

## Problem Set 4

**Deliverable:** Submit your responses as a single PDF file on the collab site before **6:29pm** on **Friday, 23 September**. The PDF you submit can be a scanned handwritten file (please check the scan is readable), or a typeset PDF file (e.g., generated by LaTeX or Word).

### Collaboration Policy - Read Carefully

For this assignment, you should work in groups of *one* to *four* students of your choice with no restrictions. The rest of the collaboration policy is identical to what it was on PS3, and is not repeated here.

### Preparation

This problem set focuses on Chapter 4 (up to Section 4.4) of the MCS book, and Class 7 and Class 8.

### Directions

Solve as many of the 9 problems as you can. For maximum credit, your answers should be correct, clear, well-written, and convincing. The problems marked with (\*) are believed to be challenging enough that it is not necessary to solve them well to get a “green-star level” grade on this assignment (although we certainly hope you will try and some will succeed!)

### Sets

1. For each set  $S$  defined below, indicate whether or not it is equivalent to  $A$ , where  $A$  and  $B$  are any sets. Support your answer with a brief explanation.

- a.  $S = A \cup \emptyset$ .
- b.  $S ::= \{x | x \in A \wedge x \in \overline{B}\}$
- c.  $S ::= \{x | x \in A \wedge x \notin \overline{A}\}$
- d.  $S ::= A \cap (B \cup A)$ .
- e.  $S ::= A - (B \cap \overline{B})$ .

2. Use the definitions of the set operations to prove that for all sets  $A$  and  $B$ ,

$$A = (A \cap B) \cup (A - B).$$

3. In Class 7, we defined set difference as:

$$\forall x. x \in A - B \iff x \in A \wedge x \notin B.$$

Provide an alternate (but equivalent in meaning) definition of set difference using only the other defined set operations (you may use any of the union ( $\cup$ ), intersection ( $\cap$ ), and complement ( $\bar{\phantom{x}}$ ) operations in your definition, but no other operations or qualifiers). A good answer will include a proof that shows your definition is equivalent to the original set difference definition.

## Functions and Relations

4. For each function described below, identify a *domain* and *codomain* that make the function *total*. For example, for  $f(x) ::= 1/x$  you could correctly answer that the domain is  $\mathbb{R} - \{0\}$  and codomain is  $\mathbb{R}$ .

- $f(x) ::= x + 1$
- $f(x) ::= \frac{x}{(x-1)}$
- $f(S) ::= \text{minimum}_{<}(S \cap \mathbb{N})$  where  $\text{minimum}_{<}$  is defined for all sets  $A$  that are well-ordered by  $<$  as:

$$\text{minimum}_{<}(A) = x \in A \text{ such that } \forall a \in A - \{x\}. x < a.$$

and  $<$  is a binary relation on the real numbers.

- Consider the relation,  $<$ , with the domain set,  $\{1, 2, 3\}$  and codomain set,  $\{0, 1, 2\}$ .
  - Describe the graph of the relation. Your description can be a picture showing the graph, or some other clear way of defining that graph.
  - Which of these properties does the relation have: *function*, *total*, *injective*, *surjective*, *bijective*. (You do not need to provide a detailed proof, but should support your answer with a very brief explanation.)
- Consider the relation,  $\leq$ , with the domain set,  $\mathbb{N}$  and codomain set  $\mathbb{N}$ .
  - Describe the graph of the relation. (For this one, you won't be able to draw a complete picture since the domain set is infinite. Instead, your description can be a picture illustrating the graph in a clear way, or some other clear way of defining that graph.)
  - Which of these properties does the relation have: *function*, *total*, *injective*, *surjective*, *bijective*. (You do not need to provide a detailed proof, but should support your answer with a very brief explanation.)
- Give an example of a relation  $R : A \rightarrow \bar{A}$  that is bijective, for any set  $A$ . (You should specify carefully the domain of discourse, needed for  $\bar{A}$  to be meaningfully defined.)

8. (Extracted from MCS Problem 4.23) Five basic properties of binary relations  $R : A \rightarrow B$  are:

- (1)  $R$  is a surjection [ $\geq 1$  in]
- (2)  $R$  is an injection [ $\leq 1$  in]
- (3)  $R$  is a function [ $\geq 1$  out]
- (4)  $R$  is total [ $\geq 1$  out]
- (5)  $R$  is empty [= 0 out]

Below are some assertions about a relation  $R$ . For each assertion, write the numbers (1, 2, 3, 4, 5 from above) of all properties above that the relation  $R$  *must* have (that is, the properties that are implied by the stated assertion); write “none” if  $R$  might not have any of these properties.

Variables  $a, a_1, a_2, \dots$  are elements of  $A$ , and  $b, b_1, b_2, \dots$  are elements of  $B$ .

The first answer is provided as an example.

- a.  $\forall a. \forall b. aRb$ .                      Answer: (1), (4)
- b.  $\neg(\forall a. \forall b. aRb)$ .
- c.  $\forall a. \exists b. aRb$ .
- d.  $\forall b. \exists a. aRb$ .
- e.  $\forall b. \exists a. aRb$ .
- f.  $R$  is a bijection.
- g.  $\forall a_1, a_2, b. (a_1Rb \wedge a_2Rb) \implies a_1 = a_2$ .
- h.  $\forall a_1, a_2, b_1, b_2. (a_1Rb_1 \wedge a_2Rb_2 \wedge b_1 \neq b_2) \implies a_1 \neq a_2$ .

9. (★) Consider the sets  $A$  and  $B$  where  $|A| = n$  and  $|B| = m$ .

- a. Assuming  $n = m$  (just for this sub-part), how many *bijective* relations are there  $R : A \rightarrow B$ .
- b. How many *total* functions are there  $f : A \rightarrow B$ .
- c. How many *partial* functions are there  $f : A \rightarrow B$ . (Note that the set of *partial* functions includes all *total* functions; partial means there *may* be domain elements with no associated codomain element.)
- d. How many *injective* relations are there  $R : A \rightarrow B$ .