University of Amsterdam
Informatics Institute

AMLAB Amsterdam Machine Learning Lab

Machine Learning 1

Lecture 5.2 - Supervised Learning
Bayesian Linear Regression - Bayesian Model Comparison

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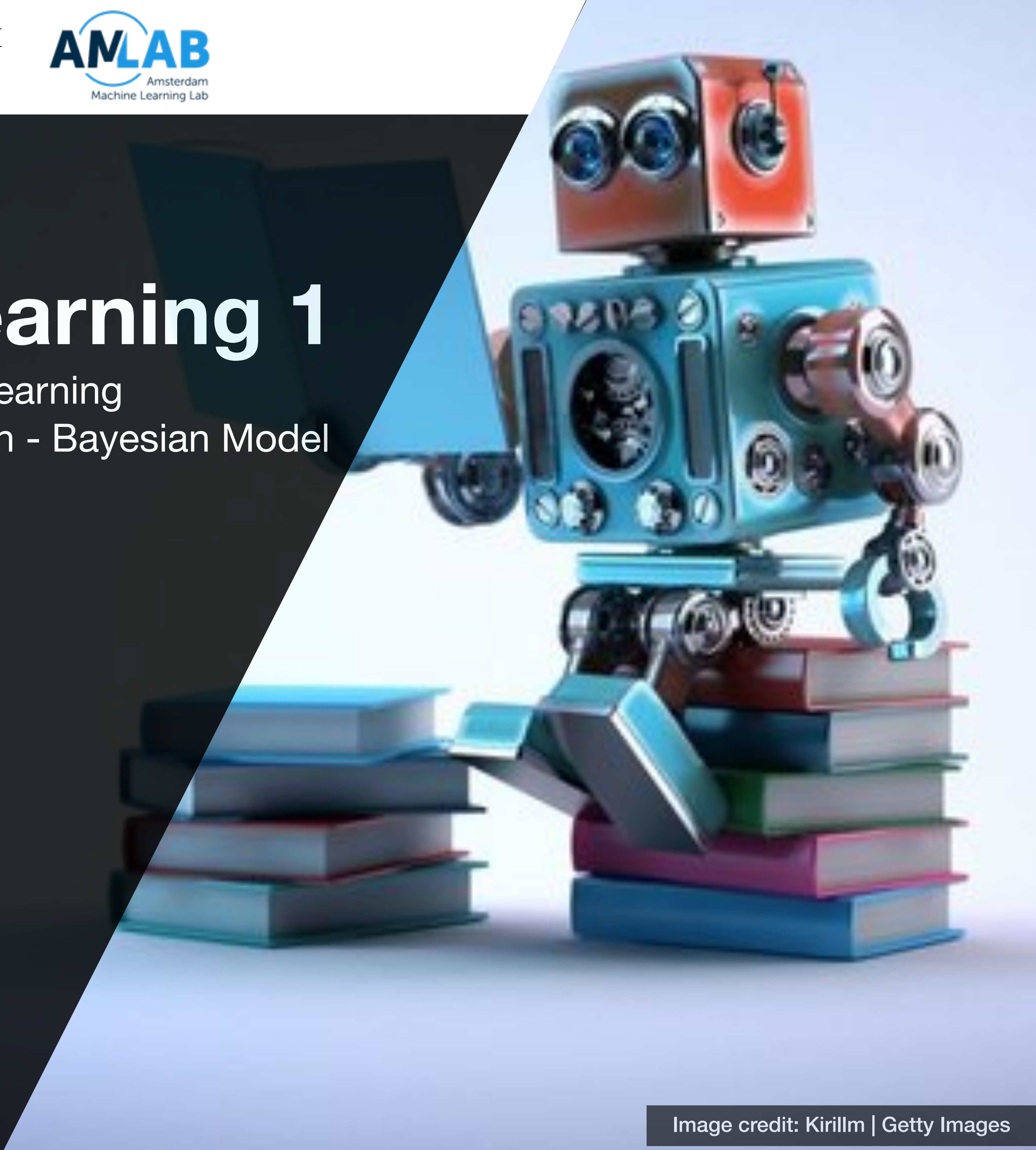
(Bishop 3.4)



Slide credits: Patrick Forré and Rianne van den Berg

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Bayesian Model Selection

- Given L models $\{\mathcal{M}_i\}_{i=1}^{L}$ with prior belief $p(\mathcal{M}_i)$

- Update prior knowledge with observations on the data D :

$$p(\mathcal{M}_i|D) = \frac{p(D|\mathcal{M}_i)\,p(\mathcal{M}_i)}{p(D)}$$

- Predictive distribution / mixture distribution / model average:

$$p(t'|\mathbf{x}', D) = \sum_{i=1}^{L} p(t'|x', \mathcal{M}_i)\, p(\mathcal{M}_i|D)$$

- Approximation: Use most probable model for predictions

flat prior

$$\mathcal{M}^* = \arg\max_{\mathcal{M}_i} p(\mathcal{M}_i|D) = \arg\max_{\mathcal{M}_i} \boxed{p(D|\mathcal{M}_i)}\, \cancel{p(\mathcal{M}_i)}$$

model selection

$$p(t'|\mathbf{x}', D, \mathcal{M}^*)$$

Bayesian Model Comparison

- Model selection

$$\mathcal{M}^* = \arg\max_{\mathcal{M}_i} p(\mathcal{M}_i|D) = \arg\max_{\mathcal{M}_i} p(D|\mathcal{M}_i)p(\mathcal{M}_i)$$

- Comparing two models $\mathcal{M}_1$ and $\mathcal{M}_2$: $\frac{p(\mathcal{M}_1|D)}{p(\mathcal{M}_2|D)} = \boxed{\frac{p(D|\mathcal{M}_1)p(\mathcal{M}_1)}{p(D|\mathcal{M}_2)p(\mathcal{M}_2)}}$

- When quotient of priors $\frac{p(\mathcal{M}_1)}{p(\mathcal{M}_2)}$ is known or close to 1, then we need

$$\boxed{\frac{p(D|\mathcal{M}_1)}{p(D|\mathcal{M}_2)}}$$

Bayes factor

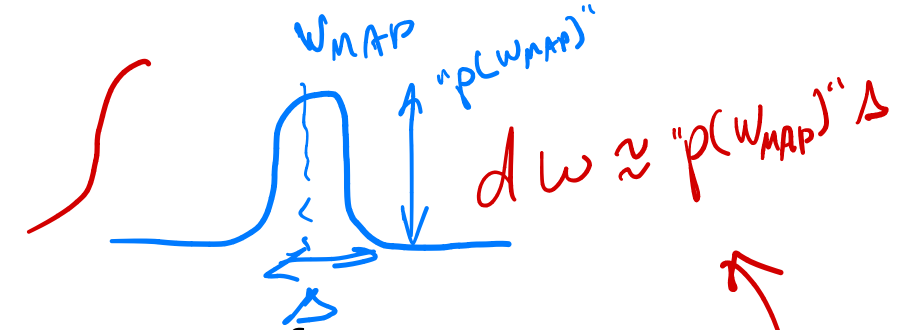
- Model evidence / marginal likelihood:

$$p(D|\mathcal{M}_i) = \int p(D|\mathbf{w}, \mathcal{M}_i)p(\mathbf{w}|\mathcal{M}_i)\mathrm{d}\mathbf{w}$$

Approximated Model Evidence

- Model evidence / marginal likelihood for single parameter w

$$p(D|\mathcal{M}_i) = \int \overbrace{p(D|w,\mathcal{M}_i)p(w|\mathcal{M}_i)}^{\text{unnormalized post}}\mathrm{d}w$$



- Note that $p(D|M_i)$ is the normalization constant of $p(w|D,M_i)$

- If posterior $p(w|D,\mathcal{M}_i)$ is sharply peaked at w_{MAP} with width $\Delta w_{\mathrm{posterior}}$

$$p(w|\mathcal{M}_i) = 1/\Delta w_{\mathrm{prior}}$$

$$p(D|\mathcal{M}_i) = \int p(D|\underset{w_{MAP}}{w},\mathcal{M}_i)p(\underset{w_{MAP}}{w}|\mathcal{M}_i)\mathrm{d}w \approx \frac{p(D|w_{MAP},\mathcal{M}_i)}{\Delta w_{prior}}\Delta w_{post}$$

- $$\ln p(D|\mathcal{M}_i) \approx \ln p(D|w_{\mathrm{MAP}},\mathcal{M}_i) + \ln\frac{\Delta w_{\mathrm{posterior}}}{\Delta w_{\mathrm{prior}}}$$

penalizes complexity

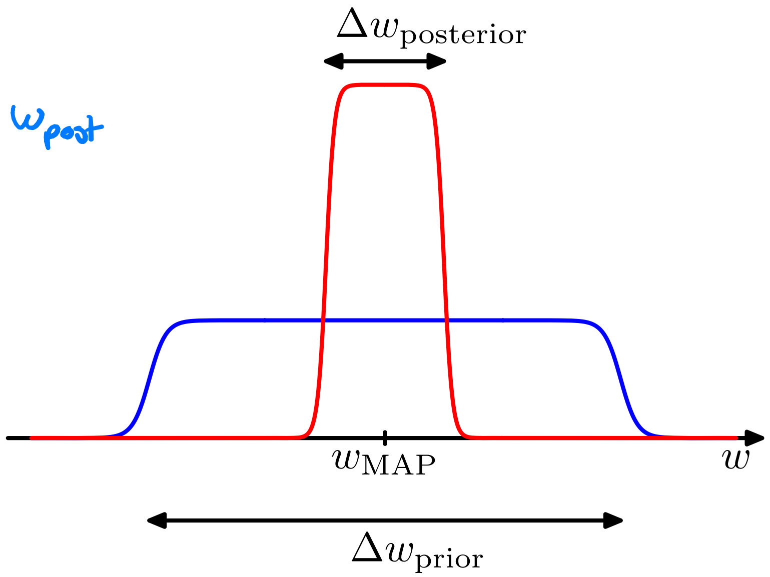


Figure: model evidence (Bishop 3.12)

Approximated Model Evidence

- $\ln p(D|\mathcal{M}_i) \approx \ln p(D|w_{\mathrm{MAP}}, \mathcal{M}_i) + \ln \frac{\Delta w_{\mathrm{posterior}}}{\Delta w_{\mathrm{prior}}}$
- if $\Delta w_{\mathrm{posterior}} < \Delta w_{\mathrm{prior}}$ then $\ln \left(\frac{\Delta w_{\mathrm{posterior}}}{\Delta w_{\mathrm{prior}}}\right)^{M} < 0$
- M parameters: $\mathbf{w} \in \mathbb{R}^M$

$$p(D|\mathcal{M}_i) = \int p(D|\mathbf{w}, \mathcal{M}_i)p(\mathbf{w}|\mathcal{M}_i)\mathrm{d}\mathbf{w} \approx p(D|w_{MAP}, M_i)\left(\frac{\Delta_{post}}{\Delta_{prior}}\right)^M$$

$$\ln p(D|M_i) \approx \ln p(D|w_{MAP}, M_i) + M \ln \frac{\Delta_{post}}{\Delta_{prior}}$$

- Model evidence favors models of medium complexity!

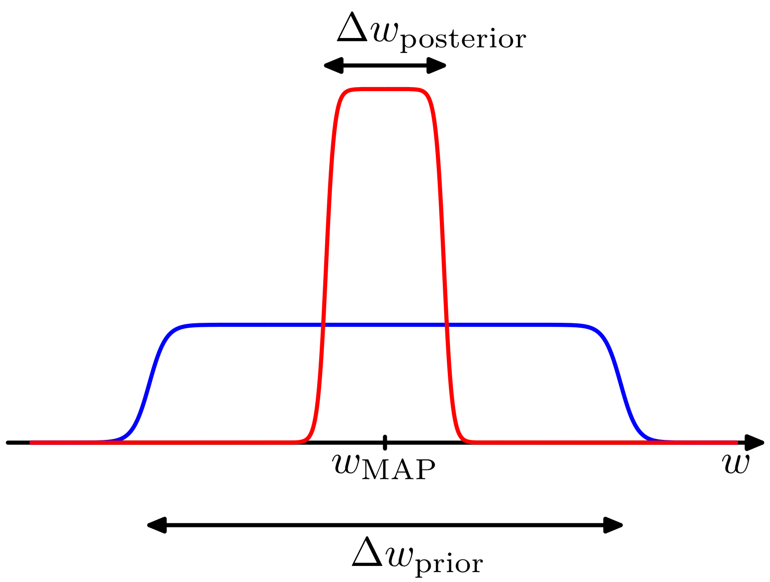


Figure: model evidence (Bishop 3.12)

Model evidence: medium complexity

✦ 3 models: M_1 is simplest, M_3 is most complex

✦ Generate datasets D from $p(D \mid M_i)$

1. sample model parameters from model prior:

$$\mathbf{w} \sim p(\mathbf{w} \mid M_i)$$

2. Sample dataset

$$D \sim p(D \mid \mathbf{w}, M_i)$$

✦ Note:

$$\int p(D \mid M_i)\, \mathrm{d}D = 1$$

✦ dataset D_0: model M_2 has highest model evidence

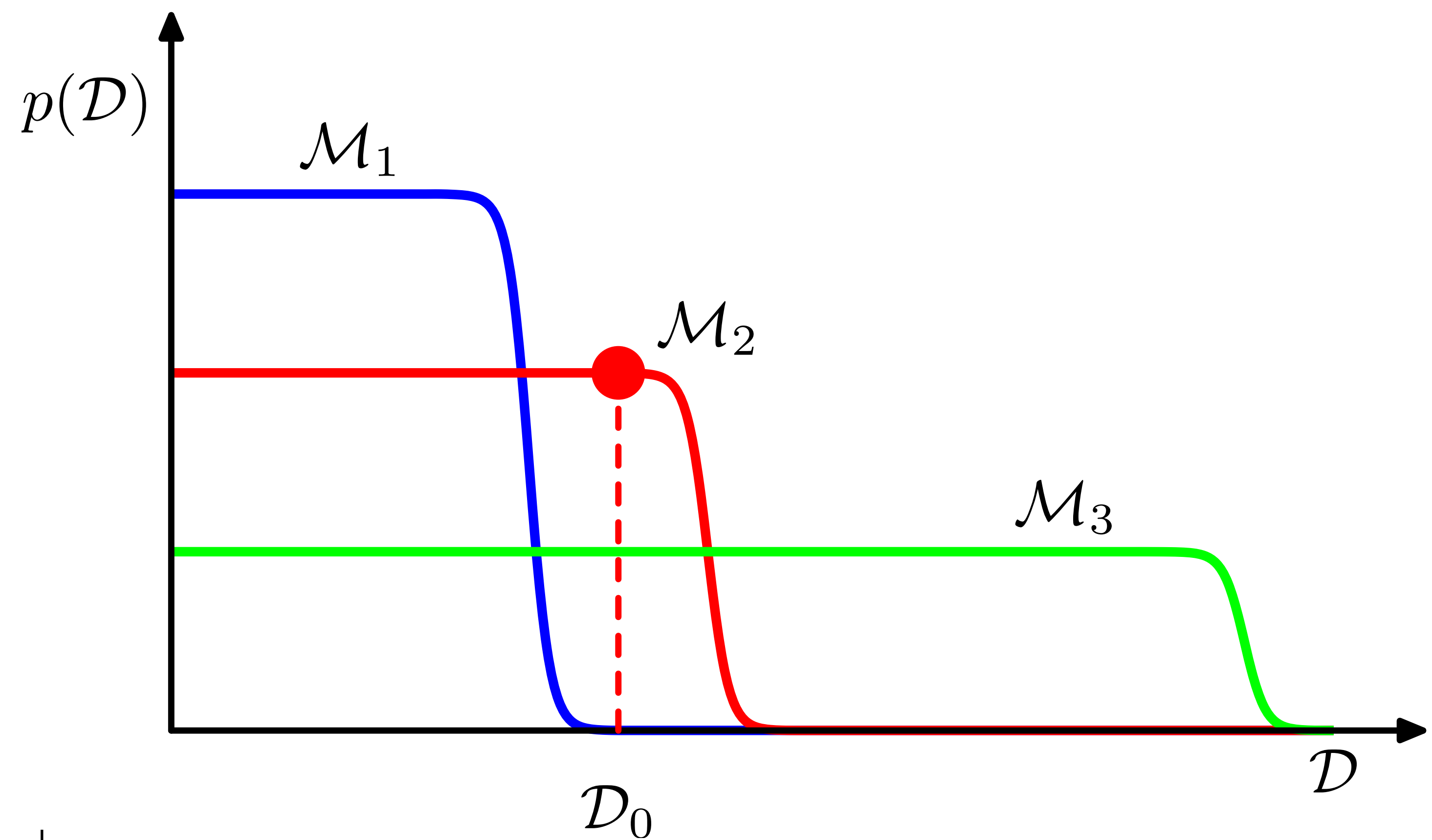


Figure: model evidence (Bishop 3.12)