OTFS Linear Receivers and MAP

OTFS System model

- Recall, the end-to-end DD-domain relationship.

$$\bar{y}_{DD} = H_{DD}\,\bar{x}_{DD} + \bar{v}_{DD}$$

$\bar{y}_{DD}$: $MN \times 1$, $\bar{x}_{DD}$: $MN \times 1$

where,
$$H = \sum_{i=1}^{L_p} h_i (\Pi)^{l_i} (\Delta)^{k_i}$$

- For rectangular pulse : $P_{rx} = P_{tx} = I_M$

Problem 1 : What is the expression for H ?

$$H = \sum_{i=1}^{L_p} h_i (\Pi)^{l_i} (\Delta)^{k_i}$$

where,

$\tau_i = l_i \Delta\tau \rightarrow$ Integer multiple of Delay resolution ($\Delta\tau$)

$\nu_i = k_i \Delta\nu \rightarrow$ Integer multiple of Doppler resolution ($\Delta\nu$)

Problem 2 :

Net end-to-end system model is

$$\bar{y}_{DD} = H_{DD}\,\bar{x}_{DD} + \bar{\nu}_{DD}$$

What is the expression for H_{DD} ?

$$H_{DD} = (F_N \otimes I_M)\, H \,(F_N^H \otimes I_M)$$

Signal Detection in OTFS

- Linear Receivers
  - ZF Receiver
  - LMMSE Receiver
- Non-linear Receiver
  - Message-Passing detector

ZF Receiver

- ZF Receiver minimizes Least-Squares (LS) cost function.

$$\min_{\hat{x}_{DD}} \left\| \bar{y}_{DD} - H_{DD}\, \hat{x}_{DD} \right\|^2$$

- The ZF estimate is

$$\boxed{\hat{x}_{DD}^{ZF} = \underbrace{\left(H_{DD}^{H} H_{DD}\right)^{-1} H_{DD}^{H}}_{\text{Pseudo inverse of } H_{DD}} \bar{y}_{DD}}$$

$\left(H_{DD}^{H} H_{DD}\right)^{-1} H_{DD}^{H}$ is the pseudo-inverse of H_{DD}.

(ie) $\left(H_{DD}^{H} H_{DD}\right)^{-1} H_{DD}^{H} \times H_{DD} = I$

LMMSE Receiver:

- LMMSE Receiver minimizes the cost function

$$E\left\{ \left\| \hat{x}_{DD}^{LMMSE} - \bar{x}_{DD} \right\|^2 \right\}$$

- consider independent information symbols, with average power P. (ie)

$$R_{xx} = \mathbb{E}\left[\bar{x}_{DD}\, \bar{x}_{DD}^{H}\right] = P\, I_{MN}$$

- LMMSE Receiver is

$$\hat{x}_{DD}^{LMMSE} = \left(H_{DD}^{H} H_{DD} + \frac{\sigma^2}{P} I_{MN}\right)^{-1} H_{DD}^{H} \bar{y}_{DD}$$

$$\boxed{\hat{x}_{DD}^{LMMSE} = \left(H_{DD}^{H} H_{DD} + \frac{1}{SNR} I_{MN}\right)^{-1} H_{DD}^{H} \bar{y}_{DD}}$$

- As $SNR \to \infty$, LMMSE $\to$ ZF receiver.

Non-Linear Receiver

- Note that the matrix $H = \sum_{i=1}^{L_p} h_i (\Pi)^{l_i} (\Delta)^{k_i}$ is SPARSE, since it has only L_p non-zero entries in each row and column, and $L_p << MN$.

Note: SPARSE $\rightarrow$	More entries are Zeros Few entries are non-zeros	} in Matrix

- Hence, H_{DD} is also SPARSE.
- ZF and LMMSE based linear receivers do not exploit this sparsity, which can significantly simplify detection.

Message Passing (MP) Detector

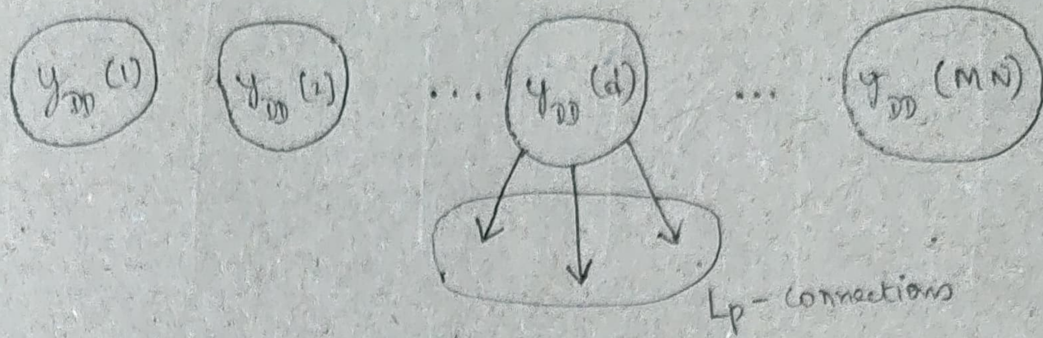
- The MP detector exploits the sparsity of the matrix H_{DD}, employs the Maximum aPosteriori (MAP) detection rule.
- symbol-by-Symbol MAP detection rule

$$\hat{x}_{DD}(c) = \underset{a_j \in A}{\arg\max}\ P_r\left(x_{DD}(c) = a_j \,\middle|\, \bar{y}_{DD}, H_{DD}\right)$$

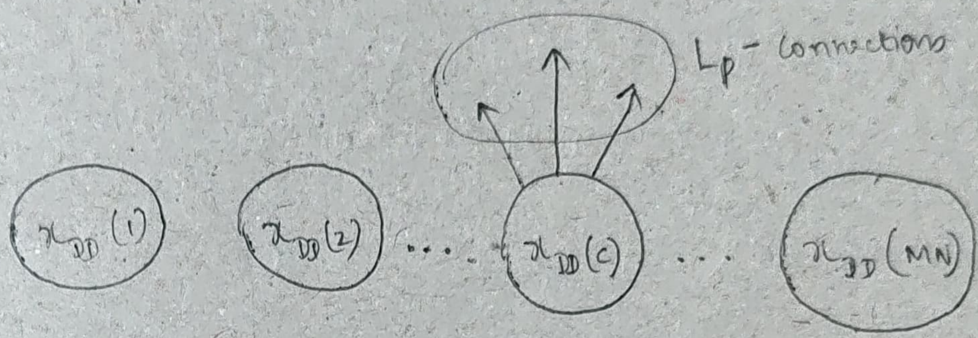
- $\mathbb{I}(d)$ denotes the set of indices of the non-zero elements in the d^{th} row of H_{DD}.
- $\mathbb{J}(c)$ denotes the set of indices of the non-zero elements in the c-th column of H_{DD}.
- Number of elements in $\mathbb{I}(d)$ and $\mathbb{J}(c)$ is L_p each.

Note: $\mathbb{I}(d) = \{c_1, c_2, \dots c_{L_p}\} \rightarrow$ Observation Node messages $\mathbb{J}(c) = \{d_1, d_2, \dots, d_{L_p}\} \rightarrow$ Variable Node messages

- Elements of $\bar{y}_{DD}$ become <u>observation nodes</u>.



- Elements of $\bar{x}_{DD}$ become <u>variable nodes</u>.



- The <u>factor graph</u> is sparsely connected.
  - Any <u>Observation node</u> $y_{DD}(d)$ is connected to only L_p variable node $x_{DD}(c)$.
  - Any <u>variable node</u> $x_{DD}(c)$ is connected to only L_p observation nodes $y_{DD}(d)$.

- MP detector can solve this symbol-by-symbol MAP detection problem with <u>linear complexity</u> in MN.

- End-to-end model

$$\begin{bmatrix} y_{DD}(1) \\ y_{DD}(2) \\ \vdots \\ y_{DD}(MN) \end{bmatrix} = \sum_{c=1}^{MN} \begin{bmatrix} H_{DD}(1,c) \\ H_{DD}(2,c) \\ \vdots \\ H_{DD}(MN,c) \end{bmatrix} x_{DD}(c) + \begin{bmatrix} v_{DD}(1) \\ v_{DD}(2) \\ \vdots \\ v_{DD}(MN) \end{bmatrix}$$

- d th equation

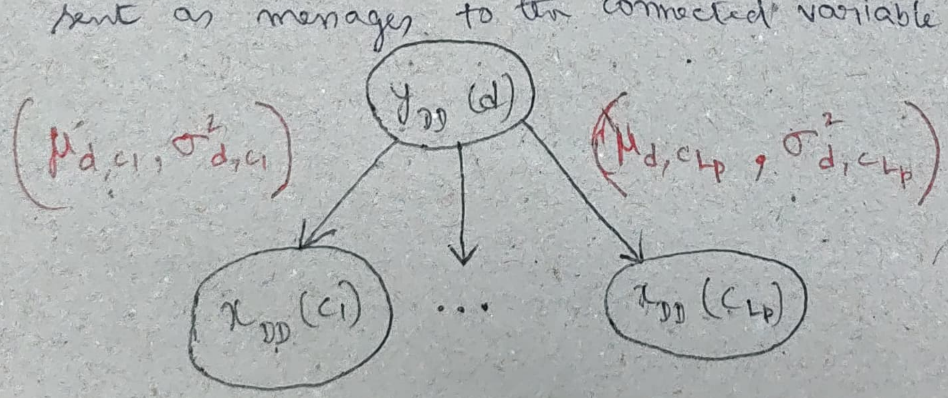
$$y_{DD}(d) = H_{DD}(d,1)\, x_{DD}(1) + H_{DD}(d,2)\, x_{DD}(2) + \dots + H_{DD}(d,MN)\, x_{DD}(MN) + v_{DD}(d)$$

$$= \sum_{c=1}^{MN} H_{DD}(d,c)\, x_{DD}(c) + v_{DD}(d)$$

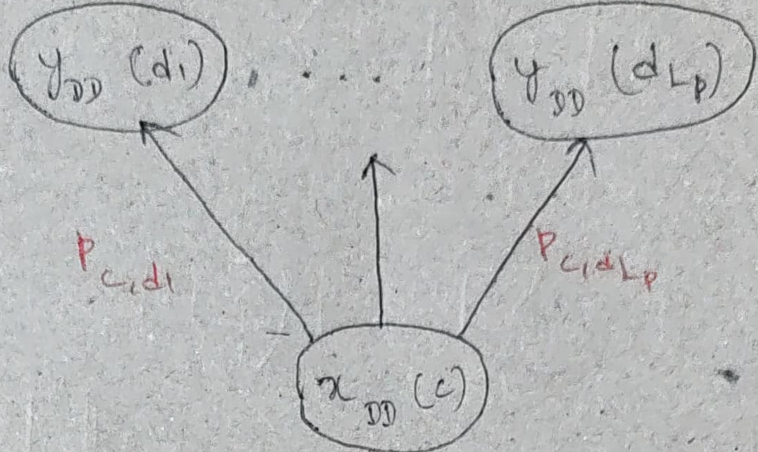
- For a given $x_{DD}(c)$, the interference term $\zeta_{d,c}$ is assumed to be Gaussian.

$$y_{DD}(d) = H_{DD}(d,c)\, x_{DD}(c) \underbrace{+ \sum_{\substack{e \neq c \\ e \in \mathcal{I}(d)}} H_{DD}(d,e)\, x_{DD}(e) + v_{DD}(d)}_{\zeta_{d,c}}$$

- From any observation node $y_{DD}(d)$, the mean $\mu_{d,c}$ and variance $\sigma^2_{d,c}$ of the interference terms are sent as messages to the connected variable nodes $x_{DD}(c)$.



⊙ Messages from any variable node $x_{DD}(c)$ (a) the <u>probability mass function</u> (pmf) of the alphabet, is passed as message to the observation node $y_{DD}(d)$.



⊙ <u>Observation node messages</u>

- Message from the observation node $y_{DD}(d)$ to any connected variable node $x_{DD}(c)$, are the mean $\mu^{(i)}_{d,c}$ and variance $\left(\sigma^{(i)}_{d,c}\right)^2$ of the interference $\zeta_{d,c}$.

- Recall,

$$y_{DD}(d) = H_{DD}(d,c)\, x_{DD}(c) + \underbrace{\sum_{\substack{e \neq c \\ e \in \mathcal{I}(d)}} H_{DD}(d,e)\, x_{DD}(e) + v_{DD}(d)}_{\zeta_{d,c}}$$

- Let i denote the iteration. It follows that

$$\mathbb{E}\left[\zeta^{(i)}_{d,c}\right] = \mu^{(i)}_{d,c} = \sum_{\substack{e \neq c \\ e \in \mathcal{I}(d)}} H_{DD}(d,e)\, \mathbb{E}\left[x_{DD}(e)\right]$$

- Let $p_{e,d}^{(i-1)}(a_j)$ denote the probability of symbol a_j, passed by the e^{th} variable node $x_{DD}(e)$ to the d^{th} observation node $y_{DD}(d)$ after iteration $i-1$.

- Hence, $$\mathbb{E}\left[x_{DD}(e)\right] = \sum_{j=1}^{Q} a_j \, p_{e,d}^{(i-1)}(a_j)$$

- The variance of interference term is

$$\left(\sigma_{d,c}^{(i)}\right)^2 = \sum_{e \neq c} |H_{DD}(d,e)|^2 \left(\sum_{j=1}^{Q} |a_j|^2 p_{e,d}^{(i-1)}(a_j) - \mathbb{E}^2\left[x_{DD}(e)\right] \right)$$

⊙ Variable Node Messages

- Message from variable node $x_{DD}(c)$ to any connected observation node $y_{DD}(d)$, is the pmf vector $p_{c,d}^{(i)}$, which contains the probabilities $p_{c,d}^{(i)}(a_j)$ of the symbols a_j.

- Note that, for any $e \in \mathcal{J}(c)$

$$p\left(y_{DD}(e) \mid x_{DD}(c) = a_j, H_{DD}\right)$$

$$\propto \exp\left(\frac{-\left|y_{DD}(e) - \mu_{e,c}^{(i)} - H_{DD}(e,c)\, a_j\right|^2}{\left(\sigma_{e,c}^{(i)}\right)^2} \right)$$

$$= \xi^{(i)}(e,c,j)$$

- Upon normalization

$$p\left(y_{DD}(e) \mid x_{DD}(c) = a_j, H_{DD}\right) = \frac{\xi^{(i)}(e,c,j)}{\sum_{k=1}^{Q} \xi^{(i)}(e,c,k)}$$

- Hence,

$$\tilde{p}^{(i)}_{c,d}(a_j) \doteq \frac{\prod_{\substack{e \neq d \\ e \in J(c)}} \frac{\xi^{(i)}(e,c,j)}{\sum_{k=1}^{Q} \xi(e,c,k)}}{\sum_{j=1}^{Q} \prod_{\substack{e \neq d \\ e \in J(c)}} \frac{\xi^{(i)}(e,c,j)}{\sum_{k=1}^{Q} \xi^{(i)}(e,c,k)}}$$

- The j^{th} element $p^{(i)}_{c,d}(a_j)$ of the pmf vector $p^{(i)}_{c,d}$ is updated as

$$p^{(i)}_{c,d}(a_j) = \Delta\, \tilde{p}^{(i)}_{c,d}(a_j) + (1-\Delta)\, p^{(i-1)}_{c,d}(a_j)$$

where, $\Delta \in (0,1]$ is the damping factor.

- Convergence Indicator $\eta^{(i)}$

- At node $x(c)$ define

$$p^{(i)}_{c}(a_j) = \frac{\prod_{e \in J(c)} \left(\frac{\xi^{(i)}(e,c,j)}{\sum_{k=1}^{Q} \xi^{(i)}(e,c,k)} \right)}{\sum_{j=1}^{Q} \prod_{e \in J(c)} \left(\frac{\xi^{(i)}(e,c,j)}{\sum_{k=1}^{Q} \xi^{(i)}(e,c,k)} \right)}$$

- Convergence indicator : For some small $\gamma > 0$

$$\eta^{(i)} = \frac{1}{NM} \sum_{c=1}^{NM} \mathrm{Ind}\left(\max p^{(i)}_{c}(a_j) \geq 1-\gamma \right)$$

Update Decision

When $\eta^{(i)} > \eta^{(i-1)}$, the decision $\hat{x}(c)$ of each transmitted symbol is updated as

$$\hat{x}(c) = \arg\max_{a_j} p_c^{(i)}(a_j)$$

Stopping Rule

- The MP algorithm stops when any of the following conditions holds

$$\eta^{(i)} \approx 1$$

- Maximum number of iterations is reached
- $\eta^{(i)} < \eta^{(i^*)} - \epsilon$, where $0 \leq i^* \leq i-1$ is the iteration such that $\eta^{(i^*)}$ is maximum.