Week 2 **AFFINE SETS**

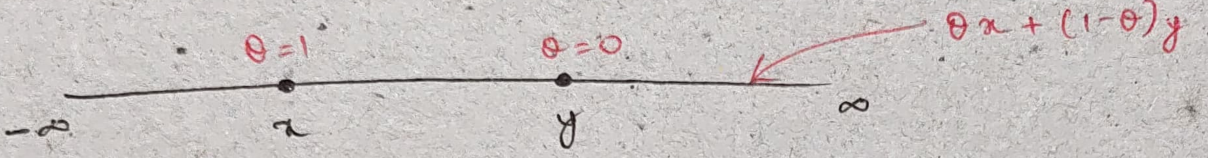
Let C be a set of vectors :

C is Affine , if the line through any 2 points x, y also lies in C.

(ie) $x, y \in C$.

Mathematically,

$$\underbrace{\theta x + (1-\theta) y}_{\text{Line through } x \text{ and } y} \in C \quad \forall\ \theta \in \mathbb{R}.$$



Affine Set - Example

$$C = \{ x \in \mathbb{R}^n \mid Ax = b \}$$

Solution Set of a System of Linear equations

$A \in \mathbb{R}^{m \times n}$ $\quad b \in \mathbb{R}^m$

The Solution is not obviously unique. (ie) it is not a single point. It can be any space. And that space is said to be Affine.

Proof : Suppose $x, y \in C \Rightarrow Ax = b$
$Ay = b$

Suppose , $z = \theta x + (1-\theta) y$, where $\theta \rightarrow$ Parameter

Does $z \in C$? (or) $Az = b$?

$Az = b$.

$\Rightarrow A(\theta x + (1-\theta) y) = b$

$\Rightarrow \theta (Ax) + (1-\theta) Ay = b$

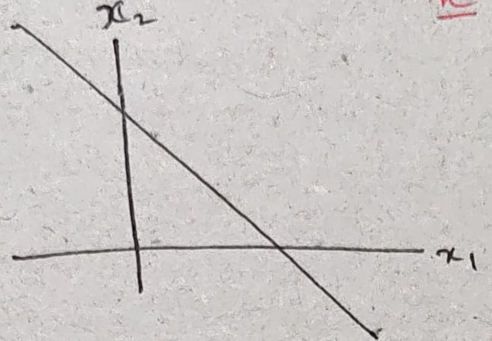
$\Rightarrow \theta b + (1-\theta) b = b$ ✓

$\Rightarrow z \in C. \ \forall\ \theta \in \mathbb{R}$.

(ie) z also belong to C for any $\theta \in \mathbb{R}$.
There cannot be any restriction on θ.

Example - Sets which are Affine.

* $n = 2$ (ie) $\left\{ \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} \in \mathbb{R}^2 \;\middle|\; \underbrace{x_1 + x_2 = 1}_{\text{Line}} \right\}$



$\Rightarrow$ Line is Affine Set

* $n = 3$ (ie) $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x_3 = 0 \right\}$

$\Rightarrow$ Plane is Affine Set

Example - Sets which are not Affine.

$D = \{ x \in \mathbb{R}^n \mid Ax \leq b \}$ ← Solution Set

$A \in \mathbb{R}^{m \times n}$ $\quad$ $b \in \mathbb{R}^m$

This set is not Affine.

Proof.

$x \in D \Rightarrow Ax \leq b$

$y \in D \Rightarrow Ay \leq b$

Consider the point $z = \theta x + (1-\theta)y \in D$?

$Az = A(\theta x + (1-\theta)y = \theta(Ax) + (1-\theta)(Ay)$

In general, $Ax \leq b \not\Rightarrow \theta(Ax) \leq \theta b$, coz this is not true for $\theta < 0$.

$\Rightarrow Az \leq b$ only when $\theta > 0$ and $1-\theta > 0$

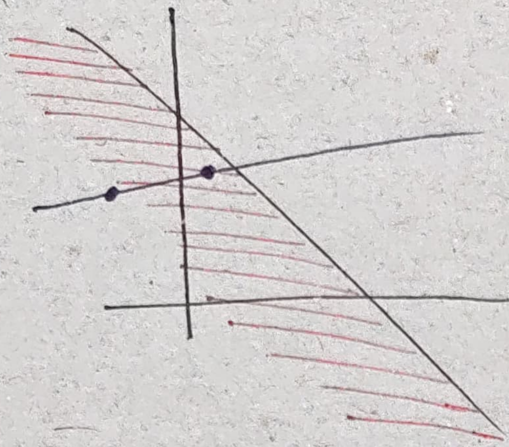
$\Rightarrow z \notin D$ when $\theta < 0$ and $1-\theta < 0$ (ie) $\theta > 1$.

(ie) Not true for arbitrary θ.

$\Rightarrow D$ is not affine.

⊙ $n=2, m=1$ (ie) $\left\{ x \in \mathbb{R}^2 \mid \underbrace{x_1 + x_2 \le 1}_{} \right\}$

Shaded region is Affine or not.



The line drawn through 2 points, where half-a-line lies inside, another half line lies outside.

As the full line does not align inside the set, therefore this is not obviously an Affine Set.

This kind of set is called Half Space (ie) whole space is divided into 2 parts.

⊙ $n=1, m=1$. (ie) $\left\{ x \in \mathbb{R} \mid x \le 1 \right\}$

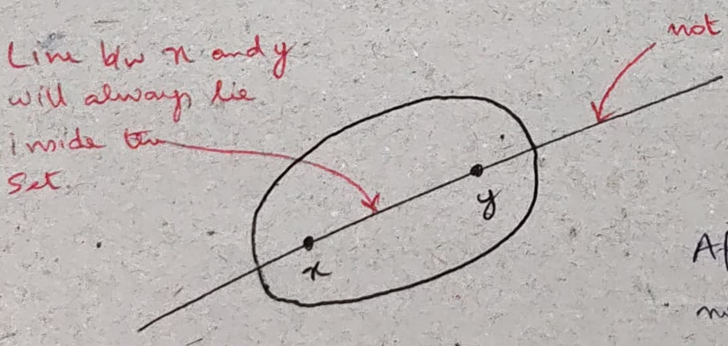
This is a line segment that ends at 1. For any 2 points, if we draw a line, some parts of the line will be outside and some will be inside. As we cannot have any part of the line outside, so therefore not Affine.

CONVEX SETS

Set C is said to be Convex, if the line between two points say x, y lies in C. (ie) $x, y \in C$.

(ie) $C_{convex} \Leftrightarrow \underbrace{\theta x + (1-\theta) y}_{\text{Line between } x \text{ and } y} \in C$

Hence there is restriction on θ. (ie) not $\forall\, \theta$.



This set is Convex, but not Affine. (Coz the whole line is not inside the set).

So, A Convex Set need not be Affine.

Question: Are Affine sets always convex ?

Consider set A, which is Affine $\Leftrightarrow$ $x, y \in A$ then

$$\theta x + (1-\theta) y \in A \quad \forall \, \theta \in \mathbb{R}$$

$$\Rightarrow \theta x + (1-\theta) y \in A \quad \forall \, \theta \in [0,1]$$

$$\Rightarrow A \text{ is convex.}$$

Thus, All Affine sets are also convex.

Example.

consider a set $\{x \mid Ax = b\}$.

This is an Affine set, which means this is also Convex.

Now, consider set $C = \{x \mid Ax \leq b\}$, $x, y \in C$.

this is not Affine. Is it convex ?

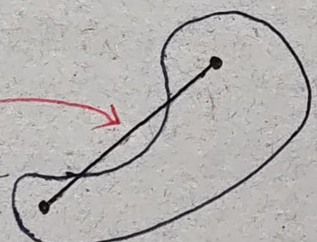
$$A(\theta x + (1-\theta) y) = \theta(Ax) + (1-\theta)(Ay)$$

$$\leq \theta b + (1-\theta) b \quad \because \theta \geq 0, \; 1-\theta \geq 0.$$

$$= b$$

$\Rightarrow$ C is convex, but not Affine.

Example for Non-convex set.



Line segment outside the set

$\Rightarrow$ Non-Convex set.

Convex (or) not ?

$$\{\underbrace{x \in \mathbb{R}_{++}}_{x > 0} \mid \underbrace{\log(x) \leq 2}_{0 < x < e^2}\}$$

Line segment b/w 0 and e^2 (open interval)

All intervals like this are convex.

This is not convex set coz, some part of the line segment is outside.

SOME EXAMPLES

① CONVEX CONES

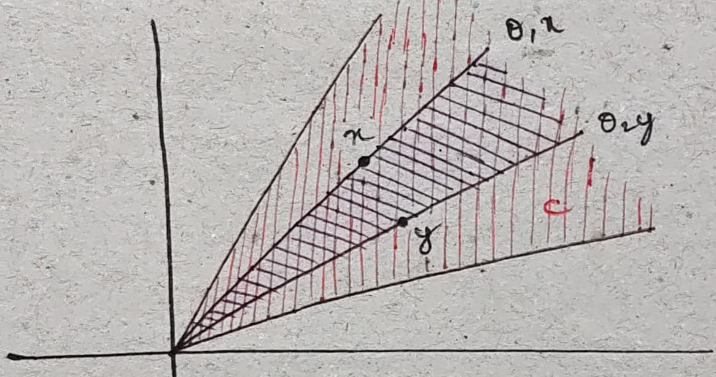
Let us look at a particular class of convex sets called as convex cones. Definition:

Suppose there are 2 points x and y in C. (i.e.) $x, y \in C$, then C is convex cone $\Leftrightarrow \theta_1 x + \theta_2 y \in C \ \forall \ \theta_1, \theta_2 \geq 0$.

Comparison with other definitions:

C is ____ when $\theta_1 x + \theta_2 y \in C$		Affine	Convex	Convex Cone
	θ_1	$\in \mathbb{R}$	$\in [0,1]$	≥ 0
	θ_2	$= 1-\theta_1$	$= 1-\theta_1$	≥ 0

Convex cone - Example



$$\underbrace{\theta_1 x + \theta_2 y}_{\text{Blue shaded Region}} \in \underbrace{C}_{\text{Red shaded Region}}$$

Blue shaded region $\theta_1 x + \theta_2 y$ lies inside the Red shaded region, C.

$\Rightarrow$ This is a Convex Cone.

Question: Is Convex Cones Convex? Yes.

Let C is Convex cone. The points x, y lies in C. (i.e.) $x, y \in C$.

then $\theta_1 x + \theta_2 y \in C \ \forall \ \theta_1, \theta_2 \geq 0$.

We $\theta_1 \in [0,1]$

$\theta_2 = 1-\theta_1 \in [0,1]$.

$\Rightarrow \theta_1 x + (1-\theta_1) y \in C$

$\Rightarrow$ C is convex.

Therefore, Convex cones are special case / special example of convex sets.

Likewise, Affine sets are special case of the convex sets.

Recap. A is Affine & convex

C is Convex

CC is Convex cone & Convex, not Affine

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② HYPERPLANE

Take n-dimensional plane and put just one restriction (linear restriction). Then it'd be a Hyperplane.

(i) $\{x \in \mathbb{R}^n \mid a^T x = b\}$

Example

⊙ $n=2$: $\{\underline{x} \in \mathbb{R}^2 \mid x_1 + x_2 = 1\} \rightarrow$ Line

⊙ $n=3$: $\{\underline{x} \in \mathbb{R}^3 \mid x_3 = 0\} \rightarrow$ Plane

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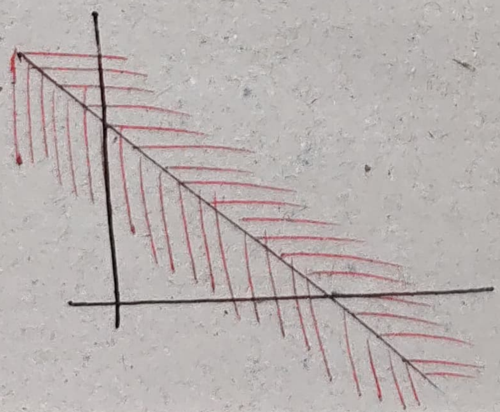
③ HALF-SPACE

Hyperplane divides n-dimensional space into two half spaces.

(ii) $\{x \in \mathbb{R}^n \mid a^T x \leq b\}$

(i) $a^T x = b$

(ii) $a^T x \leq b$ (or) $a^T x \geq b$ (or) $-a^T x \leq -b$.



Half space is

⊙ not Affine

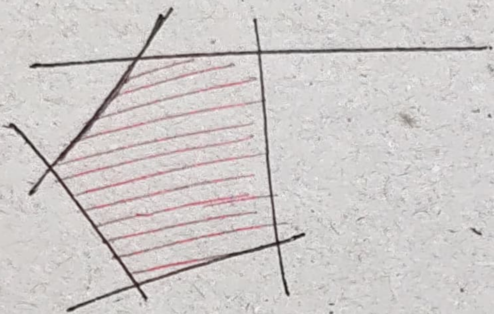
⊙ convex

⊙ generally not a convex cone unless $b=0$. (can be proved).

④ POLYHEDRON

Polyhedron is a finite intersection of Half spaces and hyperplanes.

$$\left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} a_1^T x = b_1,\ a_2^T x = b_2, \ldots, a_m^T x = b_m \\ c_1^T x \le d_1,\ c_2^T x \le d_2, \ldots, c_p^T x \le d_p \end{array} \right\}$$



Pic. 2D polyhedron.

⑤ NORM BALL

The Norm Ball, $B = \left\{ x \in \mathbb{R}^n \;\middle|\; \underbrace{\|x - x_c\|}_{\text{any Norm}} \le r \right\}$, $r \ge 0$, $x_c \in \mathbb{R}^n$.

Notation:

$B(x_c, r)$

Radius of the Norm Ball

Center of the Norm Ball

Alternate way of writing:

$$B(x_c; r) = \left\{ x_c + ru \in \mathbb{R}^n \;\middle|\; \|u\| \le 1 \right\}, \quad \text{where } x = x_c + ru$$

standard unit Norm Ball at 0, by scaling it to r and shifting it to x.

$\Rightarrow$ The Norm Ball is shifted to x_c and scaled by r.

⑥ ELLIPSOID

This is the generalization of Norm Ball.

Notation:

Center of the Ellipsoid

$\mathcal{E}(x_c, P)$

Positive Definite Matrix ($P \succ 0$), which is playing the role of radius, but it is much more than the Radius.

$$\mathcal{E}(x_c, P) = \left\{ x \in \mathbb{R}^n \;\middle|\; (x - x_c)^T P^{-1} (x - x_c) \le 1 \right\}$$

Special case of $\mathcal{E}$

$$\mathcal{E}(x_c, P) = B(x_c, r), \text{ when } P = rI.$$

Aside : Square Root Decomposition.

Let's say, the Positive Definite Matrix is Symmetric..

(ii) $P \in S^n$, $P > 0$

then its Eigen value decomposition is

$$P = E \Lambda E^T, \text{ where } EE^T = I = E^T E.$$

can we decompose in such a way that P is equal to Square root of another matrix ?

$$\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}, \quad \sqrt{\Lambda} = \begin{bmatrix} \sqrt{\lambda_1} & & & 0 \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ 0 & & & \sqrt{\lambda_n} \end{bmatrix}$$

since $\sqrt{\Lambda}\sqrt{\Lambda} = \Lambda$

$$\text{So, } P = E\Lambda E^T = E\sqrt{\Lambda}\sqrt{\Lambda}E^T$$
$$= \underbrace{E\sqrt{\Lambda}E^T}\underbrace{E\sqrt{\Lambda}E^T}$$
$$= \sqrt{P}\sqrt{P} \leftarrow \text{Matrix Square Root}$$

For any Positive Definite Matrix $(P > 0)$, we can write $P = \sqrt{P}\sqrt{P} = \sqrt{P}^2$.
It is found by just taking Square root of Eigen values. Eigen vectors remain exactly the same.

MATLAB : sqrtm() ← Square root of a Matrix

So, another way of writing the Ellipsoid set is

$$\mathcal{E}(x_c; P) = \{x_c + \sqrt{P}u \mid \underbrace{\|u\| \le 1}\}$$

↖ This definition is valid for any Norm.

l_2 case : $x = x_c + \sqrt{P}u$

$$\Rightarrow \sqrt{P}^{-1}(x - x_c) = u$$

$$\|u\|_2^2 \le 1 \Leftrightarrow (x - x_c)^T P^{-1}(x - x_c)$$

since $\underbrace{\sqrt{P}^{-1}\sqrt{P}^{-1}} = P^{-1}$

↖ Can be proved using EVD method.

Note: P is positive definite.
$\sqrt{P}$ is also positive definite.
Sqrt of Eigen Values is also positive,
$\sqrt{P}$ is therefore invertible also.

Is $B(x_c, r)$ convex ?

Let us see whether $B(0,1)$ is convex or not.

$$B(0,1) = \{x \mid \|x\| \le 1\}$$

Let the two points $x, y \in B(0,1) \Rightarrow \|x\| \le 1, \|y\| \le 1$.

consider the point $z = \theta x + (1-\theta)y$.
This is the point on the line segment between x and y.
whether z belong to $B(0,1)$ or not ?
z would belong to $B(0,1)$ if $\|z\| \leq 1$.

$$\|z\| = \underbrace{\|\theta x + (1-\theta)y\| \leq \|\theta x\| + \|(1-\theta)y\|}_{\text{Triangle inequality}}$$

$$= \underbrace{|\theta|\|x\| + |1-\theta|\|y\|}_{\text{Homogeneity property of Norms}}$$

$0 \leq \theta \leq 1$
θ and $1-\theta$ are positive
So, absolute can be removed

$$= \underbrace{\theta\|x\| + (1-\theta)\|y\|}$$

$$\leq \underbrace{\theta + (1-\theta)} = 1$$

Coz
$\|x\| \leq 1$, $\|y\| \leq 1$
$x, y \in B(0,1)$

Therefore, we proved that $\|z\| \leq 1 \Rightarrow z \in B(0,1)$.
Similar fashion, can be proved for $\mathcal{E}(x_c, P)$ also.

7 NORM CONE

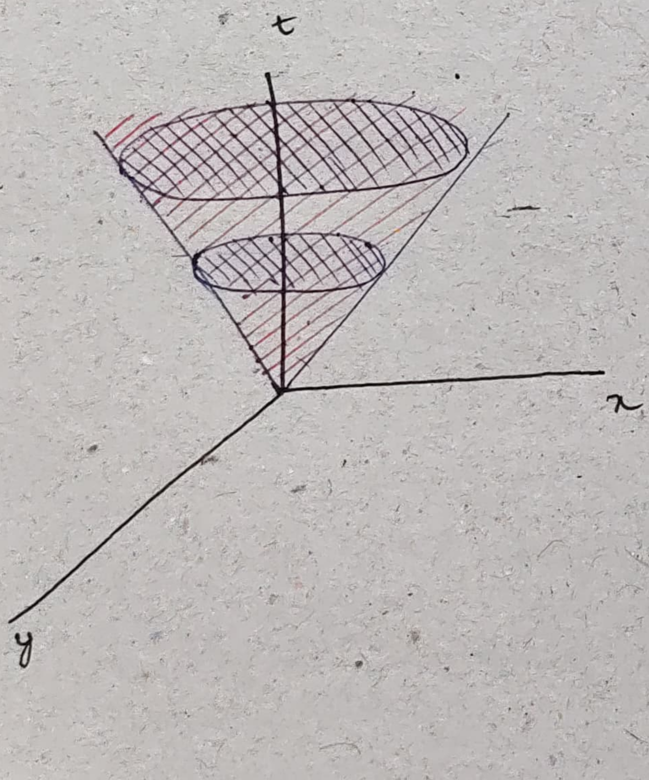
Norm Cone is defined as

$$C = \left\{ \begin{bmatrix} x \\ t \end{bmatrix} \in \mathbb{R}^{n+1} \;\middle|\; \overbrace{\underbrace{\|x\|_2 \leq t}}^{\text{implies } t \geq 0} \right\},$$

where,
x - vector $\in \mathbb{R}^n$
t - Scalar $\in \mathbb{R}$

This is a Norm Ball.
In 2D, this is a disc for fixed value of t.
If t is large, the size of disc is large
If t is small, the size of disc is small.

- n = 2 case,



- AKA,
  - Ice-cream cone
  (or)
  - Lorentz cone.

- It can be proved that this is convex cone.

SETS SPECIFIED IN TERMS OF MATRICES

Consider a set of symmetric matrices,

$$S^n = \{ X \in \mathbb{R}^{n \times n} \mid \underbrace{X = X^T}_{\text{Linear Restriction}} \}$$

$$= \left\{ X \in \mathbb{R}^{n\times n} \;\middle|\; \begin{matrix} X_{12} = X_{21} \\ X_{13} = X_{31} \\ \vdots \end{matrix} \right\}$$

What kind of set is this?

This is the intersection of hyperplane in $\mathbb{R}^{n^2}$ (i.e.) $\mathbb{R}^{n \times n}$.

Hyperplanes

- $a^T x = b$ for vectors $x \in \mathbb{R}^n$
- $\langle A, X \rangle = b$ for matrices $x \in \mathbb{R}^{n \times n}$

For example, if the restriction is $X_{12} = X_{21}$, then

$$A = \left[\begin{array}{c|c} \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right]$$

$A \in \mathbb{R}^{n \times n}$

$\Rightarrow AX = X_{12} - X_{21} = 0$

$\Rightarrow b = 0.$

This is an example where Hyperplane is specified in terms of Matrix, instead of in terms of Set.

Let us look into a bit more complicated Example.

Consider a set of Positive Semidefinite Matrices.

$$S_+^n = \{ X \in S^n \mid \underbrace{X \succeq 0}_{X \text{ is PSD}} \}$$

Let us prove that S_+^n is convex cone.

$X, Y \in S_+^n \Rightarrow X$ is PSD, Y is PSD

(i.e) $X \succeq 0$, (i.e) $Y \succeq 0$.

$\Rightarrow u^T X u \geq 0 \; \forall u$, $u^T Y u \geq 0 \; \forall u$.

Let the Matrix, $Z = \theta_1 X + \theta_2 Y$.

Z is PSD or not?

- $Z^T = \theta_1 X^T + \theta_2 Y^T = \underbrace{\theta_1 X + \theta_2 Y}$ $= Z$.

X and Y are PSD. So, naturally they are symmetric as well.

$\Rightarrow Z \in S^n$

- $v^T Z v = v^T (\theta_1 X + \theta_2 Y) v$
$= \theta_1 (v^T X v) + \theta_2 (v^T Y v)$
$\geq 0 \quad \forall v.$

$\because \theta_1, \theta_2 \geq 0$,
$v^T X v \geq 0 \ \forall v$
$v^T Y v \geq 0 \ \forall v$
$\therefore X, Y$ are PSD

$\Rightarrow Z \in S^n_+$

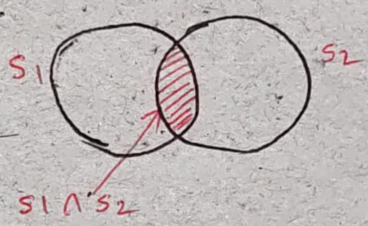
Therefore, S^n_+ is a convex cone.

OPERATIONS

Let us look at various operations that we can perform on convex sets, and these operations preserve the convexity when applied to a certain set.

① Intersection:

Two sets S_1, S_2 are convex sets. $\Rightarrow S_1 \cap S_2$ is also a convex set.



$X \in S_1 \cap S_2 \Rightarrow X \in S_1, \ X \in S_2$

$Y \in S_1 \cap S_2 \Rightarrow Y \in S_1, \ Y \in S_2$

$\Rightarrow \underbrace{\theta X + (1-\theta) Y \in S_1, \quad \theta X + (1-\theta) Y \in S_2}$

$\Downarrow$

$\theta X + (1-\theta) Y \in S_1 \cap S_2$

$\Rightarrow S_1 \cap S_2$ is a convex set.

This is also valid for infinite number of sets.

Let us consider a set $C(u)$, which is a function of u.

$$C(u) = \{X \in S^n \mid u^T X u \geq 0\}$$

If a particular value of u is fixed, then what kind of set it is?

Given u, this is an intersection of hyperplanes.

If $u^T X u \geq 0 \Rightarrow \sum X_{ij} u_i u_j \geq 0$, which is essentially a Half space, coz the elements are linearly combined, and are greater than or equal to 0.

So, essentially there are several hyperplanes, and single halfspace and we are intersecting all of them. So, the result is a POLYHEDRON.

$\Rightarrow$ $C(u)$ is Polyhedron.

Let us define C as an intersection of all the polyhedrons $C(u)$, over all u in $\mathbb{R}^n$.

$$C = \underbrace{\bigcap_{u \in \mathbb{R}^n} C(u)}$$

intersection of several polyhedrons.

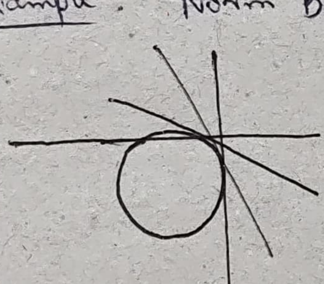
$$= \left\{ X \in \mathbb{R}^n \,\middle|\, u^T X u \geq 0 \right\} \quad \forall\, u \in \mathbb{R}^n$$

$$= \underbrace{S^n_+}$$

PSD cone.

This is an example of infinite intersection of convex sets. And the result is also convex.

Example: Norm Ball is an intersection of Half spaces.



Therefore, the operation of intersection preserves the convexity.

② Affine Transformation:

$$a(x) = Ax + b.$$

where, $a : \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

(a is a vector valued function, which goes from $\mathbb{R}^n$ to $\mathbb{R}^m$)

$b \in \mathbb{R}^m$, $x \in \mathbb{R}^n$

$A \in \mathbb{R}^{m \times n}$

$Ax + b \in \mathbb{R}^m$.

Suppose, there is a set C, which is a subset of $\mathbb{R}^n$ (Convex Set Say)

$$C \subseteq \mathbb{R}^n \text{, then the transformation of } C,$$

$$A(C) = B = \{a(\underline{x}) \mid \underline{x} \in C\}$$

This is called Affine Transformation.

$$C_{\text{convex}} \Rightarrow A(C) \text{ is Convex}$$

$$\Rightarrow (\text{image } (C) \text{ under } A)$$

Examples of Affine Transformation are

(i) <u>SCALING</u>: $A(C) = \{\alpha \underline{x} \mid \underline{x} \in C\} \in \mathbb{R}^n$

$$A = \alpha I, \quad b = 0$$

Even after scaling by a factor of α, the dimension remains exactly same.

(ii) <u>TRANSLATION</u>

$$A(C) = \{x + x_0 \mid x \in C\}$$

where, $x_0 \rightarrow$ Fixed coordinate in $\mathbb{R}^n$.

③ <u>Projection</u>

$$A(C) = \{P\underline{x} \mid \underline{x} \in C\} \subseteq \mathbb{R}^k, \quad x \in \mathbb{R}^n$$

$$P = [I_k \mid 0]_{k \times n}$$

Identity Matrix of size $k \times k$

Remaining elements of size $k \times (n-k)$

$$\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \\ x_{k+1} \\ \vdots \\ x_n \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{Retained} \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{Discarded} \end{matrix}$$

- Picks $x_1, \dots x_k$ but drops $x_{k+1}, \dots, x_n$.
- Some components of x discarded.

Example: $\underbrace{B(0,1)}_{\text{Norm Ball, centred at 0, and Radius 1}} \longrightarrow \underbrace{\mathcal{E}(x_0, P)}_{\text{General Ellipsoid centred at } x_0 \text{ with Matrix } P \text{ which is Positive Definite.}}$

We convert a Norm Ball into an Ellipsoid, by simply doing Affine Transformation.

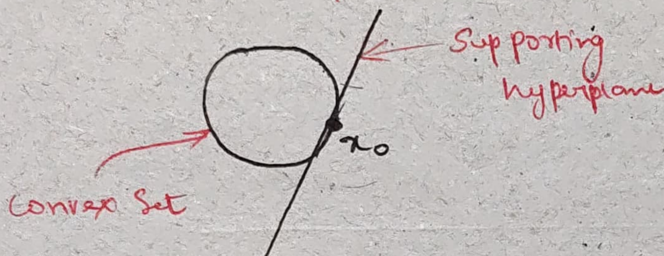
$$C = \{u \mid \|u\| \leq 1\} = B(0,1)$$

$$A(C) = \{\sqrt{P}\, u + x_0 \mid \|u\| \leq 1\} = \mathcal{E}(x_0, P)$$

SUPPORTING AND SEPARATING HYPERPLANES (in the context of convex sets).

(i) SUPPORTING HYPERPLANE

Consider a convex set C, which is not empty. Let x_0 be a point on the boundary of C. Then the supporting hyperplane is the hyperplane which passes through x_0.



The Supporting Hyperplane exists $\forall\, x_0$ in the boundary of C.

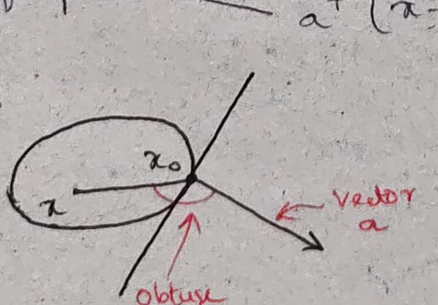
$$\text{Supporting Hyperplane} = \{a^T x = b\}.$$

C lies entirely on one side of the Hyperplane.

When the sets are not convex, it is not necessary that there is a Supporting Hyperplane. But for convex set, for every point, there is a Supporting Hyperplane so that the entire set lies on one side of that Hyperplane.

Basically, the Supporting Hyperplane will split this half space into two half spaces. One half space will contain the convex set, the other half space will not contain it.

Half Spaces $\begin{cases} a^T(x - x_0) = 0 \\ a^T(x - x_0) \leq 0 \end{cases}$

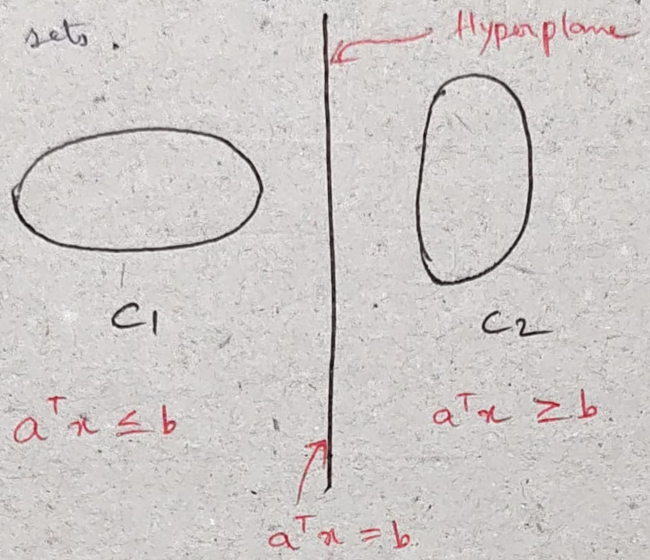


Angle between a and $(x - x_0)$ is obtuse. (i.e.) $\angle(a, x - x_0)$ is obtuse.

Any convex set is basically intersection of possibly infinite number of Half spaces.

(ii) SEPARATING HYPERPLANE

Consider there are two convex sets C_1, C_2, which are disjoint. When two convex sets are disjoint, then there always exists a hyperplane which separates these two convex sets.



Disjoint

$C_1 \cap C_2 = \emptyset$.

This Hyperplane divides the space into two half spaces. One Half space contains C_1, other Half space contains C_2.

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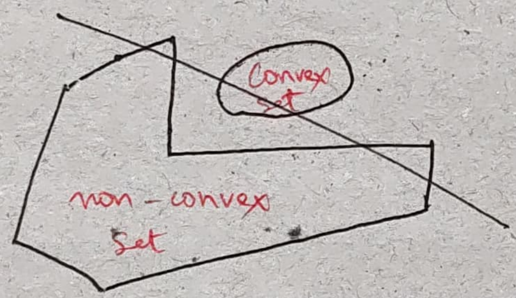
consider one of the sets is non-convex.



We cannot find any Hyperplane which separates these two Sets.

(c) No Separating Hyperplane

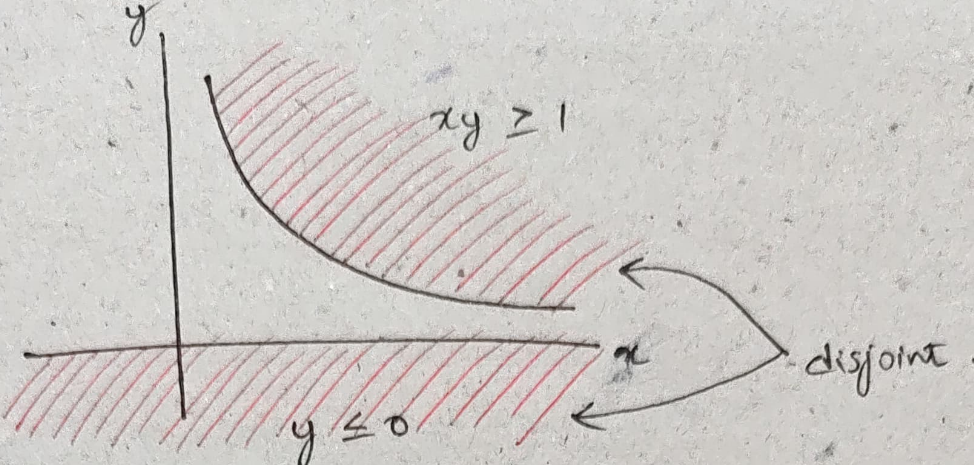
Another example in 2D space.

$$C = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2_+ \;\middle|\; xy \ge 1 \right\}$$

$$D = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; y \le 0 \right\}$$

These are two Sets in $\mathbb{R}^2$.

Let us see the pictorial representation and its Separating Hyperplane.



$\therefore$ The Separating Hyperplane is $\{y = 0\}$.

The Separating Hyperplane in this case is already included in one of the sets, which is also possible.