# Test 1

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|---|---|
| **Name:** | |
| **UVA Computing ID:** | |
| **Enrolled Section** (Fill in the correct circle): | ◯ 10AM (David Evans)<br>◯ 2PM (Aidan San) |
| **Honor Pledge** (Sign your name): | On my honor as a student, I have neither given nor received unauthorized aid on this exam. |

## Instructions

– This exam is to be taken individually. You will have 50 minutes for this test.

– You are permitted to use a single paper page of notes (no larger than 8.5 x 11 inches) you prepared. You may not use any special devices (e.g., magnifying glasses) to read your page.

– No other resources, other than your own brain, body, writing instrument, and single note sheet are permitted during the exam.

– Write your UVA Computing ID (e.g., mst2k) clearly on the top of each page.

– Print clearly. We can only give credit for what we recognize as correct.

– Write your answers in the provided boxes. You are free to use the rest of each page for scratch paper. If you need more space for your answer (which you shouldn't), make sure it is clearly marked from the answer box where to find your answer.

– Please check all sides of the all the papers before you submit your exam.

## Definitions and Theorems

This page provides definitions we use in the exam problems and theorems that you are free to use without needing to restate or prove.

### Odds and Evens

**Definition of Odd:** An integer, $z$, is odd if and only if there exists an integer $k$ such that $z = 2k + 1$.

**Definition of Even:** An integer, $z$, is even if and only if there exists an integer $k$ such that $z = 2k$.

**Definition of Square:** The square of an integer $x$ is the result of multiplying $x$ by itself.

**Odd Square Theorem:** If $n$ is an odd integer, then $n^2$ is odd.

**Even Square Theorem:** If $n^2$ is even, then $n$ must be even.

**Even-Not-Odd Lemma:** For any natural number $n$, if $n$ is even, then $n$ is not odd.

**Odd-Not-Even Lemma:** For any natural number $n$, if $n$ is odd, then $n$ is not even.

**Not-Even-Odd Lemma:** For any natural number $n$, if $n$ is not even, then $n$ is odd.

**Not-Odd-Even Lemma:** For any natural number $n$, if $n$ is not odd, then $n$ is even.

### Rationality

**Definition of Rational:** A number is rational if and only if it can be written as a ratio of two integers.

**Definition of Irrational:** A real number is irrational if and only if it cannot be written as a ratio of two integers.

**Definition of Prime:** A positive integer is prime if it has exactly one positive divisor other than 1.

**Definition of Divides:** An integer $a$ divides another integer $b$ if and only if there exists an integer $d$ such that $ad = b$.

### Logic

**Contrapositive Inference Rule:**

$$\frac{\text{NOT}(B) \implies \text{NOT}(A)}{A \implies B}$$

**De Morgan's Laws:**

$$\text{NOT}(A \text{ AND } B) = \text{NOT}(A) \text{ OR } \text{NOT}(B)$$

$$\text{NOT}(A \text{ OR } B) = \text{NOT}(A) \text{ AND } \text{NOT}(B)$$

  

**Problem 1** *Inference Rules*

For each candidate rule below, indicate whether or not the rule is sound. Support your answer with a convincing argument. The variables $P$ and $Q$ are Boolean propositions (either true or false).

**(a)** $$\frac{\text{NOT}(\text{NOT}(\text{NOT}(P)))}{P}$$

- ◯ Sound
- ◯ Not Sound
- ◯ Cannot be determined if it is sound or not sound

**(b)** $$\frac{P \implies Q}{\text{NOT}(P) \text{ OR } Q}$$

- ◯ Sound
- ◯ Not Sound
- ◯ Cannot be determined if it is sound or not sound

  

**Problem 2** *Spot the Proof Bugs*

**(a)** Bogus Proof that 2102 = 2120.

1. We prove the proposition $P ::= 2102 = 2120$ using direct proof.
2. By the definition of even, $2102$, which is even, can be written as $2k$ for some integer $k$.
3. By the definition of even, $2120$, which is even, can be written as $2k$ for some integer $k$.
4. We know $2k = 2k$, and substituting from (2) on the left side and (3) on the right side gives, $2102 = 2120$.
5. Thus, we have proven $P$. $\square$

**Explain the flaw in the proof above:**

**(b)** Bogus proof that $\sqrt{5}$ is irrational:

$P$: $\sqrt{5}$ is irrational.

1. We use proof by contradiction.
2. Suppose $P$ is false: that the $\sqrt{5}$ is rational.
3. By the definition of rational, there exist two integers $a$ and $b$ such that $\sqrt{5} = \frac{a}{b}$
4. We can find an equal ratio that is in lowest terms, $\frac{c}{d} = \frac{a}{b}$
5. Using arithmetic,
   $\sqrt{5} = \frac{c}{d}$
   $5 = \frac{c^2}{d^2}$
   $5d^2 = c^2$

   So $5$ divides $c^2$.
6. Using the Even Square Theorem, since $5$ divides $c^2$, $5$ also divides $c$.
7. Since $5$ divides $c$ we can express $c = 5k$ using the definition of divides.
8. Substituting, we get $5d^2 = (5k)^2$.
9. Using arithmetic,
   $5d^2 = (5k)^2$
   $5d^2 = 25k^2$
   $d^2 = 5k^2$

   So using similar logic, $5$ divides $d$.
10. Since $5$ divides both $c$ and $d$, they have a common divisor $5$, but we said that $\frac{c}{d}$ was in lowest terms.
11. Thus we have a contradiction and $\sqrt{5}$ must be irrational. □

**Explain the flaw in the proof above:**

**Problem 3** *Product of Rationals*

Prove that the product of any two rational numbers must be a rational number.

**Problem 4** *Proof of Irrationality*

Prove that the product of a positive rational number and an irrational number must be irrational.

**End of Test 1!**
Please check that you filled in your UVA Computing ID on each page.

  