Lecture 13
(and possibly 14)

Markets, Mechanisms and Machines

David Evans and Denis Nekipelov

Incomplete information games

- In reality, assuming complete information may not be possible
- Players may have information that they do not want to (or cannot) share with other players or the designer of the game
  - Cost of manufacturing or acquisition
  - Maximum willingness to pay in an auction
  - Cognitive ability in collaborative classroom
- We need to adjust the structure of the game to allow players to have private information

Incomplete information games

- To have a self-contained game structure we need to have a model for how players acquire and use private information
- We model private information for each player *i* by a scalar or vector τ_i. We call τ_i the type (or signal) of player *i*.
- We assume that another player (Nature) assigns types to players by drawing them from distribution *D*
- As soon as each player learns her type, she conceals it and does not reveal it to other players
- It, however, remains in her information set

Incomplete information games

- Formally, the incomplete information game is $\Gamma=\{N,\{A_i,T_i,u_i\}_{i\in N},D\}$
  - Set of players N
  - Action space A_i for player i
  - Type space T_i that contains possible values of this player's type τ_i
  - Distribution over types D
  - Utility function u_i that maps action profile and type of the player into real numbers

Incomplete information games

- In complete information game if players agree to choose a specific (equilibrium) action profile, no player has an incentive to deviate from it because it decreases utility
- AND all players know it
- In incomplete information game each player can *claim* that deviation from an agreed upon action profile does not decrease her utility
- AND other players cannot verify this claim

Incomplete information games

- So the game leads to actions motivated by players' types
  - Example: Assume that with some low probability Player 1 in the Battle of the Sexes can have utility of -10 from boxing
- If a player knows D then if her opponent makes a claim about her type, you can judge if that claim is likely true
- Then each player needs to construct a probability distribution that would reflect the likely actions of their opponents

Incomplete information games

- We call this beliefs of players
- Formally, these are conditional distributions

  $D(\tau_{\text{-i}} \mid \tau_{\text{i}})$

- Knowing their own type, each player tries to predict the types of all their opponents
  - Just like in the card game each player tries to predict the cards of everyone else knowing their own cards

Incomplete information games

- Only τ_i is in the information set of each player
- The strategy of the player prescribes action that correspond to given information
- In the incomplete information game the strategy of player i is the mapping β_i from T_i into A_i
- Once the player learns her type, she makes an action
- Unlike complete information games we focus only on deterministic such functions
- The strategy profile of the incomplete information game is $\beta(\tau)=(\beta_1(\tau_1),\ldots, \beta_N(\tau_N))$

Incomplete information games

- We define *ex post* utilities of players (i.e. when the uncertainty of types was revealed after the game was played)

  $u_i(\beta(\tau), \tau_i)$

- *Interim* utilities of players (i.e. when player learns her type but does not know the types of others)

  $E[u_i(\beta(\tau), \tau_i) | \tau_i]$

- Ex ante utilities of players (i.e before nature assigns types)

  $E[u_i(\beta(\tau), \tau_i)]$

Incomplete information games

- Strategy β_i weakly dominates β_i' if for any a_{-i} and τ: $u_i(\beta_i(\tau_i), a_{-i}, \tau_i) \geq u_i(\beta_i'(\tau_i), a_{-i}, \tau_i)$
  - The inequality is strict for some of those alternative strategies
- Strategy β_i is dominant if it dominates any other strategy β_i'
- Strategy β_i is undominated if no strategy dominates it

Incomplete information games

Definition: A dominant strategy equilibrium of an incomplete information game Γ is a strategy profile β_i such that for any $i \in N$ and $\tau \in T$ and $a_{-i} \in A_{-i}$ and any strategy β_i‘:

$$u_i(\beta_i(\tau_i), a_{-i}, \tau_i) \geq u_i(\beta_i'(\tau_i), a_{-i}, \tau_i)$$

Definition: A Bayes-Nash equilibrium of an incomplete information game Γ is a strategy profile β_i such that for any $i \in N$ and $\tau_i \in T_i$ satisfies:

$$E[u_i(\beta(\tau), \tau_i) | \tau_i] \geq E[u_i(a_i, \beta_{-i}(\tau_{-i}), \tau_i) | \tau_i] \text{ for any } a_i \in A_i$$

- If a given strategy profile is a dominant strategy equilibrium, it is also BNE

Incomplete information games

- Properties of BNE:
  1. Strategy of each player is interim-optimal
  2. Strategy of each player is ex ante optimal
  3. We can define ex post equilibrium as an equilibrium where after observing each other's types players do not want to deviate from the BNE profile (important for settings with repeatedly interacting players)
  4. Ex post equilibria are a subset of BNE and BNE is a subset of ex ante equilibria

Example

- Buyer and seller want to trade an object
- Buyer’s value for object is \$3
- Seller’s value is either \$0 or \$2 depending on type {L,H}
- Buyer can offer either \$1 or \$3 for the object
- Seller chooses whether to sell or not

Example

Seller's type =L

	sale	no sale
\$3	0,3	0,0
\$1	2,1	0,0

Seller's type =H

	sale	no sale
\$3	0,3	0,2
\$1	2,1	0,2

- Regardless of type distribution, this game has BNE where $\beta_S(L)$=sale, $\beta_S(H)$=no sale, and β_B=\$1
- Selling is weakly dominant when seller has type L
- Offering \$1 is weakly dominant for buyer

Auctions: Why Economists talk so much about them?

- Explains price formation
  - Walrasian Auction
- Widely used selling game

Explore strategic behavior of:

- Bidders (usually buyers)
  - What bid to submit?
- Sellers
  - Which auction format to use?
  - Which selling game
  - Whether to restrict participation
  - Whether to charge entry fees

Examples

- Auctions used for many transactions in the Ancient world (marriage auctions in Mesopotamia, auctions for debt claims in ancient Greece)
- Art auctions for the last 500 years, (Christie's, Sotheby's)
- Real estate, treasury bills, electricity, livestock
- Large corporations are sold at auction
- Government procurement (highway construction), spectrum licenses
- Online advertising auctions

Auction formats

- A variety of formats are used to sell items
- Single item auction formats
  - English auction
    - Bidders call out successively higher prices until one bidder remains (Sotheby's and Christy's: Hammer auctions)
    - Japanese auction: seller continuously increases price, bidders drop out gradually and irrevocably by pressing a button
  - Vickrey or 2nd price auction:
    - Bidders submit sealed bids; high bidder wins and pays second highest bid
  - Dutch or descending price auction
    - opposite of English auction, Price falls until one bidder presses button, bidder gets object at the current price (Dutch flower auction)

Auction formats

- First Price sealed bid auction
  - Bidders submit sealed bids; high bidder wins and pays his bid
  - Construction contracts, governmental procurement
- Multi items auction formats
- Discriminatory Auction
  - A seller has an supply of items (possibly increasing in *p*)
  - Buyers submit downward sloping demand schedules (*p*; *q* combinations)
  - Equilibrium supply where aggregate demand equals supply
  - Buyers pay their bid for sold items

Auction formats

- Uniform Price
  - A seller has an supply of items (possibly increasing in *p*)
  - Buyers submit downward sloping demand schedules (*p*; *q* combinations)
  - Equilibrium supply where aggregate demand equals supply
  - Buyers pay the equilibrium price (where aggregate demand equals supply)
- Vickrey Auction
  - Win k units, then pay k highest opponents' losing bids (first highest losing bid for top unit, second highest losing bid for second unit, ...)

Auction formats

- Simultaneous ascending price auction (Milgrom (2000))
  - Each bidder demands one unit,
  - Bids are raised in multiple rounds,
  - In each round bidders specify which object that they are bidding for, and may switch from bidding for one object to bidding for another object
  - Auction closes when no further bids are raised
- Combinatorial Auction
  - Submit bids for stand-alone items and also for combination of items
  - Most expensive bidder/item allocation wins

Strategic equivalence

- When do auctions yield the same outcome? When are the bidding strategies identical?
- Example:
  - First price and Dutch auctions
  - Rational bidders, think about bidder giving instructions to an agent
  - In Dutch auction: a price at which to jump in.
  - Would do the same in a first price auction
  - Intuition: no information is revealed in a Dutch auction.
- Under some conditions there is also a strategic equivalence between the 2nd price and the English auction

Informational environment

- Private values:
  - Each bidder *i* values the item at a (privately) known value v_i
  - Other bidders do not know vi but know that v_i is drawn from some probability distribution
  - Example: construction contract in which firms know their own cost but not other firms' costs
- Common values
  - Same value for all bidders
  - Each bidder has a signal of the true value
  - Example: oil field as the value of oil is the same to everyone

Informational environment

- Affiliated values
  - a mixture between private and common values
- Interdependent values
- Reserve price: R
  - seller announces a minimum price prior to the auction, $b \geq R$
  - Reserve price may be kept secret

Vickrey (2nd price) auction

- Rule: High bidder wins and pays the second highest bid
- N bidders
- Common model: private values, each bidder's value $v_i \in [0, V]$ known to bidder i but not known to other bidders
- Bidder i wins if her bid is the highest
  - Gets payoff $v_i - b_{(2)}$ ($b_{(1)}, b_{(2)}, \ldots, b_{(N)}$ are order statistics of the set of submitted bids)
- Otherwise she gets 0

Vickrey (2nd price) auction

Theorem: *Every bidder bids their true value is a dominant strategy equilibrium.*

Proof:

- If $v_i < R$ (R is the reserve price): optimal to bid your value as otherwise pay at least R and v_i - $R < 0$
- *I*f $v_i > R$: strategy $b_i = v_i$
- Consider a deviation:
  - $b_i > v_i$, for $b_{(2)} \leq v_i$ pay b(2) and get the same payoff or for $\mathrm{b_i} > \mathrm{b_{(2)}} > v_i$ make a loss.
  - $b_i < v_i$, for $b_{(2)} < b_i$ pay $b_{(2)}$ and get the same payoff or for $b_i < b_{(2)} < v_i$ loose the auction

Vickrey (2^{nd} price) auction

- Vickrey auction is efficient (item is allocated to bidder with the highest value)
- Expected Revenues (for the seller) of the Vickrey auction equal the expected second highest valuation
- Other equilibria?
  - yes
- Suppose $b_1 = V$ and all other bidders bid $b_i = 0$
- This is an equilibrium as nobody benets from deviating, but it is not a dominant strategy equilibrium

English auction

Setup (with private values):

- Button auction with continuously increasing prices
- Observe other bidders drop out prices
- No bidding costs

- Strategy: Press button until the price reaches your value v_i
- Is this an equilibrium?
  - Follow the proof for the Vickrey auction
- Other variants of the English auction may feature:
  - Discrete price increases; Open access: bidders may re-enter later-on; Bidding costs.

First price sealed bid auction

Setup:

- One object auctioned off and R=0
- N bidders with private values $v_i \in [0,1]$ (normalization)
- Beliefs of other bidders about vi drawn from distribution with density f: $[0,1] \longmapsto \boldsymbol{R}_+$
- Note: v_i are independently drawn across players!
- Pay-your-bid payment rule:
  - If bidder i bids b, her payoff is $v_i - b$ (if she wins)
  - She wins if b is the highest bid

First price sealed bid auction

- Expected payoff (interim utility)

  $U_i(b_i\,;\, v_i) = (v_i - b_i)\Pr(b_i > b_j\ \forall j \neq i)$
- Strategy $\beta_i : [0, 1] \rightarrow \mathbf{R}_+$ (maps values to bids)
- In BNE: each bidder chooses b_i that maximizes expected payoff given v_i and beliefs regarding values of other bidders
- Symmetric equilibrium: bidder i with value b_i picks $b_i = \beta(v_i)$
- Claim: if density $f > 0$ on its support then $\beta(v)$ is strictly monotone

First price sealed bid auction

- Determining the probability of winning

$$\Pr(b_i > b_j\ \forall j \neq i) = \Pr(b_i > \beta(v_j)\ b_j\ \forall j \neq i)$$

$$= \Pr(\beta^{-1}(b_i) > \beta^{-1}(\beta(v_j))\ \forall j \neq i)$$

$$= \Pr(\beta^{-1}(b_i) > v_j\ \forall j \neq i) \qquad \text{(monotonicity)}$$

$$= \Pr(\beta^{-1}(b_i) > v_1, \beta^{-1}(b_i) > v_2, \ldots, \beta^{-1}(b_i) > v_N)$$

$$= \Pr(\beta^{-1}(b_i) > v_1)\ \ldots\ \Pr(\beta^{-1}(b_i) > v_N) \qquad \text{(independence)}$$

$$= F(\beta^{-1}(b_i))\ldots F(\beta^{-1}(b_i)) = F^{N-1}(\beta^{-1}(b_i)) \quad \text{(symmetry+beliefs)}$$

- Expected payoff (interim utility)

$$U_i(b_i\,;\,v_i) = (v_i - b_i)\Pr(b_i > b_j\ \forall j \neq i) = (v_i - b_i)\ F^{N-1}(\beta^{-1}(b_i))$$

First price sealed bid auction

- Utility is determined by interim allocation rule $x_i(b_i)= F^{N\text{-}1}(\beta^{-1}(b_i))$ (allocation probability as a function of bid)
- In a symmetric BNE: Bids maximize utilities; same values should lead to same bids
- Thus
  - $\frac{d}{db_i} U_i(b_i\,; v_i)=0$, which produces $b_i= \beta(v_i)$
  - $\beta^{-1}(b_i)= v_i$ and $F^{N\text{-}1}(\beta^{-1}(b_i))= F^{N\text{-}1}(v_i)$

First price sealed bid auction

- Now

$$\frac{d}{dv_i} U_i(\beta(v_i); v_i) = \frac{\partial}{\partial v_i} U_i(\beta(v_i); v_i) + \frac{\partial}{\partial b_i} U_i(\beta(v_i); v_i) \frac{\partial \beta(v_i)}{\partial v_i}$$

$$= x_i(\beta(v_i)) = F^{N-1}(v_i) \quad \text{(allocation for values)}$$

  - Because, FOC holds and $\frac{\partial}{\partial b_i} U_i(b_i; v_i)=0$
  - Because, $U_i(b_i; v_i)=x_i(b_i)(v_i - b_i)$ and $\frac{\partial}{\partial v_i} U_i(b_i; v_i)=x_i(b_i)$

- This means that $U_i(\beta(v_i); v_i) = \int_0^{v_i} \frac{\partial U_i(\beta(z);z)}{\partial z} dz$

First price sealed bid auction

- Collecting information
  - $U_i(\beta(v_i); v_i) = (v_i - b_i)\, F^{N-1}(v_i)$
  - $U_i(\beta(v_i); v_i) = \int_0^{v_i} F^{N-1}(z)\, dz$
- This produces explicit expression for bid function

$$\beta(v_i) = v_i - \frac{\int_0^{v_i} F^{N-1}(z) dz}{F^{N-1}(v_i)}$$

- This demonstrates that bid function is indeed differentiable and monotone

First price sealed bid auction

- **Example**: 2 bidders with values uniformly distributed on [0,1]
- **Theorem:** $\beta(v)=\mathrm{E}[v_{(2)} \mid v_{(2)} < v]$
- Note that $\Pr(v_{(2)} < v) = G(v) = F^{N\text{-}1}(v_i)$
- The density $f_{v(2)}(v) = G'(v)$ and

$$f_{v(2)}(v \mid v_{(2)} < v) = \frac{f_{v(2)}(v)}{\Pr(v_{(2)}<v)}$$

Revenue equivalence

- How do auction formats compare?

Theorem: *The expected revenue to the seller is the same in a first-price and second-price auction.*

- First price auction bid equals the expected second highest valuation
- Second price auction payment equals the second highest valuation
- In expectation they are the same. Thus, seller's expected revenues are identical

Bidder surplus equivalence

- How do auction formats compare?
- Note that both first and second price auctions allocate to the highest value bidder (since the bid function in the first price auction is monotone in values)
- Thus, both auction formats lead to the same welfare
- Auction welfare is equal to the sum of revenue of the auctioneer and the surplus of bidders
- Since the revenue of the auctioneer is the same for both formats (by revenue equivalence theorem), the surplus of bidders has to be the same too

Revenue optimization

- First and second price auctions are efficient: they maximize social welfare by allocating the item to the bidder with the highest value
- Can auctioneer optimize her revenue by, possibly, sacrificing efficiency
- Yes, by setting a reserve price
  - Reserve price excludes some bidders (whose values are below the reserve)
  - BUT it also incentivizes the remaining bidders to raise their bids
  - This results in an increased revenue of the auctioneer

Revenue optimization

- Total social welfare with the reserve price R

$$N\int_R^1 zF^{N\text{-}1}(z)f(z)dz = \text{Revenue}(R)$$
$$+ N\int_R^1 \left(\int_0^z F^{N-1}(t)dt\right) f(z)dz$$

- Therefore

$$\text{Revenue}(R) = N\int_R^1 \left(v - \frac{1-F(v)}{f(v)}\right) F^{N\text{-}1}(v)f(v)dv$$

Revenue optimization

$$\text{Revenue}(R) = N\int_R^1 \left(v - \frac{1-F(v)}{f(v)}\right) F^{N\text{-}1}(v)f(v)dv$$

- Revenue is maximized when $R = (1 - F(R))/f(R)$
- Function $(1 - F(v))/f(v)$ is called Myerson's "virtual value"
- We maximize the revenue by discarding all bidders whose values are less than their virtual values
- Nobel prize, 2007