Week 4 OPTIMIZATION PROBLEMS

GENERAL OPTIMIZATION PROBLEMS

Let us first look into the general optimization problem.

General form : $\underbrace{x^*}_{\text{optimum}} = \arg\min_{\underbrace{x}_{\text{optimization variable}}} \underbrace{f_0(\underline{x})}_{\text{objective}}$

subject to the constraints

$\underbrace{f_i(\underline{x}) \leq 0}_{\text{inequality constraint}} \quad , \; i = 1, \ldots, m$

$\underbrace{h_j(\underline{x}) = 0}_{\text{equality constraint}} \quad , \; j = 1, \ldots, p$

Note : Apart from the inequality / equality constraints, there is another constraint which is called as Implicit constraints.

$$\underbrace{\underline{x} \in \mathrm{dom}(f_0), \; \mathrm{dom}(f_i), \; \mathrm{dom}(h_i).}_{\text{implicit constraint}}$$

$x \in D$, where

$$D = \mathrm{dom}(f_0) \cap \left(\bigcap_{i=1}^{m} \mathrm{dom}(f_i) \right) \cap \left(\bigcap_{j=1}^{p} \mathrm{dom}(h_j) \right)$$

which is usually not written explicitly.

Example : Let us take an example where the domain restrictions play an important role, so that the implicit constraint is not trivial. (ie) if $\underline{x} \in \mathbb{R}^n$, then the implicit constraint is trivial.

$$\min_{\underline{x} \in \mathbb{R}^n} \quad c^T x - \sum_{i=1}^{m} \log (a_i^T x - b_i)$$

we see that there are no explicit constraints, but the problem should be well defined.

$$\Rightarrow x \in \mathrm{dom}(f_0) = \{ \underline{x} \mid \underbrace{a_i^T x - b_i > 0}_{\text{Half space}}, \; i = 1, 2, \ldots, m \}$$

FEASIBLE SOLUTION

$\tilde{\underline{x}}$ is feasible if $\quad \tilde{\underline{x}} \in \underbrace{\mathcal{D}}_{\text{Domain of the functions.}}$

$f_i(\tilde{\underline{x}}) \leq 0 \quad , \; i = 1, 2, \ldots, m$

$h_j(\tilde{\underline{x}}) = 0 \quad , \; j = 1, 2, \ldots, p$

also: $f_0(x^*) = \boxed{\begin{array}{l}\min\limits_{x} f_0(x) \\ \text{s.t. } f_i(x) \leq 0,\ i=1,\dots,m \\ \quad h_j(x) = 0,\ j=1\dots p\end{array}} = \begin{cases} \text{finite (solvable)} \\ -\infty \text{ (unbounded below)} \\ \infty \text{ (infeasible)} \end{cases}$

Example.

$$\min_{x \in \mathbb{R}^n} \sum_{i=1}^{n} a_i x_i = \begin{cases} 0, & a_i > 0\ \forall i \\ & (x^* = 0) \\ \boxed{-\infty,} & \boxed{a_1 < 0 \\ a_2 \dots a_n \geq 0} \end{cases}$$

$$\text{st. } x_i \geq 0,\ i=1,\dots,n$$

Assume $x_1 \to \infty$

$$\underbrace{a_1 x_1}_{} + \underbrace{a_2 x_2 + \dots + a_n x_n}_{},$$

unbounded below
$a_1 < 0,\ x_1 > 0,$
$x_1 \to \infty$

Minimized for
$x_2 = x_3 = \dots = 0$

FEASIBILITY PROBLEM.

Find x, such that the following constraints are satisfied.

$$f_i(x) \leq 0,\ i=1\dots,m$$
$$h_j(x) = 0,\ j=1\dots p$$

(or)

$$\begin{array}{l}\min\ 0 \\ \text{s.t. } f_i(x) \leq 0 \\ \quad h_j(x) = 0\end{array} = \begin{cases} 0, & \text{if feasible solution exists} \\ \infty, & \text{if infeasible} \end{cases}$$

Any optimization problem can be written in standard form.

Example: $\min f_0(x)$ $\Longleftrightarrow$ $\min f_0(x)$

$l_i \leq x_i \leq u_i$ $\qquad$ $x_i - u_i \leq 0,\ i=1,\dots,n$

$\qquad\qquad\qquad\qquad$ $l_i - x_i \leq 0,\ i=1,\dots,n.$

CONVEX OPTIMIZATION PROBLEMS

Now, let us look into the convex optimization problem specifically.

Standard form of Convex optimization problem : $x^* = \arg\min_x f_0(x)$

These two functions have to be convex.

$$\text{s.t. } f_i(x) \le 0, \quad i = 1, 2, \ldots, m$$

$$h_j(x) = a_j^T x - b_j = 0, \quad j = 1, \ldots, p.$$

Equality function has to be Affine.

Note:

- Seemingly non-convex problems may be expressed/converted into convex optimization problems (standard form).
- The software used for solving these problems may only recognize the standard form.

Example

$$\min\ x_1^2 + x_2^2$$

$$\text{s.t } \frac{x_1}{1+x_2^2} \le 0$$

$$(x_1 + x_2)^2 = 0$$

Non convex constraint

Non-Affine equality

Software will throw error : not convex!

$\Rightarrow$

$$\min\ x_1^2 + x_2^2$$

$$\text{s.t. } x_1 \le 0$$

$$x_1 + x_2 = 0$$

Standard form.

Many problems thought to be non-convex for many years, turned out to be convex.

Example.

$$\begin{aligned} &\min f_0(x) \\ &f_i(x) \le 0 \\ &a_i^T x - b_i = 0 \end{aligned} \quad \underset{\text{Equivalent}}{\Longleftrightarrow} \quad \begin{aligned} &\min\ \alpha f_0(x) \\ &\beta_i f_i(x) \le 0 \\ &\gamma_i (a_i^T x - b_i) = 0 \\ &\boxed{\alpha, \beta > 0,\ \gamma_i \ne 0} \end{aligned}$$

Here, we obtain solution of one from the other.

Here, same x^* but objective value will be different.

CHANGE OF VARIABLES

Let us look into the technique of "Change of variables", which is used to transform problems from one form to another form.

$z = \phi(x)$, where ϕ is one-to-one function, meaning unique z for each x.

$\Rightarrow x = \phi^{-1}(z)$.

More concretely,

$$\boxed{\begin{aligned} z^* &= \arg\min_z f_0(z) \\ f_i(z) &\le 0 \\ h_j(z) &= 0 \\ z &\in \mathcal{D} \end{aligned}} \Rightarrow \boxed{\begin{aligned} x^* &= \arg\min f_0(\phi(x)) \\ f_i(\phi(x)) &\le 0 \\ h_j(\phi(x)) &= 0 \\ \phi(x) &\in \mathcal{D} \end{aligned}} \Rightarrow \boxed{\begin{aligned} x^* &= \arg\min \tilde{f}_0(x) \\ \tilde{f}_i(x) &\le 0 \\ \tilde{h}_j(x) &= 0 \\ x &\in \tilde{\mathcal{D}} \end{aligned}}$$

$\{x \mid \phi(x) \in \mathcal{D}\}$

Could be convex

Note: $x^* = \phi^{-1}(z^*)$
$z^* = \phi(x)$

Example:

$$\min_{x \in \mathbb{R}^3} \; c_1 \log x_1 + c_2 \log x_2 + c_3 \log x_3$$
$$\text{s.t } a_1 \log x_1 + a_2 \log x_2 + a_3 \log x_3 \le b$$
$$x_i > 0, \; i = 1,2,3$$
$$c_i, a_i > 0$$

$\because$ there exists log, this is concave if all c_i's are positive.

Not convex, as objective and constraints are not convex.

Consider $\phi(\cdot) = \log(\cdot)$, $z_i = \log(x_i)$
$\Rightarrow x_i = e^{z_i}$

$$z^* = \arg\min_z \; c_1 z_1 + c_2 z_2 + c_3 z_3$$
$$= a_1 z_1 + a_2 z_2 + a_3 z_3 \le b,$$
$$z_i \ge 0$$

Linear Program (LP)

$$x_i^* = e^{z_i^*}$$

INTRODUCE NEW VARIABLES

Let us look at another technique of converting problems from one form to another, which is "Introduction of new variables".

(i) Epigraph trick

consider an optimization problem of form

$$\begin{array}{ll} \min\limits_{x} & f_0(\underline{x}) \\ \text{s.t} & f_i(x) \le 0 \end{array} \Rightarrow \begin{array}{ll} \min\limits_{x,t} & t \\ & f_0(x) \le t \\ & f_i(x) \le 0 \end{array}$$

Note: When $f_0(x)$ is convex, then

$f_0(x) - t$ is also convex in $(\underline{x}, t)$

convex   Affine function.

(ii) Slack variables.

$$a_i^T x - b_i \le 0 \Rightarrow a_i^T x - b_i + s_i = 0, \quad s_i \ge 0$$

(iii) Equality constraints can be eliminated.

$$\begin{array}{ll} \min\limits_{x_1, x_2} & f_1(x_1) + f_2(x_2) \\ & x_1 + x_2 = 1 \end{array} \Rightarrow \min_{\theta} f_1(\theta) + f_2(1-\theta)$$

$x_1 = \theta, \ x_2 = 1-\theta$

General approach

$$\begin{array}{l} \min f(x) \\ Ax = b \end{array}, \quad x \in \mathbb{R}^n, \ b \in \mathbb{R}^m, \ m < n, \ A \in \mathbb{R}^{m \times n}$$

Suppose that $b \in R(A)$, then the solution exists.

(a) There exists x_0 s.t $Ax_0 = b$.

Feasible region: $\mathcal{X} = \{x \mid Ax = b\}$

Let $x = x - x_0 + x_0$

$$Ax = A(x - x_0) + Ax_0 = b$$

$$\Rightarrow A(x - x_0) = 0$$

$$\Rightarrow x - x_0 \in \mathcal{N}(A)$$

Null Space of A

Suppose $N(A)$ has basis vectors $c_1, c_2, \dots, c_{m-r}$

then $x - x_0 = Cu$, $u \in \mathbb{R}^{m-r}$

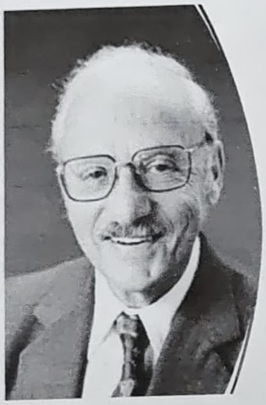
$$x = Cu + x_0$$

Therefore, the original problem becomes

$$\min_{u \in \mathbb{R}^{m-r}} f(Cu + x_0)$$

Note: Problem complexity may not reduce.

LINEAR PROGRAM (LP) HISTORY



George Dantzig

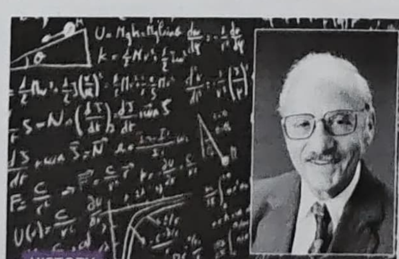
- Invented the Simplex Algorithm (1947)
- Dominant algorithm 1950s to 80s
- Father of Optimization

https://mathshistory.st-andrews.ac.uk/Biographies/Dantzig_George/

Jerzy Neyman

(prof. of George Dantzig)

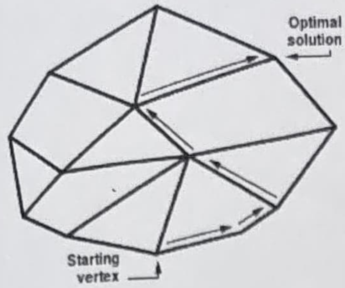
https://opc.mfo.de/detail?photo_id=3044



After getting late to class, **George Dantzig** copied from the blackboard two problems thinking they were homework, and then **solved them**. They were actually **two famous unsolved statistics problems**, which earned him his **PhD**.

https://i.redd.it/yiuxlfdttbt51.jpg

Simplex Algorithm

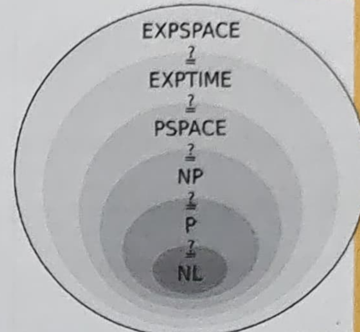


https://commons.wikimedia.org/wiki/File:Simplex_description.png

- Fastest Algorithm from 1950s to 1980s
- Worst case could be slow

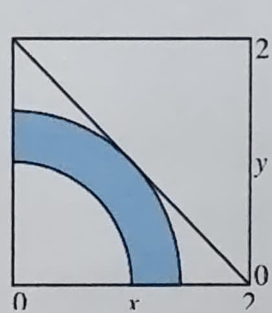
Computational Complexity of Simplex

- Worst case complexity exp(n)
- Stress on polynomial time algorithms



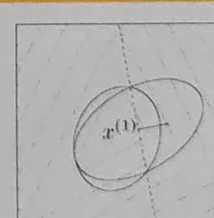
Non-linear programming

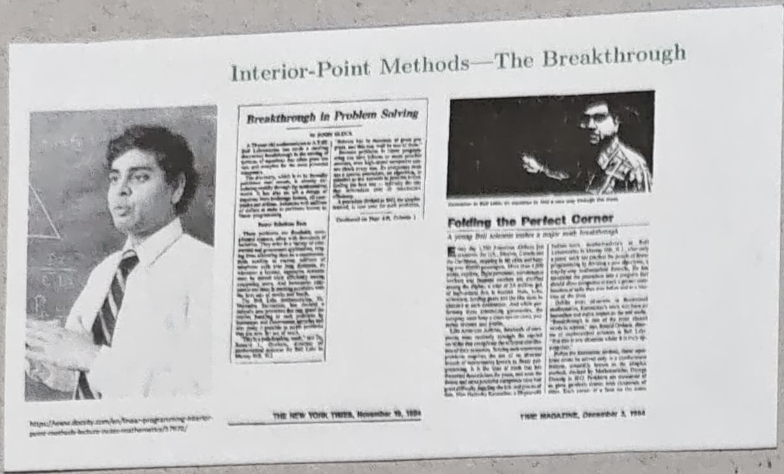
- Considered to be more difficult
- No counterpart to simplex
- Slow algorithms

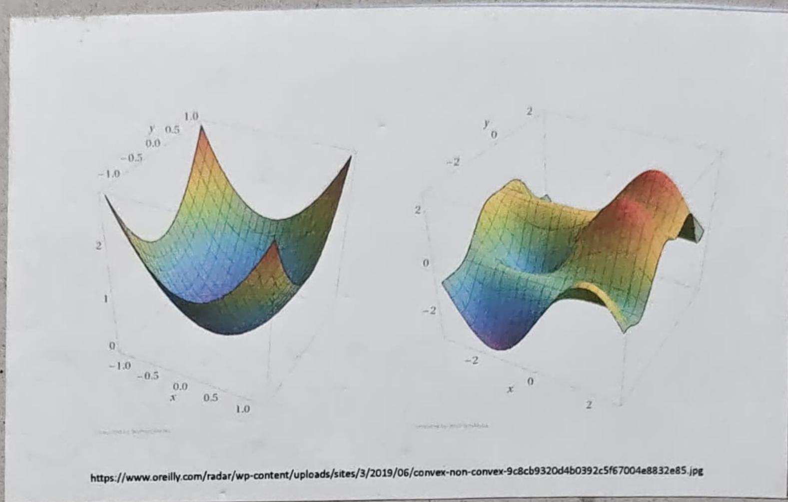


Leonid Khachiyan: Ellipsoid method

http://www.cs.rutgers.edu/Khachiyan/





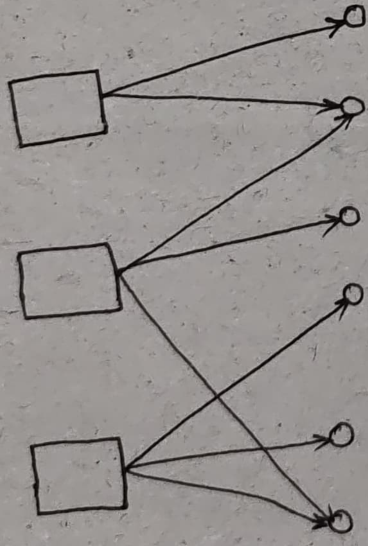


LINEAR PROGRAMS (LP)

History : LP is used in Aircraft scheduling during World War 2.

Applications : Expenditure planning, Logistics supply chain, e-commerce, Financial planning, Portfolio optimization.

Example : Realistic problem to appreciate LP (LOGISTICS)



Warehouse $i = 1, 2, \ldots, W$

Retail outlet (shops) $j = 1, 2, \ldots, O$

Let

c_{ij} = Cost of transport (per unit) from warehouse i to outlet j

X_{ij} = Quantity sent from $i \to j$

S_i = Supply / availability at warehouse i.

d_j = Demand at outlet j

N_i = Neighbors of i (Outlets connected to i)

N_j = Neighbors of j (Warehouses connected to outlet j)

With these notations, let us write down the optimization problem.

objective : Minimize the cost.

(LP)
$$\min_{\{x_{ij}\}} \sum_{i=1}^{W} \sum_{j \in N_i} c_{ij} x_{ij}$$

subject to constraints

- $x_{ij} \geq 0$
- Total quantity shipped to outlet j $\rightarrow$ $\displaystyle\sum_{k \in N_j} x_{kj} \geq d_j$ $\leftarrow$ demand at outlet j
- Total quantity shipped by i $\rightarrow$ $\displaystyle\sum_{j \in N_i} x_{ij} \leq S_i$ $\leftarrow$ Supply at i

This is one of the oldest algorithms (ie) Simplex algorithm and is still widely used.

$$\Rightarrow \min \ c^T x$$
$$\text{s.t } Gx \leq h \quad \leftarrow \text{inequality constraint}$$
$$Ax = b \quad \leftarrow \text{Equality constraint}$$

Can we possibly converge like this? Yes.

$$\Rightarrow \boxed{\begin{array}{ll} \min & c^T y \\ \text{s.t} & Hy = d \\ & y \geq 0 \end{array}}$$

Non-negative constraint

This is the general form for LP problems.

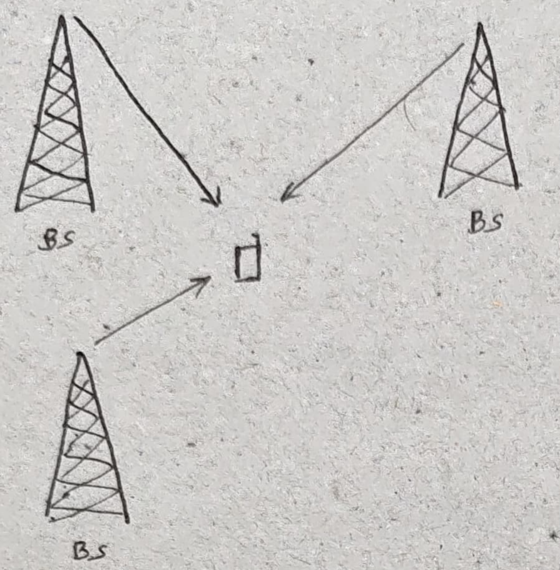
(a): Slack variable

$$Gx \leq h \iff Gx + s = h, \quad s \geq 0$$
$$\begin{bmatrix} G & I \end{bmatrix} \begin{bmatrix} x \\ s \end{bmatrix} = h$$

x could be positive.

(b) $x = u - v$, where $u \geq 0$, $v \geq 0$.

COMMUNICATION EXAMPLE



Assume that, user can combine signals received from multiple B.S.

N = No. of B.S

M = No. of users

- Tx. power (optimization variable) $\rightarrow p_i^j$

  where, $j \rightarrow$ B.S index ($j = 1, \dots, N$)

  $i \rightarrow$ User index ($i = 1, \dots, M$)

- Channel gain (measured) $\rightarrow g_i^j$

The Rx. power at the i^{th} user $= \sum_{j=1}^{N} p_i^j g_i^j$

This is called coherent combining of signals from multiple BS.

Let's formulate the problems.

P1 :

Minimize Tx power but ensure Rx power is not too low.

$$(LP) \quad \min_{\{p_i^j\}} \sum_{i=1}^{M} \sum_{j=1}^{N} p_i^j$$

$$\text{s.t.} \quad \circ \ \sum_{j=1}^{N} p_i^j g_i^j \geq \gamma_i$$

$$\circ \ p_i^j \geq 0 \quad \forall\, i, j$$

Rx power should be above the threshold to decode the signal

P2 :

Maximize Rx power at the worst user

User receiving least power

Fair allocation.

$$\max_{\{p_i^j\}} \left(\min_{1 \leq i \leq M} \underbrace{\overbrace{\sum_{j=1}^{N} p_i^j g_i^j}^{\text{Rx. power at the worst user}}}_{\text{Rx. power (Concave function)}} \right)$$

$$\text{s.t.} \quad \circ \ \sum p_i^j \leq p_{max}^j$$

$$\circ \ p_i^j \geq 0.$$

Power budget at BS j.

Is it LP ? Apply Epigraph trick.

Aside :

$$F = \max f(x) = -\min -f(x)$$
$$G = \min g(x) = -\max -g(x)$$

So,

$$-\min_{\{p_i^j\}} -\left(\min_i \sum_{j=1}^{N} p_i^j g_i^j\right)$$

$$= -\min_{\{p_i^j\}} \underbrace{\left(\max_i -\sum_{j=1}^{N} p_i^j g_i^j\right)}_{\text{This is convex}}$$

$$= -\min_{\{p_i^j, t\}} t$$
$$\text{s.t} \left(\max_{1 \le i \le M} -\sum_{j=1}^{N} p_i^j g_i^j\right) \le t$$

$\min f(x)$

$\Updownarrow$

$\min t$

$\text{s.t. } f(x) \le t$

$$= -\min_{\{p_i^j, t\}} t$$

Affine inequality

$$\text{s.t} \quad ① \; -\sum_{j=1}^{N} p_i^j g_i^j \le t \;, \quad i = 1, 2, \ldots, M.$$

$$② \; \sum p_i^j \ge 0$$

This is the final LP.

- LP are usually simpler to solve than general convex.

QUADRATIC PROBLEMS (QP)

Quadratic optimization problems / Quadratic programs takes the form

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^T P x + q^T x \quad \leftarrow \text{Quadratic objective.}$$

$$\left.\begin{aligned} \text{s.t.} \quad Gx &\le h \\ Ax &= b \end{aligned}\right\} \leftarrow \text{Linear constraints}$$

QP is convex when $P \succeq 0$.

Note : QP are easy to solve when there is no constraints.
(i.e.) $A=0,\ G=0,\ h=0,\ b=0$.

then $\min_{x} \frac{1}{2} x^T P x + q^T x$

$$\Rightarrow \nabla\left(\frac{1}{2} x^T P x + q^T x\right) = 0$$

(or) $Px + q = 0$.

QCQP :

Quadratically constrained QP.

$$\min \frac{1}{2} x^T P_0 x + q_0^T x$$

$$\text{s.t. } \frac{1}{2} x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, \dots, m$$

$$Ax = b, \quad x \in \mathbb{R}^n.$$

QCQP is convex when $P_0, P_i \succeq 0$ (i.e.) PSD.

Example : $\omega = \min_{x} x^T A x$

where $A \nsucceq 0$ (i.e.) A is not PSD

$A \in S^n$ (i.e.) A is still symmetric

$\Rightarrow \lambda_{min}(A) < 0$ (i.e.) a negative eigen value

Suppose $Au = \underbrace{\lambda_{min}(A)}\, u$

Minimum Eigen value.

Let us take $\underline{x} = \alpha \underline{u}$

$\omega = \min x^T A x$, $\quad v = \min x^T A x$

$\underline{x} = \alpha . \underline{u}$ Additional constraint

$\omega \leq v$ More constraints may hurt.

$$v = \min_{\alpha} \alpha^2 (u^T A u)$$

$$= \min_{\alpha} \underbrace{\alpha^2}_{\geq 0} \underbrace{\lambda_{min}(A)}_{<0} \underbrace{\|u\|^2}_{\geq 0}$$

λ_{min} is Negative

$\|u\|^2$ is Positive

α^2 is Positive

As $\alpha \to \infty$, $\min\limits_{\alpha} \alpha^2 \lambda_{min}(A) \|u\|^2 \to -\infty$

Therefore, this problem is definitely unbounded below.

$v \to -\infty$, $w \le v$

$\Rightarrow w \to -\infty$.