Class 8: Sequences, Relations, Functions

Schedule

Problem Set 3 is due **Friday at 6:29pm**.

Sequences

A **sequence** is a mathematical datatype that bears similarities to sets. A sequence S also contains some elements, but we usually refer to them as *components*. There are two major differences between sets and sequences:

1. Components of a sequence are **ordered**. There is either 0, or 1 or 2, or $\ldots$ n components, when the sequence if *finite* or it could be an infinite sequence that has an i'th component for any non-zero natural number i.
2. Different components of a sequence could be equal. For example (a, b, a) has a repeating, and this is a different sequence compared to (a, b, b) even though they both have 3 componetns. If we interprete them as *sets*, then they will be equal sets with 2 elements each.

Cartesian Product

We can use set products to get new sets whose elements are sequences. Cartesian product is a very useful way of doing it.

Set Products. A *Cartesian product* of sets $S_1, S_2, \cdots, S_n$ is a set consisting of all possible sequences of n elements where the i^{th} element is chosen from S_i.

$$S_1 \times S_2 \times \cdots \times S_n = \{(s_1, s_2, \cdots, s_n) | s_i \in S_i\}$$

How many elements are in $A \times B$?

Relations and Binary Relations

A *relation* between some elements from set A and some elements from set B could be represented by putting all such pairs (a, b) in a set P. As you can see, this way, P would be a subset of the cartesian product $A \times B$, namely $P \subseteq A \times B$. More formally we have the following definition.

A **binary relation**, P, is defined with respect to a *domain* set, A, and a *codomain* set, B, and it holds that P is $P \subseteq A \times B$. When we draw P by connecting elements of A to B based on their membership in P, we call this the *graph* of R.

The notion of relations could be generalized to having relations between elements coming from multiple sets A, B, C, and we can also talk about relations of the form $P \subseteq A \times B \times C$, but the binary relation remains a very important data type as it allows us to define *functions*...

Functions

The concept of a function F models is a special form of a binary relation R between A and B where for every element $a \in A$ there is a *unique* element in $b \in B$ that is in relation with a (i.e. $(a, b) \in R$). More formally, we use a direct new notation just reserved for working with functions.

A **function** is a mathematical datatype that associates elements from one set, called the *domain*, with elements from another set, called a *codomain*.

$$f : \textit{domain} \rightarrow \textit{codomain}$$

Defining a function. To define a function, we need to describe how elements in the domain are associated with elements in the codomain.

What are the (sensible) domains and codomains of each function below:

$f(n) ::= |n|$ $\qquad f(x) ::= x^2$ $\qquad f(n) ::= n + 1$ $\qquad f(a, b) ::= a/b$

$f(x) ::= \sqrt{x}$ $\qquad f(S) ::= \text{minimum}_<(S)$

Properties of Relations and Functions

If the function is *total*, every element of the domain has one associated codomain element; if the function is *partial*, there may be elements of the domain that do not have an associated codomain element. Note that, unless specified otherwise, we assume that a function is *total* (this is how we defined a function) but we use the term *partial* to denote cases where some inputs do not have an output, and reserve the word *total* to emphasize that no such thing exists.

For which of the above-listed functions are *total*?

We call a function F from A to B *surjective* if every element in B is "covered", namely,

$$\forall y \in B. \exists x \in A. F(x) = y.$$

A similar definition could be defined for any binary relation. We also call a function F *injective* if for any element y in the codomain B there is *at most* one x for which $F(x) = y$. How do you write this formally using quantifiers?

When a function is total, surjective, and injective, then we say this is a *bijection*.

For each statement below, and using the notion of graphs for binary relations and functions, give the name of the described case, and give at least one example.

– A binary relation where no element of A has more than one outgoing edge:

– A binary relation where every element of A has exactly one outgoing edge:

– A binary relation where every element of B has exactly one incoming edge:

– A binary relation where every element of A has exactly one outgoing edge and every element of B has exactly one incoming edge:

If there exists a *bijective* relation between S and T defined by the graph G which of these *must* be true:

a. there exists some *injective* relation between S and T.
b. there exists some *bijective* relation between T and S.
c. there exists a *total* function, $f : S \to T$.
d. $S - T = \emptyset$.
e. the number of elements of S is equal to the number of elements of T.

