Week 6 : Session 1

Detection over Wireless Channel

So far we've seen several detection problems, but we have not yet incorporated the effect of the wireless channel in these problems. So, what happens when we try to detect a signal over the wireless channel.

Let us start considering the impact of the wireless channel in the detection problems that we've seen so far.

Previously, we've seen the BPSK modulation, which is one of the most basic and popular modulation schemes, where $x \in \{-A, A\}$, $s = A$. Under Hypothesis (Wireline/AWGN)

$$\left.\begin{aligned} H_0 &: y = -A + v \\ H_1 &: y = A + v \end{aligned}\right\}$$

where 'v' is the Gaussian noise with zero mean and variance $= \sigma^2$.

(i.e.) $v \sim \mathcal{N}(0, \sigma^2)$

H_1 — Alternative Hypothesis

H_0 — NULL Hypothesis.

$\sigma^2 = \frac{N_0}{2}$ (For the comm. System)

Under Hypothesis H_0, the symbol is $-A$. Under Hypothesis H_1, the symbol is A. There are two phases ($0°$ and $180°$). Thus the phase difference is $180°$. No. of bits per symbol is $\log_2 2 = 1$. So, this is Binary Phase Shift Keying (BPSK).

The probability of error, P_e is

$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

$$= Q\left(\sqrt{\frac{E_b}{N_0/2}}\right)$$

For the system $y = x + v$,

$E_b = E\{|x|^2\}$ = Signal Power

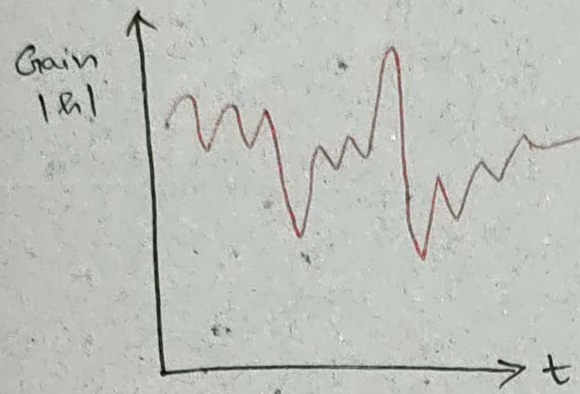
$\frac{N_0}{2} = E\{|v|^2\}$ = Noise Power

$$= Q\left(\sqrt{SNR}\right)$$

$$= Q(\rho). \quad \rho = SNR = \frac{E_b}{N_0/2} = \text{Signal to Noise Power Ratio}$$

The probability with which a bit is in Error (P_e) is also termed as Bit Error Rate (BER)

In case of Wireless channel, the channel is not Static, instead the channel is FADING. So, there will be the impact of channel coefficient which is changing, and is termed as the Fading channel coefficient 'h'. The Wireless channel is varying / fluctuating which results in output power fluctuation.



- Gain of wireless channel is rapidly fluctuating. This is termed as a Fading channel.
- The Fading channel coefficient (h) is RANDOM in nature

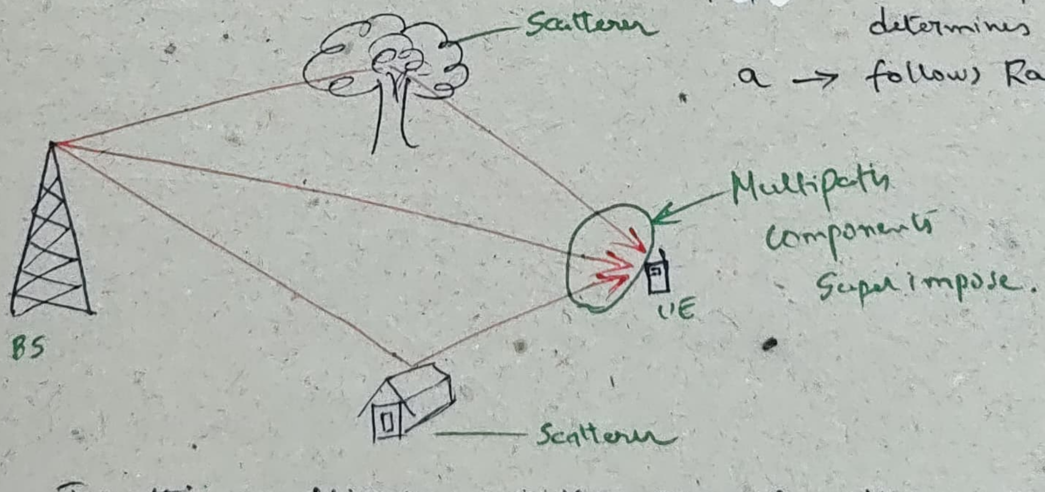
Plot: Gain of Wireless channel

- The Gain is complex. Both the magnitude/amplitude and phase are are random in nature (rapidly varying/fluctuating). But of course, the phase does not influence the power. The square of the magnitude/amplitude influences the power.

So, considering BPSK over Wireless channel, the Hypotheses

$$\left.\begin{aligned} H_0 &: y = -hA + v \\ H_1 &: y = hA + v \end{aligned}\right\}$$

where,
$h \rightarrow$ Fading channel Coefficient
$|h| = a \rightarrow$ Gain of channel, which determines Output power
$a \rightarrow$ follows Rayleigh PDF.



In this multipath wireless channel, there will be one LOS component and several reflected NLOS components. These multiple signal copies superimpose at the receiver, which results in interference which can be either constructive or destructive, as the interference is changing and hence the Gain / output power is also changing. This is the reason why the output power fluctuates, which is termed as Fading.

The gain of the channel '$|h| = a$' is Random in nature, which follows the Rayleigh PDF given as

$$f_A(a) = 2 a e^{-a^2}, \quad a \geq 0$$

Hence the channel is termed as "Rayleigh Fading Channel"

The Output SNR is given as

For the system $y = hx + v$, $x_i \in \{-A, A\}$

$E_b = E\{|x|^2\} = A^2 =$ Signal Power

$\frac{N_0}{2} = E\{|v|^2\} =$ Noise power

$$SNR_o = |h|^2 \frac{E\{|x|^2\}}{E\{|v|^2\}}$$

$$= |h|^2 \cdot \frac{E_b}{N_0/2}$$

$$= a^2 \cdot SNR$$

$$= a^2 \cdot \rho$$

where,

$$a = |h|$$

$$\rho = \frac{E_b}{N_0/2} = \frac{2E_b}{N_0}.$$

The Instantaneous BER is

Tail Probability of Standard Gaussian Random Variable

$$Q(\alpha) = \int_{\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$Q\left(\sqrt{SNR_o}\right) = Q\left(\sqrt{a^2\rho}\right)$$

$$= \int_{\sqrt{a^2\rho}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{a\sqrt{\rho}}^{\infty} e^{-\frac{x^2}{2}} dx$$

The Average BER is

$$P_e = \int_0^\infty Q\left(\sqrt{SNR_0}\right) \cdot f_A(a)\, da$$

inner integral

outer integral

$$= \int_0^\infty Q\left(\sqrt{a^2\rho}\right) \cdot 2a e^{-a^2}\, da$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi}} \int_{a\sqrt{\rho}}^\infty e^{-\frac{x^2}{2}} \cdot 2a e^{-a^2}\, dx\, da$$

Let $\frac{x}{a\sqrt{\rho}} = u \Rightarrow x = a\sqrt{\rho} \cdot u$

$dx = a\sqrt{\rho}\, du$

$$\Rightarrow P_e = \frac{1}{\sqrt{2\pi}} \int_0^\infty \int_{1}^\infty e^{-\frac{1}{2}a^2\rho u^2} \cdot 2a e^{-a^2} \cdot a\sqrt{\rho}\, du\, da$$

w.r.t a

w.r.t u

$$= \frac{\sqrt{\rho}}{\sqrt{2\pi}} \int_0^\infty \int_1^\infty 2a^2 e^{-\frac{1}{2}a^2\rho u^2 - a^2}\, du\, da$$

Interchanging the order of integration

inner integral

outer integral

$$\Rightarrow P_e = \frac{\sqrt{\rho}}{\sqrt{2\pi}} \int_1^\infty \int_0^\infty 2a^2 e^{-a^2\left(\frac{\rho u^2}{2}+1\right)}\, da\, du$$

w.r.t a

w.r.t u

Note:

$a \rightarrow$ Gaussian Random variable with mean $= 0$, Variance $= \sigma^2$.

Then,

$$\int_{-\infty}^\infty a^2 * \begin{pmatrix}\text{PDF which is Gaussian with} \\ \text{mean} = 0,\ \text{variance} = \sigma^2\end{pmatrix} = \sigma^2$$

$$\int_{-\infty}^\infty a^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-a^2/2\sigma^2}\, da = \sigma^2$$

$$2\int_0^\infty \frac{1}{\sqrt{2\pi}} a^2 e^{-a^2/2\sigma^2}\, da = \sigma^3$$

$$\boxed{\int_0^\infty \frac{2}{\sqrt{2\pi}} a^2 e^{-a^2/2\sigma^2}\, da = \sigma^3}$$

Use this Property in P_e

$$\Rightarrow P_e = \frac{\sqrt{\rho}}{\sqrt{2\pi}} \int_1^\infty \boxed{\int_0^\infty 2}\, a^2 e^{-a^2\left(\frac{\rho u^2+2}{2}\right)} \, da \, du$$

$$\Rightarrow P_e = \sqrt{\rho} \int_1^\infty \boxed{\int_{-\infty}^{\infty}} \frac{1}{\sqrt{2\pi}} a^2 e^{-\frac{a^2}{2}(\rho u^2+2)} \, da \, du$$

$$\Rightarrow P_e = \sqrt{\rho} \int_1^\infty \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} a^2 e^{-\frac{a^2}{2} \cdot \frac{1}{\left(1/\sqrt{\rho u^2+2}\right)^2}} \, da \, du$$

$$x = \frac{1}{\left(1/\sqrt{x}\right)^2}$$

$$\Rightarrow P_e = \sqrt{\rho} \int_1^\infty \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} a^2 e^{-a^2/2 \underbrace{\left(1/\sqrt{\rho u^2+2}\right)^2}_{\sigma^2}} \, da}_{\sigma^3} \, du$$

$$\Rightarrow P_e = \sqrt{\rho} \int_1^\infty \left(\frac{1}{\sqrt{\rho u^2+2}}\right)^3 du$$

$$\Rightarrow P_e = \sqrt{\rho} \int_1^\infty \left(\frac{1}{\rho u^2+2}\right)^{3/2} du$$

Let $u = \sqrt{\frac{2}{\rho}} \tan\theta \Rightarrow du = \sqrt{\frac{2}{\rho}} \sec^2\theta \, d\theta$

$u^2 = \frac{2}{\rho} \tan^2\theta$

$$\Rightarrow P_e = \sqrt{\rho} \int_{\tan^{-1}\sqrt{\rho/2}}^{\pi/2} \left(\frac{1}{\rho \cdot \frac{2}{\rho} \tan^2\theta + 2}\right)^{3/2} \sqrt{\frac{2}{\rho}} \sec^2\theta \, d\theta$$

$$\Rightarrow P_e = \cancel{\sqrt{\rho}} \int_{\tan^{-1}\sqrt{\rho/2}}^{\pi/2} \frac{1}{2^{3/2}} \left(\frac{1}{\tan^2\theta+1}\right)^{3/2} \frac{\sqrt{2}}{\cancel{\sqrt{\rho}}} \sec^2\theta \, d\theta$$

$$\Rightarrow P_e = \int_{\tan^{-1}\sqrt{\rho/2}}^{\pi/2} \frac{\cancel{\sqrt{2}}}{\cancel{\sqrt{2}} \cdot \sqrt{2} \cdot \sqrt{2}} \left(\frac{1}{\sec^2\theta}\right)^{3/2} \sec^2\theta \, d\theta$$

$$\Rightarrow P_e = \int_{\tan^{-1}\sqrt{\rho/2}}^{\pi/2} \frac{1}{2} \left(\frac{1}{\cancel{\sqrt{\sec^2\theta}\sqrt{\sec^2\theta}}\sqrt{\sec^2\theta}}\right) \cancel{\sec^2\theta} \, d\theta$$

$$\Rightarrow P_e = \frac{1}{2}\int_{\tan^{-1}\sqrt{\rho/2}}^{\pi/2} \frac{1}{\sqrt{1/\cos^2\theta}}\, d\theta$$

$$\Rightarrow P_e = \frac{1}{2}\int_{\tan^{-1}\sqrt{\rho/2}}^{\pi/2} \cos\theta \, d\theta$$

$$\Rightarrow P_e = \frac{1}{2}\sin\theta \Big|_{\tan^{-1}\sqrt{\rho/2}}^{\pi/2}$$

$$\Rightarrow P_e = \frac{1}{2}\left(\sin\frac{\pi}{2} - \underbrace{\sin\left(\tan^{-1}\sqrt{\rho/2}\right)}\right)$$

Now, $\sin^2 x = \dfrac{\sin^2 x}{\sin^2 x + \cos^2 x}$

$$= \frac{\frac{\sin^2 x}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x} + 1} = \frac{\tan^2 x}{1+\tan^2 x}$$

$$\therefore \sin x = \sqrt{\frac{\tan^2 x}{1+\tan^2 x}} = \frac{\tan x}{\sqrt{1+\tan^2 x}}$$

Put $x = \tan^{-1}\sqrt{\rho/2}$

$$\therefore \sin\left(\tan^{-1}\sqrt{\rho/2}\right) = \frac{\tan\left(\tan^{-1}\left(\sqrt{\rho/2}\right)\right)}{\sqrt{1+\tan^2\left(\tan^{-1}\left(\sqrt{\rho/2}\right)\right)}}$$

$$= \frac{\sqrt{\rho/2}}{\sqrt{1+\left(\sqrt{\rho/2}\right)^2}}$$

$\tan(\tan^{-1}x) = x$
$\tan^2(\tan^{-1}x) = x^2$

$$\boxed{\sin\left(\tan^{-1}\sqrt{\rho/2}\right) = \frac{\sqrt{\rho/2}}{\sqrt{1+\rho/2}}}$$

$$\Rightarrow P_e = \frac{1}{2}\left(1 - \sqrt{\frac{\rho/2}{1+\rho/2}}\right) = \frac{1}{2}\left(1 - \sqrt{\frac{\rho}{2+\rho}}\right)$$

Average BER Expressions!

$$= \frac{1}{2}\left(1 - \sqrt{\frac{SNR}{2+SNR}}\right)$$

In fact, we don't need to say Average BER coz, when we talk about fading in wireless channel, we don't talk about the instantaneous coz instantaneous doesn't have any meaning here, as it is fluctuating. So, when we talk about BER, we're to talk about Average performance only.

The BER of Fading wireless channel is

$$P_e = \frac{1}{2}\left(1 - \sqrt{\frac{\rho}{2+\rho}}\right)$$

This can be approximated as follows.

$$P_e = \frac{1}{2}\left(1 - \left(\frac{2+\rho}{\rho}\right)^{-1/2}\right)$$

$$\Rightarrow P_e = \frac{1}{2}\left(1 - \left(1 + \frac{2}{\rho}\right)^{-1/2}\right)$$

At high SNR,
ρ is high
$\frac{2}{\rho}$ is very small
we use the property,
$(1+x)^{-1/2} \approx 1 - \frac{1}{2}x$

$$\Rightarrow P_e = \frac{1}{2}\left[1 - \left(1 - \frac{1}{2}\left(\frac{2}{\rho}\right)\right)\right]$$

$$\Rightarrow P_e = \frac{1}{2}\left(1 - 1 + \frac{1}{\rho}\right)$$

$$\Rightarrow P_e = \frac{1}{2\rho} \leftarrow \text{High SNR approximation}$$

Example Find BER of Wireless and Wireline channels, SNR = 20dB

$$SNR = 20\,dB = 10^{20/10} = 10^{2} = 100$$

- BER of wireline $= Q(\sqrt{SNR}) = Q(\sqrt{100}) = Q(10) = 7.6 \times 10^{-24}$
- BER of wireless $= \frac{1}{2\rho} = \frac{1}{2 \times 100} = 5 \times 10^{-3}$

BER of wireless is SIGNIFICANTLY HIGHER than wireline!

- Wireline BER decreases exponentially!

$$BER_{wireline} = Q(\sqrt{\rho}) = Q(\sqrt{SNR}) \leq \frac{1}{2} e^{-\frac{1}{2}SNR}$$

- Wireless BER decreases only as $\frac{1}{SNR}$! (very slow rate)

$$BER_{wireless} = \frac{1}{2 \times SNR} \propto \frac{1}{SNR}$$

(e^{-x} decreases much faster than $\frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3} \ldots$)

QAM

- BER for QAM in Wireline channel is

$$P_e = 4\left(1-\frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3E_s}{(M-1)N_0}}\right)$$

where, M = No. of Symbols in QAM

$$\frac{E_s}{N_0} = SNR = \rho$$

$$\Rightarrow P_e = 4\left(1-\frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3\rho}{(M-1)}}\right)$$

- For wireless, Output SNR $= a^2\rho$.

Instantaneous SER is

$$P_e^{inst} = 4\left(1-\frac{1}{\sqrt{M}}\right) Q\left(\sqrt{a^2 \frac{3\rho}{(M-1)}}\right)$$

Effective SNR

Average the instantaneous SER w.r.t a,

$$\text{Average } P_e = \overset{2}{\cancel{4}}\left(1-\frac{1}{\sqrt{M}}\right) \cdot \frac{1}{\cancel{2}\times \frac{3\rho}{(M-1)}}$$

$$= \frac{2}{3\rho}\left(1-\frac{1}{\sqrt{M}}\right)(M-1)$$

therefore, the Average SER decreases as $\frac{1}{\rho}$.

$$P_e = \frac{2}{3\rho}\left(1-\frac{1}{\sqrt{M}}\right)(M-1) \propto \frac{1}{\rho} = \frac{1}{SNR}.$$