Problem Set 4

Deliverable: Submit your responses as a single PDF file on the collab site before **6:29pm** on **Friday, 23 September**. Follow the directions on Generating PDFs posted on the course site.

Collaboration Policy - Read Carefully

For this assignment, you should work in groups of *one* to *four* students of your choice with no restrictions. The rest of the collaboration policy is identical to what it was on PS3, and is not repeated here.

Preparation

This problem set focuses on Chapter 4 (up to Section 4.4) of the MCS book, and Class 7 and Class 8.

Directions

Solve as many of the 9 problems as you can. For maximum credit, your answers should be correct, clear, well-written, and convincing. The problems marked with ($\star$) are believed to be challenging enough that it is not necessary to solve them well to get a "green-star level" grade on this assignment (although we certainly hope you will try and some will succeed!)

Sets

For each set S defined below, indicate whether or not it is equivalent to A, where A and B are any sets. Support your answer with a brief explanation.

a. $S = A \cup \emptyset$.
b. $S ::= \{x | x \in A \land x \in \overline{B}\}$
c. $S ::= \{x | x \in A \land x \notin \overline{A}\}$
d. $S ::= A \cap (B \cup A)$.
e. $S ::= A - (B \cap \overline{B})$.

Use the definitions of the set operations to prove that for all sets A and B,

$$A = (A \cap B) \cup (A - B).$$

Prove by double containment. First prove $A \subseteq (A \cap B) \cup (A - B)$.

Let $x \in A$ We have two cases, either $x \in B$ or $x \notin B$

If $x \in B$ then $x \in A \cap B$ so $x \in (A \cap B) \cup (A - B)$

If $x \notin B$ then $x \in (A - B)$ so $x \in (A \cap B) \cup (A - B)$.

Thus $A \subseteq (A \cap B) \cup (A - B)$

The proof $(A \cap B) \cup (A - B) \subseteq A$ is similar.

In Class 7, we defined set difference as:

$$\forall x . x \in A - B \iff x \in A \land x \notin B.$$

Provide an alternate (but equivalent in meaning) definition of set different using only the other defined set operations (you may use any of the union ($\cup$), intersection ($\cap$), and complement ($\overline{S}$) operations in your definition, but no other operations or qualifiers). A good answer will include a proof that shows your definition is equivalent to the original set difference definition. Answer: $(A - B) = A \cap \overline{B}$

Prove by double containment, its like one line each way.

Functions and Relations

For each function described below, identify a *domain* and *codomain* that make the function *total*. For example, for $f(x) ::= 1/x$ you could correctly answer that the domain is $\mathbb{R} - \{0\}$ and codomain is $\mathbb{R}$.

a. $f(x) ::= x + 1$ domain: (R)
   codomain; (R)
b. $f(x) ::= \frac{x}{(x-1)}$ domain: $(R) - \{1\}$
   codomain: (R)
c. $f(S)$ ::= $\text{minimum}_<(S \cap \mathbb{N})$ where $\text{minimum}_<$ is defined for all sets A that are well-ordered by $<$ as:

$$\text{minimum}_<(A) = x \in A \text{ such that } \forall a \in A - \{x\}.\, x < a.$$

   and $<$ is a binary relation on the real numbers.
   domain: $2^{\mathbb{N}} - \{\}$ the powerset of the natural numbers minus the empty set.
   The domain just has to be sets that have non empty intersections with the natural numbers.
   codomain: $\mathbb{N}$

Consider the relation, $<$, with the domain set, $\{1, 2, 3\}$ and codomain set, $\{0, 1, 2\}$.

a. Describe the graph of the relation. Your description can be a picture showing the graph, or some other clear way of defining that graph.

b. Which of these properties does the relation have: *function, total, injective, surjective, bijective.* (You do not need to provide a detailed proof, but should support your answer with a very brief explanation.)

Consider the relation, $\leq$, with the domain set, $\mathbb{N}$ and codomain set $\mathbb{N}$.

Describe the graph of the relation. (For this one, you won't be able to draw a complete picture since the domain set is infinite. Instead, your description can be a picture illustrating the graph in a clear way, or some other clear way of defining that graph.)

Which of these properties does the relation have: *function*, total, *injective*, *surjective*, *bijective*. (You do not need to provide a detailed proof, but should support your answer with a very brief explanation.)

Give an example of a relation $R : A \rightarrow \overline{A}$ that is bijective, for any set A. (You should specify carefully the domain of discourse, needed for $\overline{A}$ to be meaningfully defined.)

(Extracted from MCS Problem 4.23) Five basic properties of binary relations $R : A \to B$ are:

(1) R is a surjection [≥ 1 in]
(2) R is an injection [≤ 1 in]
(3) R is a function [≥ 1 out]
(4) R is total [≥ 1 out]
(5) R is empty [$= 0$ out]

Below are some assertions about a relation R. For each assertion, write the numbers (1, 2, 3, 4, 5 from above) of all properties above that the relation R *must* have (that is, the properties that are implied by the stated assertion); write "none" if R might not have any of these properties.

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Variables $a, a_1, a_2, \cdots$ are elements of $A$, and $b, b_1, b_2, \cdots$ are elements of $B$
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The first answer is provided as an example.
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a. $\forall a. \forall b.\, aRb.$ Answer: (1), (4)
b. $\neg(\forall a. \forall b.\, aRb.$
c. $\forall a. \exists b.\, aRb.$
d. $\forall b. \exists a.\, aRb.$
e. $\forall b. \exists a.\, aRb.$
f. R is a bijection.
g. $\forall a_1, a_2, b. (a_1 R b \land a_2 R b) \implies a_1 = a_2.$
h. $\forall a_1, a_2, b_1, b_2. (a_1 R b_1 \land a_2 R b_2 \land b_1 \neq b_2) \implies a_1 \neq a_2.$

($\star$) Consider the sets A and B where $|A| = n$ and $|B| = m$.

a. Assuming $n = m$ (just for this sub-part), how many *bijective* relations are there $R : A \to B$.
   $n!$ First element has n options, next has $n-1$ and so on.
b. How many *total* functions are there $f : A \to B$.
   m^n each of the n elements in A can point to m different options.
c. How many *partial* functions are there $f : A \to B$. (Note that the set of *partial* functions includes all *total* functions; partial means there *may* be domain elements with no associated codomain element.)
   $(m+1)^n$ One more option for not pointing to any.
d. How many *injective* relations are there $R : A \to B$.

