Class 3: Well-Ordering Principle

Schedule

This week, you should read MCS Chapter 2 and MCS Chapter 3 (at least through the end of Section 3.4).

Problem Set 1 is due **Friday at 6:29pm**.

Office hours started Monday afternoon. See the course calendar for the full office hours schedule.

Notes and Questions

What properties must a sensible ordering function have?

Definition. A set is *ordered* with respect to an ordering relation (e.g., $<$), if two things hold. First: every pair of inequal elements a, b in A either satisfy $a < b$ or $b < a$. Second $a < b$ and $b < a$ should always imply that $a < b$ (this is called the transitivity).

Definition. An ordered set,with respect to an ordering relation (e.g., $<$), is *well-ordered* if all of its non-empty subsets has a minimum element.

Which of these are well-ordered?

– The set of non-negative integers, comparator $<$.
– The set of integers, comparator $<$.
– The set of integers, comparator $|a|< |b|$.
– The set of integers, comparator if |a| = |b|: $a < b$, else: $|a|< |b|$.
– The set of national soccer teams, comparator winning games.

Prove the set of positive rationals is *not* well-ordered under $<$.

Provide a comparison function that can be used to well-order the positive rationals.

Template for Well-Ordering Proofs (Section 2.2)

To prove that $P(n)$ is true for all $n \in \mathbb{N}$:

1. Define the set of counterexamples, $C ::= \{n \in \mathbb{N} | NOT(P(n))\}$.
2. Assume for contradiction that C is ______________.
3. By the well-ordering principle, there must be __________________ , $m \in C$.
4. Reach a contradiction (this is the creative part!). One way to reach a contradiction would be to show $P(m)$. Another way is to show there must be an element $m' \in C$ where $m' < m$.
5. Conclude that C must be empty, hence there are no counter-examples and $P(n)$ always holds.

Example: Betable Numbers. A number is *betable* if it can be produced using some combination of $2 and $5 chips. Prove that all integer values greater than $3 are betable.

Example: Division Property. Given integer a and positive integer b, there exist integers q and r such that: $a = qb + r$ and $0 \leq r < b$.

The whole problem with the world is that fools and fanatics are always so certain of themselves, and wiser people so full of doubts. Bertrand Russell

