Problem Set 3

Deliverable: Submit your responses as a single PDF file on the collab site before **6:29pm** on **Friday, 16 September**. The PDF you submit can be a scanned handwritten file (please check the scan is readable), or a typeset PDF file (e.g., generated by LaTeX or Word).

Collaboration Policy - Read Carefully

For this assignment, you should work in groups of *one* to *four* students of your choice. The only constraint on teams for PS3 is that if you worked with the same team for PS1 and PS2, you must not work with any of the same people for PS3.

The rest of the collaboration policy is identical to what it was on PS2, and is not repeated here.

Preparation

This problem set focuses on Chapter 3 (especially 3.4-3.6) of the MCS book, and Class 5 and Class 6 (which include some material not in Chapter 3).

Directions

Solve all 10 problems. For maximum credit, your answers should be correct, clear, well-written, and convincing. The problems marked with ($\star$) are believed to be challenging enough that it is not necessary to solve them well to get a "green-star level" grade on this assignment (although we certainly hope you will try and some will succeed!)

Quantified Formulas

For each pair of logical formula given, state if the the formula is *valid* and if it is *satisfiable*. You should provide a brief argument supporting your answer (enough to convince a reader you are not just guessing!), but do not need to provide a thorough proof.

$\mathbb{N}$ represents the non-negative integers = $\{0, 1, 2, \cdots\}$.

1. $\forall x \in \mathbb{N}. \exists y \in \mathbb{N}. x = y + 1.$
2. $\forall x \in \mathbb{N}. \forall y \in \mathbb{N}. \exists z \in \mathbb{N}. z = x + y.$
3. $\forall x \in \mathbb{N}. \exists y \in \mathbb{N}. \exists b \in \{0, 1\}. x = 2y + b.$
4. $\forall F \in \text{CNF}. \exists G \in \text{3CNF}. F \equiv G$. (This means F and G are logically equivalent.)

Conjunctive Normal Form

5. Write a logical formula in Conjuctive Normal Form that is equivalent to:

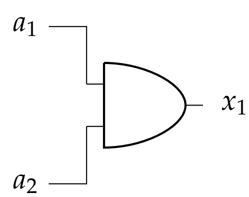
$$(A \vee B) \implies C$$

6. Write a logical formula in 3CNF form that is equivalent to:

$$A \vee \overline{B} \vee C \vee (\overline{D} \wedge E)$$

Use as few clauses as possible.

7. A circuit can be converted to a SAT formula by assigning a variable label to each wire in the circuit, and using clauses to constrain the variable values according to the circuit's logic. Consider a single AND gate circuit shown below with inputs labeled a_1 and a_2 and output labeled x_1. Logically, this means $x_1 = a_1 \wedge a_2$ (but we can't have an equality constraint like this in a SAT formula). Write a 3CNF formula that represents the AND gate.



SAT Solving

For each of the formula, either (1) give a satisfying assignment, or (2) state that it is not satisfiable. (If you follow a smart strategy, these can be solved by hand without a lot of tedious effort. But, you are welcome to use SAT solving programs to solve them, including the simple-sat solver from Class 6.)

8. $(x_1 \vee x_2 \vee \overline{x_3}) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (x_1 \vee \overline{x_2} \vee x_4)$

9. $(x1 \vee x2 \vee x3) \wedge (x1 \vee x2 \vee \overline{x3}) \wedge (x1 \vee \overline{x2} \vee x3) \wedge (x1 \vee \overline{x2} \vee \overline{x3}) \wedge$
$(\overline{x1} \vee x2 \vee x3) \wedge (\overline{x1} \vee x2 \vee \overline{x3}) \wedge (\overline{x1} \vee \overline{x2} \vee x3) \wedge (\overline{x1} \vee \overline{x2} \vee \overline{x3})$

Satisfaction

The length of the formula is the number of clauses, and no clause may be repeated. (The order of literals within a clause doesn't matter, so the clauses $(x_1 \vee x_2 \vee x_3)$ and $(x_3 \vee x_1 \vee x_2)$ would count as the same clauses, but $(x_1 \vee x_2 \vee x_3)$ and $(\overline{x_1} \vee x_2 \vee x_3$ are different clauses.)

10. What is the length of the *shortest* unsatisfiable formula involving 3 variables? (That is, each clause involves x_1, x_2, and x_3, and your goal is to show that there exists an unsatisfiable formula of length l, and any formula of length $< l$ is satisfiable.)

11. ($\star$) What is the length of the *longest* satisfiable formula involving v variables? An outstanding answer would include a convincing proof (hint: well-ordering principle!) that there exists a satisfiable formula with v variables of length l, but no satisfiable formula with v variables of length $l + 1$.

