Week 7 CONJUGATE FUNCTIONS

Definition : $f^*(y) = \max_{x \in dom(f)} \underbrace{\langle y, x \rangle - f(x)}_{\text{Affine in } y}$

Therefore, conjugate function is always convex regardless of what f is. (ie) even f is not convex.

$\Rightarrow f^*(y) \geq \langle y, x \rangle - f(x) \quad \forall x$

$\Rightarrow f(x) + f^*(y) \geq \langle y, x \rangle$

Examples :

① $f(x) = \frac{1}{2} x^T x$

conjugate function, $f^*(y) = \max_x y^T x - \frac{1}{2} x^T x = \frac{1}{2} y^T y$

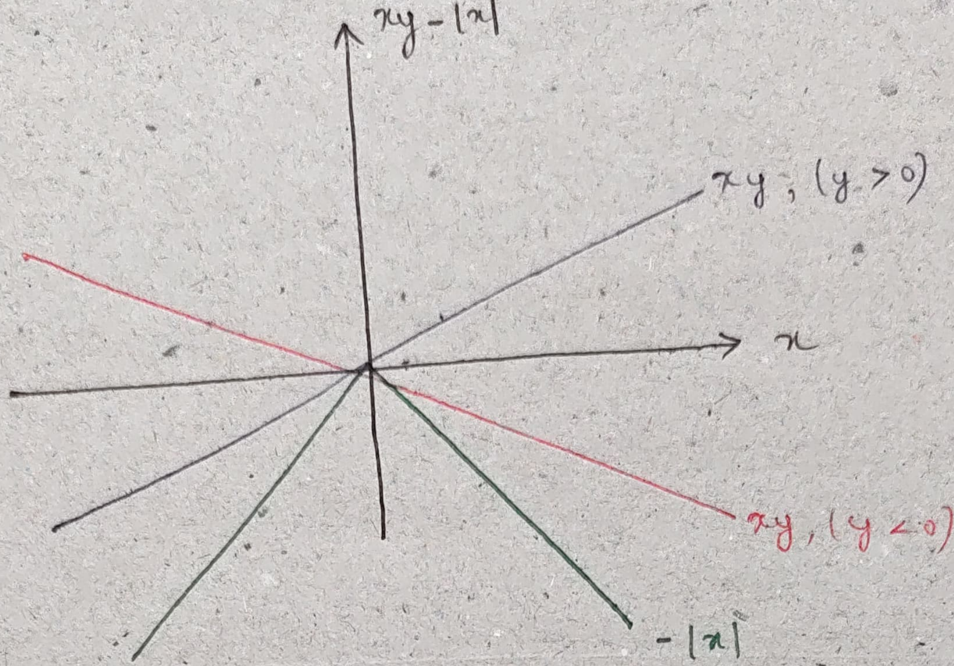
($y = x$ maximizes the RHS).

$\boxed{\frac{1}{2} x^T x + \frac{1}{2} y^T y \geq x^T y}$ inequality would take the form.

② $f(x) = \|x\|_2$

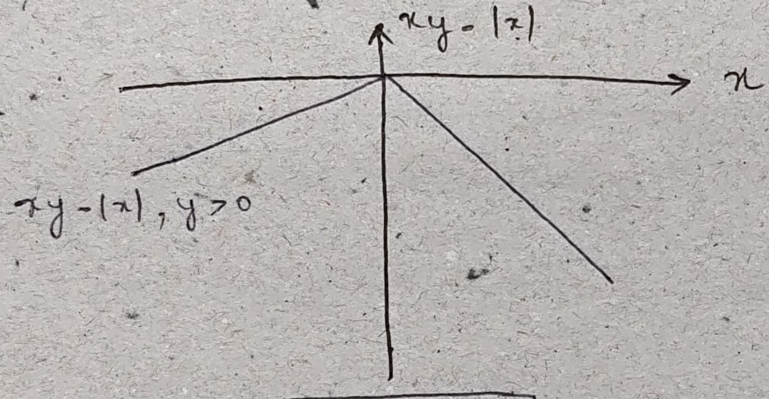
$f(y^*) = \max_x x^T y - \|x\|_2$

$n = 1$ case, $f^*(y) = \max_x xy - |x|$



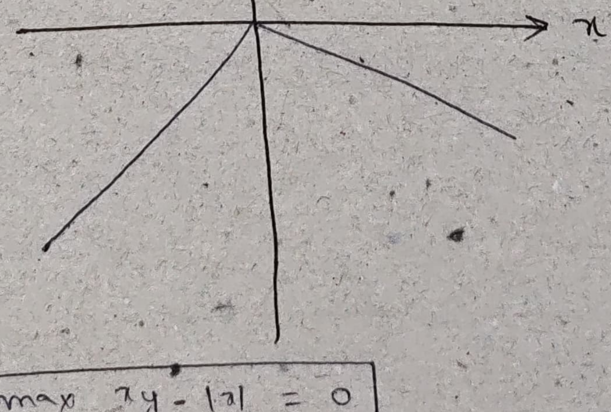
Further split into cases : (y small or large)

Case 1 (a) : $y > 0$, (i) y is small.



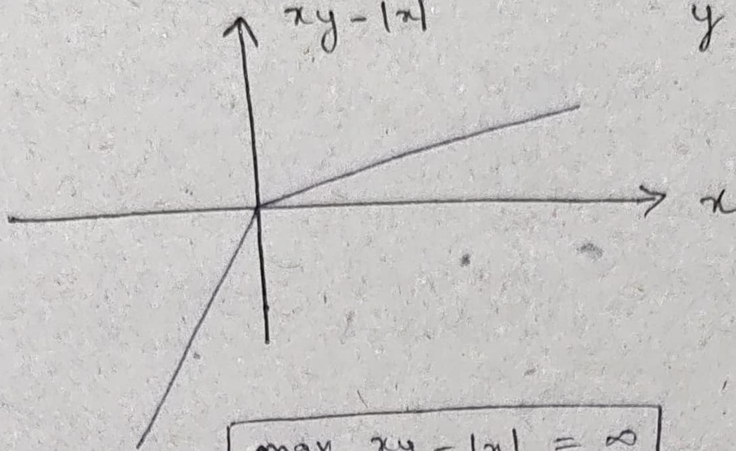
$\boxed{\max \; xy - |x| = 0}$

Case 2 (a) : $y < 0$, (i) y is small



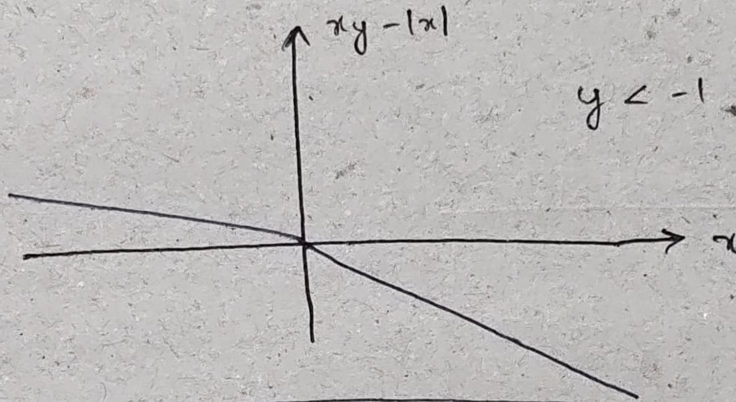
$\boxed{\max \; xy - |x| = 0}$

Case 1 (b) : $y > 0$, (ii) y is large



$$\boxed{\max\ xy - |x| = \infty}$$

Case 2 (b) : $y < 0$, (ii) y is large



$$\boxed{\max\ xy - |x| = \infty}$$

$$f^*(y) = \begin{cases} 0, & |y| \text{ is small}, \ |y| \le 1 \\ \infty, & |y| \text{ is large}, \ |y| > 1 \end{cases}$$

$$\text{dom}\,(f^*) = \{ y \mid |y| \le 1 \}.$$

General case :

Case (a) : $\|y\| \le 1$

C.S. inequality,

$x^T y \le \|x\| \, \|y\|$

$\Rightarrow x^T y - \|x\| \le \|x\| \, \|y\| - \|x\|$

$\Rightarrow x^T y - \|x\| \le \|x\| (\|y\| - 1) \le 0$

$\Rightarrow \max_x \ x^T y - \|x\| \le 0$

Also, $x = 0$ then $x^T y - \|x\| = 0$

$$\Rightarrow \max_x \; x^T y - \|x\| = 0 \quad , \text{ when } \|y\| \leq 1$$

Case (b): $\|y\| > 1$

Let us consider a special case where

$$x = \frac{\alpha y}{\|y\|}$$

$$x^T y - \|x\| = \frac{\alpha y^T y}{\|y\|} - \alpha = \alpha (\|y\|_2 - 1)$$

So, as $\alpha \to \infty$, $x^T y - \|x\| \to \infty$.

$$f^*(y) = \begin{cases} 0 & , \|y\| \leq 1 \\ \infty & ; \text{ otherwise} \end{cases}$$

Linear Fractional Program (LFP)

It is a special case of optimization problem where the convexity is hidden, and we need to apply certain Transform in order to reveal it.

$$\min f_0(x) = \frac{c^T x + d}{e^T x + f} \quad \leftarrow \text{Linear fractional (not convex function)}$$

$$\text{s.t. } Gx \leq h$$
$$Ax = b$$

$$\text{dom}(f_0) = \{ x \mid e^T x + f > 0 \}$$

Convexity is hidden!

Reveal using Charnes - Cooper transform:

$$y = \frac{x}{e^T x + f} \quad , \quad t = \frac{1}{e^T x + f}.$$

$$e^T y + f t = \frac{e^T x + f}{e^T x + f} = 1$$

Also, $x = \dfrac{y}{t}$

$Gx \leq h$ can be written as $\dfrac{Gx}{e^T x + f} \leq \dfrac{h}{e^T x + f}$

$$\Rightarrow Gy \leq ht.$$

$$Ax = b \iff Ay = bt$$

$$\frac{c^T x + d}{e^T x + f} = c^T y + dt$$

Therefore,

$$\min \; c^T y + dt$$

s.t.
- $e^T y + ft = 1$
- $Gy \leq ht$
- $Ay = bt$

This is an LP.

Other transformations also possible.

Generalized LFP

Lets look into generalized LFP which is actually not a convex problem and it cannot be converted into convex form either. However, let us see a method by which we can solve it efficiently.

So, this is to demonstrate the idea that, even if we dont have convex problem, sometimes it can be solved using convex optimization in an efficient manner.

Generalized LFP

- Not a convex problem
- There is no transform which can be used to make it into a convex problem.

Generalized LFP takes the form,

$$\min \left\{ \max_i \frac{c_i^T x + d_i}{e_i^T x + f_i} \right\}$$

$$\text{s.t.} \quad Gx \leq h$$
$$Ax = b$$

Applying epigraph trick, (Charnes – Cooper transform is not applicable).

$$\min_{x, t} \; t$$

$$\text{s.t.} \quad \frac{c_i^T x + d_i}{e_i^T x + f_i} \leq t, \quad \forall i$$
$$Gx \leq h$$
$$Ax = b$$

dom:

$e_i^T x + f_i > 0$

How to solve?

Observe : $g_i(x) = \dfrac{c_i^T x + d_i}{e_i^T x + f_i}$ (Neither convex nor Concave)
(but Quasi convex).

Given α, $C_\alpha^i = \{ x \mid g_i(x) \leq \alpha \}$ Is this convex set ?

$$= \{ x \mid c_i^T x + d_i \leq \alpha (e_i^T x + f_i) \}$$

$$= \{ x \mid \underbrace{(c_i - \alpha e_i)^T x + (d_i - \alpha f_i) \leq 0}_{\text{half space.}} \}$$

Perform line search (eg. bisection) on α.
Consider this convex feasibility problem denoted by (P_α).

$$\begin{aligned} &\underset{x}{\text{find}} \quad x \\ &\text{s.t. } g_i(x) \leq \alpha \leftarrow \text{convert into Convex for fixed } \alpha \\ &\quad Gx < h \\ &\quad Ax = b. \end{aligned}$$

for $\alpha \to \infty$, Constraint will be easily satisfied.
(ie) P_α is Feasible.

for $\alpha \to -\infty$, constraint not satisfied.
(ie) P_α is infeasible.

Alternative way of solving the problem is to solve P_α by finding the smallest α s.t. constraint satisfied. (ie) P_α is feasible.

So, the original problem has now been recast into a problem of finding α s.t. P_α is feasible.

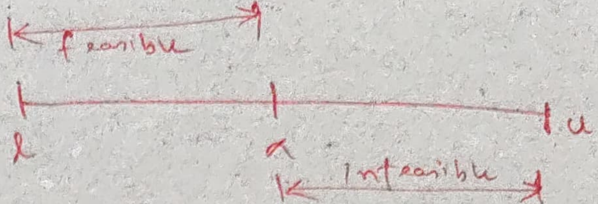
Suppose $l \leq \alpha \leq u$ (we know apriori).
For each α, solve P_α

Bisection algorithm will proceed like as follows

$\longrightarrow$

given $l \le \alpha \le u$,

set $\alpha = \frac{l+u}{2}$



Check by solving P_α

- This problem is infeasible : $u = \alpha$
- This problem is feasible : $l = \alpha$

Repeat this until $(u - l) \le t$

⊙ Takes $\sim O(\log 1/\varepsilon)$ iterations.

Solve using a series of convex problems.

Summary. (G-LFP)

P_α : find x

$(c_i - \alpha e_i)^T x + (d_i - \alpha f_i) \le 0 \quad \forall i$.

⊙ $Gx \le h$

⊙ $Ax = b$

⊙ $e_i^T x + f_i > 0 \quad \forall i$

- This is a convex problem.

Can be solved using cvx / linprog in MATLAB.

The original Quasi-convex problem can be solved using series of convex, with $O\left(\log(1/\varepsilon)\right)$.

Geometric Programs (GP)

Geometric Program is a class of problems that is not in the standard convex form, but they can be converted into convex optimization problems.

Definition

① MONOMIAL : $h(x) = d\, x_1^{a^{(1)}} x_2^{a^{(2)}} \cdots x_n^{a^{(n)}}$

where $d \ge 0$, $a^{(i)} \in \mathbb{R}$.

$h : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$

Example : $4x_1^2 x_2^{-1/2} x_3^{-3}$ (or) x_1/x_2 are Monomial

but $-x_1$ is not a monomial.

Note: $h(x)$ is monomial $\Rightarrow \frac{1}{h(x)}$ is monomial

$$\therefore \frac{1}{h(x)} = \left(\frac{1}{d}\right) x_1^{-a^{(1)}} x_2^{-a^{(2)}} \cdots x_n^{-a^{(n)}}$$

$$\frac{1}{d} \geq 0, \quad -a^{(i)} \in \mathbb{R}.$$

$\therefore \frac{1}{h(x)}$ is also monomial.

② POSYNOMIAL : Sum of monomials

$$g(x) = \sum_{k=1}^{K} d_k x_1^{a_k^{(1)}} x_2^{a_k^{(2)}} \cdots x_n^{a_k^{(n)}}$$

Example:

$x_1 + x_2$, $x_1^2 + x_2$, $x_1 x_2 + \frac{x_2}{x_3}$ are Posynomial

$x_1 - x_2$, $\frac{1}{x_1 + x_2}$ are not posynomial.

GP

$$\min \; g_0(x)$$
$$\text{s.t.} \quad \circ \; g_i(x) \leq 1, \quad i = 1, \dots, m$$
$$\circ \; h_l(x) = 1, \quad l = 1, \dots, p$$

Posynomial

Monomial

Note: Monomials / Posynomials are not convex functions.

How to get convex form of GP?

$$y_i = \log(x_i) \quad (\text{or}) \quad x_i = e^{y_i}$$

$$h_l(x) = d_l \, x_1^{a_l^{(1)}} x_2^{a_l^{(2)}} \cdots x_n^{a_l^{(n)}}$$

$$= d_l \left(\exp(a_l^{(1)} x_1) \exp(a_l^{(2)} x_2) \cdots \exp(a_l^{(n)} x_n) \right)$$

$$= d_l \left(e^{a_l^{(1)} y_1 + a_l^{(2)} y_2 + \cdots + a_l^{(n)} y_n} \right)$$

$$= \exp\left(\underline{a}_l^T \underline{y} + b_l \right), \text{ where } a_l = \begin{bmatrix} a_l^{(1)} \\ \vdots \\ a_l^{(n)} \end{bmatrix}, \; b_l = \log d_l$$

Therefore, $h(x) = 1 \Leftrightarrow a_l^T y + b_l = 0$

After this transformation, the monomial constraint gets transformed into Affine equality constraint.

Likewise, $g_i(x) = \sum_{k=1}^{K} d_{ik} \, x_1^{a_{ik}^{(1)}} x_2^{a_{ik}^{(2)}} \cdots x_n^{a_{ik}^{(n)}}$

$$= \sum_{k=1}^{K} \exp\left(\underline{a}_{ik}^T y + \underline{b}_{ik}\right)$$

$$g_i(x) \le 1 \iff \log(g_i(x)) \le 0$$

$$\Rightarrow \log\left(\sum_{k=1}^{K} \exp\left(a_{ik}^T y + b_{ik}\right)\right) \le 0$$

This is known as log-sum-exp function, which is a convex function.

Summary.

$$\min \log\left(\sum_{k=1}^{K} \exp\left(a_{0k}^T y + b_{0k}\right)\right)$$

$$\text{s.t.} \quad \log\left(\sum_{k=1}^{K} \exp\left(a_{ik}^T y + b_{ik}\right)\right) \le 0, \quad i = 1, \dots, m$$

$$a_l^T y + b_l = 0, \quad l = 1, \dots, p.$$

How to manipulate GPs.

(1) $\max h(x) \equiv \min \left(\frac{1}{h(x)}\right)$

(2) posynomial $\le$ monomial constraints

$$g_i(x) \le h_i(x) \iff \frac{g_i(x)}{h_i(x)} \le 1$$

Posynomial

(3) Product of posynomials is also posynomial

(4) $\min \dfrac{g_1(x)}{h(x) - g_2(x)} \equiv \min t$

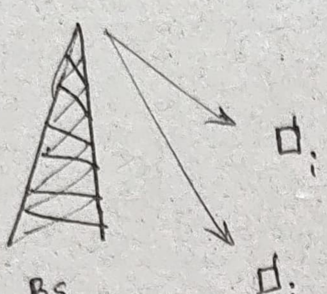
$$\frac{g_1(x)}{h(x) - g_2(x)} \le t$$

$$g_1(x) \le h(x) t - g_2(x) t$$

$$\frac{g_1(x) + t\, g_2(x)}{t\, h(x)} \le 1$$

where, $\dfrac{g_1(x)}{t\, h(x)}$ and $\dfrac{g_2(x)}{h(x)}$ are posynomials.

Power Control problem solved using GPs



BS

$g_{ii} \rightarrow$ Gain

$p_i \rightarrow$ Tx. power

$p_i g_{ii} \rightarrow$ Rx. power

$n_i \rightarrow$ Noise power

$$\sum_{j \neq i} p_i \underbrace{g_{ij}}_{\text{leakage}} \rightarrow \text{Interference}$$

Signal-to-interference-plus-Noise (SINR)

$$= \frac{\text{useful signal power}}{\text{interference + Noise power}} = \frac{p_i g_{ii}}{\sum_{j \neq i} p_i g_{ij} + n_i} = \gamma_i$$

Generally, when p_i increases, Rx power increases
Interference increases
but SINR may decrease.

Sum-rate Maximization Problem

$$\boxed{\begin{aligned} &\max_{\{p_i\}} \sum w_i \log(1+\gamma_i), \quad w_i > 0 \\ &\text{(i.e.) Maximize weighted sum of user rate} \\ &\text{s.t.} \quad \underbrace{\min_i \left[\log(1+\gamma_i)\right]}_{\text{QoS}} \geq R_{th} \quad \text{Threshold} \\ &0 \leq p_i \leq P \end{aligned}}$$

This is the Sum-rate maximization problem, as it arises in some communication systems, and we want to find out the power allocation, so as to ensure that the Sum-rate is maximized.

QoS Constraint:

$$\log(1+\gamma_i) \geq R_{th} \quad \forall i$$

$$\text{(or)} \quad \gamma_i \geq \exp(R_{Th}) - 1 = \gamma_{Th}$$

Now, the problem is of the form,

$$\left.\begin{aligned} &\max_{\{P_i\}} \sum w_i \log(1+\gamma_i) \\ &\text{s.t.} \quad \text{①}\ \gamma_i \geq \gamma_{Th} \\ &\qquad \text{②}\ 0 \leq p_i \leq P \end{aligned}\right\} \text{Not convex.}$$

Assume High SNR case : γ_{Th} is high.

(i) $\gamma_i >> 1 \Rightarrow \log(1+\gamma_i) \simeq \log(\gamma_i)$

The problem becomes,

$$\begin{aligned} &\max \sum w_i \log(\gamma_i) \\ &\text{s.t.} \quad \text{①}\ \gamma_i \geq \gamma_{Th} \\ &\qquad \text{②}\ 0 \leq p_i \leq P \end{aligned}$$

Recall, $\gamma_i = \dfrac{p_i g_{ii}}{\sum_{j \neq i} p_i g_{ij} + n_i}$

$$\Rightarrow \frac{1}{\gamma_i} = \underbrace{\sum p_j p_i^{-1} g_{ij} g_{ii}^{-1} + n_i p_i^{-1} g_{ii}^{-1}}_{\text{Sum of Monomials} \Rightarrow \text{Posynomial}}$$

$$\gamma_i \geq \gamma_{Th} \Rightarrow \underbrace{\gamma_{Th}\left(\frac{1}{\gamma_i}\right)}_{\text{This is posynomial}} \leq 1.$$

$$p_i \leq P \Rightarrow \underbrace{\frac{p_i}{P}}_{\text{This is monomial.}} \leq 1$$

$$\max \sum w_i \log(\gamma_i) \quad , \quad w_i > 0$$

$$\text{(or)} \; -\min_{\{P_i\}} -\sum w_i \log(\gamma_i) \quad \text{(or)} \; \min \sum_i \log\left(\frac{1}{\gamma_i}\right)^{w_i}$$

$$\text{(or)} \; \max \prod \gamma_i^{w_i} \quad \text{(or)} \; \min \prod \left(\frac{1}{\gamma_i}\right)^{w_i}$$

Applying Epigraph trick,

$$\min \prod t_i^{w_i}$$

$$\text{s.t. } \frac{1}{\gamma_i} \leq t_i \quad \text{(or)} \quad \frac{1}{t_i \gamma_i} \leq 1$$

Posynomial

<u>Summary</u>.

In high SNR case,

$$\min \prod t_i^{w_i}$$

$$\text{s.t.} \begin{cases} \odot \sum p_j p_i^{-1} t_i g_{ij} g_{ii}^{-1} + n_i p_i^{-1} g_{ii}^{-1} t_i \leq 1 \\ \odot \sum p_j p_i^{-1} g_{ij} g_{ii}^{-1} \gamma_{Th} + n_i p_i^{-1} g_{ii}^{-1} \gamma_{Th} \leq 1. \\ \odot \frac{p_i}{p} \leq 1. \end{cases}$$

GP constraints

$\therefore$ High SNR case $\longrightarrow$ GP.