m)$, explain how to construct a predicate, $P'(m)$, that works with regular induction.

Example Strong Induction Proof

Theorem: Every number, $n \in \mathbb{N}$ can be written as $\alpha \cdot 2 + \beta \cdot 5$ where $\alpha, \beta \in \mathbb{N}$.

Proof by Strong Induction:

1. First we need to define the predicate:

$$P(n) := \exists \alpha, \beta \in \mathbb{N}.n = \alpha \cdot 2 + \beta \cdot 5$$

.

2. Basis: we are proving for all $n > 3$:

   $P(4)$: $\alpha = 2, \beta = 0$ gives $4 = 2 \cdot 2 + 0 \cdot 5$.
   $P(5)$: $\alpha = 0, \beta = 1$ gives $5 = 0 \cdot 2 + 1 \cdot 5$.

3. Induction step: $\forall n \in \{5, 6, \ldots\}$

   By strong induction, assume $P(m)$ is true for all $m \in 4, 5, 6, \ldots, m$.

   Show $P(m+1)$: Since $P(m-1)$ is true (but the strong induction hypothesis), we know $\exists \alpha, \beta \in \mathbb{N}. m - 1 = \alpha \cdot 2 + \beta \cdot 5$. We can show $P(m+1)$ since $m + 1 = (\alpha + 1) \cdot 2 + \beta \cdot 5$.

Proof by Contra-Positive (Review)

Recall: $P \implies Q$ is equivalent to $\neg Q \implies \neg P$. (If you are shaky on this, prove it to yourself using a truth table.)

Typical use: where the negation of the proposition is easier to reason about than the original proposition (e.g., irrational is a complex property to describe, but rational (NOT irrational) is a simple one.

Proof by Contradiction (Review)

To prove P, show $\neg P \implies False$.

Example: Proving the $\mathbb{Z}$ is not well ordered.

Goal: $G := \mathbb{Z}$ has no minimum.

1. To prove by contradiction, assume $\neg G$ (that is, $\mathbb{Z}$ does have a minimum).
2. Then, $\exists m \in \mathbb{Z}$ that is the minimum of $\mathbb{Z}$.
3. But, this leads to a contradiction: $m - 1 \in \mathbb{Z}$ and $m - 1 < m$. So, the m we said was the minimum of $\mathbb{Z}$ is not the minimum.
4. Thus, we have a contradiction, so something must be wrong. All our logical inferences after step 1 are correct, so the assumption we made in step 1 must be invalid. If $\neg G$ is invalid, G must be true.

