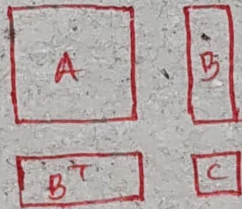
Week 8 — Schur's Complement

Consider the Block matrix ;

$$X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$




$A \in S^{n \times n}$

$C \in S^{m \times m}$

$X \in S^{n+m}$

- $A = A^T$, $C = C^T$, A, C are Symmetric

  $X = X^T \in S^n$, X is also Symmetric.

Schur's complement,

$S_1 = C - B^T A^{-1} B$, when A is invertible

$S_2 = A - B C^{-1} B^T$, when C is invertible.

$B \rightarrow$ Rectangular Matrix

Key result :

(a) $X \succ 0 \iff \begin{matrix} S_1 \succ 0 \\ A \succ 0 \end{matrix} \iff \begin{matrix} S_2 \succ 0 \\ C \succ 0 \end{matrix}$

Positive Definite (PD)

(b) $\det(X) = \det(A)\det(S_1) = \det(C)\det(S_2)$.

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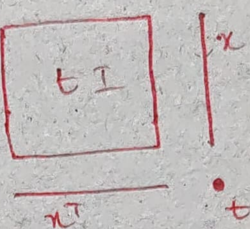
Example

$$D = \begin{bmatrix} tI & \underline{x} \\ \underline{x}^T & t \end{bmatrix}$$




$\underline{x} \rightarrow$ Column Vector

$\underline{x}^T \rightarrow$ Row vector

$t \rightarrow$ Single element.

Relati, with Block matrix,

$A = tI, \quad B = \underline{x}, \quad C = t, \quad t > 0.$

$$S_1 = t - x^T t^{-1} x = t - \frac{x^T x}{t}$$

so, $D \succeq 0 \iff \begin{matrix} x^T x \le t^2 \\ t \succeq 0 \end{matrix}$ (or) $\underbrace{\|x\| \le t}_{\text{Second order Cone (SOC) constraint.}}$

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Semi definite Programs (SDP) :

– Linear Matrix Inequalities (LMI) form

Let us look into the SDPs, in particular we'll look at a form of SDPs, which is called LMI.

The problem is of the form,

$\min \; c^T x \leftarrow$ Linear objective

$\odot \; F(\underline{x}) : \underbrace{G + F_1 x_1 + F_2 x_2 + \dots + F_n x_n \preceq 0}_{\text{Negative semidefinite}}$

$\odot \; Ax = b$

This is Linear Matrix inequality (LMI).

This is more general than SOCP.

$\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$G, F_i \in S^m$

Example: $[G]_{jj} = g_j \qquad [F_i]_{jj} = f_{ij}$

$G, F_i$ are diagonal.

$$F(x) = \begin{bmatrix} g_1 + \sum_{i=1}^{n} f_{i1} x_i & 0 & \cdots & 0 \\ 0 & g_2 + \sum_{i=1}^{n} f_{i2} x_i & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & g_m + \sum_{i=1}^{n} f_{im} x_i \end{bmatrix}$$

$$F(x) \preceq 0 \iff g_j + \sum_{i=1}^{m} f_{ij} x_i \leq 0, \quad \forall j = 1, \ldots, m.$$

This is a very special case in which SDP reduces to a Linear Program (LP).

Is SDP convex ?

$F(x) \preceq 0 \iff \lambda_i(F(x)) \leq 0$

(ie) $F(x)$ is PSD. (or) $\lambda_{max}(F(x)) \leq 0$

Maximum Eigen Value

Claim that we are going to make:

$\lambda_{max}(F(x))$ convex function in $x$, for $G, F_i \in S^m$

Note: $\lambda_{max}(A) = \max_{\substack{y \\ \|y\| \leq 1}} y^T A y$

$$\max_{\substack{y \\ \|y \leq 1\|}} y^T F(x) y = \max_{\substack{y \\ \|y \leq 1\|}} \underbrace{y^T G y + \sum_{i=1}^{n} (y^T F_i y) x_i}_{\text{Affine function of } x}$$

$$\lambda_{max}(F(x)) = \max_{\|y\|=1} y^T F(x) y$$

(pointwise maximum of affine).

$\Rightarrow \lambda_{max}(F(x))$ is convex in $x$.

$\Rightarrow$ SDP are convex.

<u>Note</u>: $\lambda_{min}(F(x))$ is concave in $x \Rightarrow F(x) \succeq 0$ also valid.

$F(x) \succeq 0 \Leftrightarrow -F(x) \preceq 0$ (or) $\lambda_{min}(F(x)) \succeq 0$

$$\left.\begin{aligned} F(x) &= G + \sum F_i x_i \preceq 0 \\ -F(x) &= -G - \sum F_i x_i = H(x) \succeq 0 \end{aligned}\right\} \text{Both are valid convex Form.}$$

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<u>SDP Examples</u>

① Express the two LMIs into single LMI.

$$\begin{aligned} F_1(x) &\preceq 0 \\ F_2(x) &\preceq 0 \end{aligned} \Rightarrow \begin{bmatrix} F_1(x) & 0 \\ 0 & F_2(x) \end{bmatrix} \preceq 0$$

$$\Rightarrow \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} + \sum x_i \begin{bmatrix} [F_1]_i & 0 \\ 0 & [F_2]_i \end{bmatrix} \preceq 0.$$

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② $\min \lambda_{max}(F(x))$

Applying epigraph trick,

$\Rightarrow \min t$

s.t. ◦ $\lambda_{max}(F(x)) \leq t$

(or)

◦ $\lambda_{max}(F(x) - tI) \leq 0$.

(or)

$\min t$

$F(x) \preceq tI$

$$\Rightarrow \underbrace{G + \sum F_i x_i - tI \preceq 0}_{\text{LMI constraint}}, \quad x_1, \ldots, x_n, t$$

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③ LP $\subseteq$ QP $\subseteq$ QCQP $\subseteq$ SOCP $\subseteq$ SDP

Most Specialized (Easiest to solve)

Most General (Hardest to solve)

$$\min \; c^T x$$

$$\text{s.t.} \; \|A_i x + b_i\| \le c_i^T x + d_i$$ (Second order Cone (SOC) constraint)

$$Gx = h.$$

Recall $\|x\| \le t \Rightarrow \begin{bmatrix} tI & x \\ x^T & t \end{bmatrix} \preceq 0.$

$$\Rightarrow \underbrace{\begin{bmatrix} (c_i^T x + d_i) I & A_i x + b_i \\ (A_i x + b_i)^T & c_i^T x + d_i \end{bmatrix} \preceq 0}_{\text{LMI}}$$

$$\Rightarrow \begin{bmatrix} d_i I & b_i \\ b_i^T & d_i \end{bmatrix} + \sum_{j=1}^{n} x_j \begin{bmatrix} c_{ij} I & a_{ij} \\ a_{ij}^T & c_{ij} \end{bmatrix} \preceq 0$$

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④ $$\min \; \frac{(c^T x)^2}{d^T x}$$

$$Ax = b.$$

Applying epigraph trick,

$$\min \; t.$$

$$(c^T x)^2 \le (d^T x) t$$

Applying Schur's complement,

$$\begin{bmatrix} t & c^T x \\ c^T x & d^T x \end{bmatrix} \succeq 0.$$

This is a LMI (SDP).

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## SDP Dual form

Let us see the Dual problem corresponding to the LMI, which yield different way of expressing the problems and also different way of taking the dual.

Consider the problem,

$$\min\ c^T x \qquad\qquad \text{(or)} \qquad\qquad \min\ c^T x$$

$$\text{s.t. } \lambda_{max}(F(x)) \le 0 \qquad\qquad F(\underline{x}) \preceq 0$$

(not in standard convex form).

Matrix Dual variable : $Y \in S^n$

$$L(X, Y) = c^T X + \langle F(X), Y \rangle$$

$$= c^T X + \sum_{i,j} [F(X)]_{ij} Y_{ij}$$

$$= c^T X + Tr(GY) + \sum_{i=1}^{n} x_i\, Tr(F_i Y)$$

Observe that $L(X, Y)$ is affine in $X$ and $Y$.

Therefore,

$$\min_{X} L(X, Y) = \sum_{i=1}^{n} \min_{x_i} x_i\left(c_i + Tr(F_i Y)\right) + Tr(GY).$$

$$\min_{x_i} x_i\left(c_i + Tr(F_i Y)\right) = \begin{cases} 0, & \text{Only if } c_i + Tr(F_i Y) \succeq 0 \\ -\infty, & \text{otherwise} \end{cases}$$

The dual of SDP is given by

$$\max\ Tr(GY)$$

s.t. ⊙ $c_i + Tr(F_i Y) = 0$

⊙ $\underbrace{Y \succeq 0}_{PSD}$.

Dual of SDP is also an SDP.

$$\underline{y} = \underbrace{vec(Y)}_{} \in \mathbb{R}^{n^2}$$

vectorized version of $Y$.

$\mathrm{Tr}(GY) = g^T y \quad , \quad g = \mathrm{vec}(G)$

$\mathrm{Tr}(F_i Y) = f_i^T y \quad , \quad f_i = \mathrm{vec}(F_i)$

Then, $Y = \underbrace{\sum_{ij} \mathbb{1}_{ij} Y_{ij} \succeq 0}_{\text{LMI form}}$

$\mathbb{1}_{ij} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & 1 & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$

$(i,j)^{th}$ entry.

$\max \; g^T y$

$f_i^T y + c_i = 0$

$H(y) = \sum \mathbb{1}_{ij} Y_{ij} \succeq 0.$ also LMI.