Problem Set 7: Countability

Response by: **TODO: replace this with your name (and computing id)**

Submit your responses as a single PDF file to gradescope before **8:29pm** on
Thursday, 10 April.

Preparation

This problem set includes material covered in Chapter 8 (*Infinite Sets*, but only through 8.1.2) of the *MCS book*, and focuses on the new concepts introduced in Classes 32–33 (but also builds upon everything we have done so far this semester).

Collaborators and Resources: TODO: replace this with your collaborators and resources (if you did not have any, replace this with *None*)

Collaboration and Resources Policy

Note: identical to previous problem sets but repeated here, but with the **modification that you should include specifics of AI tool use in each answer where you use an AI tool**.

Remember to follow the course pledge you read and signed at the beginning of the semester.

For this assignment, you may discuss the problems and work on solutions with anyone you want (including other students in this class), but you must write your own solutions and understand and be able to explain all work you submit on your own.

To confirm your own understanding, after discussing the problems with others, you should attempt to write your solutions on your own without consulting any notes from group work sessions. If you get stuck, you may visit notes from the group work sessions, but should make sure you understand things well enough to produce it on your own. You may also use any external resources you want, with the exception of solutions and comments from previous offerings of this course.

Since the staff and students benefit from being able to both reuse problems from previous years, and from being able to provide detailed solutions to students, it is important that students do not abuse these materials even if it is easy to find them. Using solutions from last year's course would be detrimental to your learning in this course, and is considered an honor violation.

If you use resources other than the class materials, lectures, and course staff, you should document this and mention it clearly on your submission. For everyone other than the course staff you work with, you should credit them clearly on your assignment. If you use any AI tools like ChatGPT or Claude (which we do encourage, so long as you are using them to learn!), you should explain how they used them and include a URL that links to a transcript of your interactions. You should describe the general AI tool use in the Collaborators and Resources box, and also include specific descriptions as part of the answer response for each question where you used an AI tool for assistance.

Directions

(Almost identical to PS6)

1. Follow the steps as in previous problem sets to create your own copy of the template in https://www.overleaf.com/read/rypcqjdgmvsy#273e12.

2. Solve all the problems and put your responses in the clearly marked answer boxes. For full credit, your answers should be correct, clear, well-written, and convincing.

3. Before submitting, make sure to list your collaborators and resources by replacing the TODO in `\collaborators{TODO: replace ...}` with your collaborators and resources. Check the policy in the pink box on the front page to make sure you understand what you need to document here.

4. Download your complete ps6.pdf file, and submit it using gradescope.

Problem 1 *Countable Sets*

(a) Prove that the set of problems you are asked to solve for this problem set is *countable*.

(b) Prove that the set of prime numbers is *countable*.

Problem 2 *Finite and Infinite Sets*

(a) Prove that every subset of a finite set is *finite*.

(b) Prove that if A is an infinite set, and $B \subseteq A$ and B is finite, that $A - B$ must be an *infinite* set.

Problem 3 *Cardinality of Union*

Prove that if A and B are infinite sets, $A \cup B$ has the same cardinality as A or has the same cardinality as B. (This is a logical or, so it could have the same cardinality as both A and B.)

Problem 4 *Cardinality of Many Unions*

($\star$) Prove that if $\forall n \in \mathbb{N} . S_n$ is a countable set, then

$$\bigcup_{n \in \mathbb{N}} S_n$$

is a countable set. (You may use the theorem from the previous problem in your proof, whether or not you were able to prove it.)

End of Problem Set 7!
Remember to follow the instructions to prepare and submit your PDF (which like PS6 should still include all of the directions and start with your answers on page 3) and remember to complete *collaborators* with information on your collaborators and the resources you used (including describing and linking to a transcript of any AI tool use for each problem where you used an AI tool).