Week 2 : Session 3

Neyman Pearson (NP) Criterion

Having completed the discussion on Optimal detector and its performance for the detection of a signal in AWGN, let us look at the theory behind. (ie) How to derive the optimal detector / optimal decision rule for a given system. And this principle is basically known as Neyman Pearson (NP) criterion.

(ie) NP criterion tells us how to determine the optimal decision rule for any given detection problem.

The detection problem can be formulated as follows.

- The optimal detector maximizes P_D for a given P_{FA}

" Recall, we can trade-off P_D vs P_{FA} by adjusting γ.
(ie) we can achieve whatever value of P_D we want.
$\gamma \to \infty$, $P_D = 0$
$\gamma \to -\infty$, $P_D = 1$.
So, it is not question of How to maximize P_D.
Rather, the question is How to achieve maximum P_D for a given fixed value of P_{FA}.

- (ie) Maximize P_D
  Subject to $P_{FA} = \alpha$
  $0 \le \alpha \le 1$.
  ← Constrained optimization Problem.

- What is the decision rule that achieves the maximum value of P_D for the given value of P_{FA} ?

Any detector has the following structure.

$\bar{y} \sim N \times 1$ ← N dimensional vector

$\in \mathbb{R}^N$ ← N dimensional space

output vector / observation vector

Based on $\bar{y}$, the optimal detector decides Hypothesis H_0 (or) H_1

- The optimal detector choose H_1 whenever $\bar{y}$ belongs to the region R_1, which is a subset of $\mathbb{R}^N$.

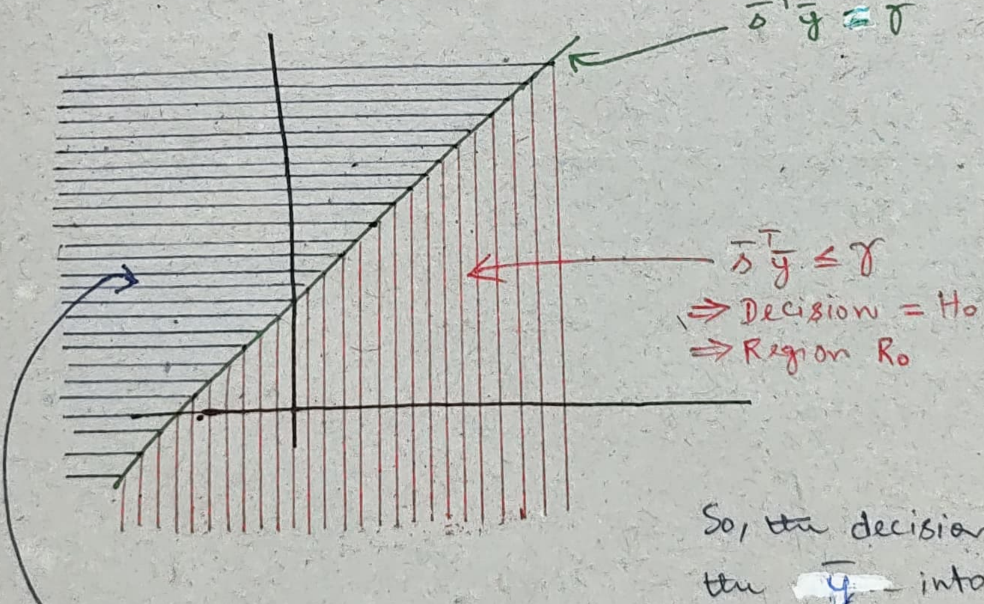
$$H_1 : \bar{y} \in R_1 \subset \mathbb{R}^N \longleftarrow \text{N dimensional space}$$

- The optimal detector choose H_0 when $\bar{y}$ lies anywhere outside of R_1. (i.e.) $\mathbb{R}^N - R_1$, which is called R_0, which is also a subset of $\mathbb{R}^N$.

$$H_0 : \bar{y} \in \mathbb{R}^N - R_1 = R_0 \subset \mathbb{R}^N$$

So, what we are doing is, we are taking N dimensional space and partitioning into two parts, and the optimal detector chooses H_1 if $\bar{y} \in R_1$, and chooses H_0 if $\bar{y} \in$ complement of R_1. (i.e.) $\mathbb{R}^N - R_1$

For instance, if we take a look at the previous detector (i.e.) H_0 if $\bar{s}^T\bar{y} \leq \gamma$ in 2-dimensional space, this corresponds to a line



$\bar{s}^T\bar{y} > \gamma$
$\Rightarrow$ Decision $= H_1$
$\Rightarrow$ Region R_1

So, the decision rule divides the $\bar{y}$ into two parts.

(i) $\bar{y} \in R_1$

(ii) $\bar{y} \in R_0$

Optimization problem.

How to partition $\mathbb{R}^N$ into R_0, R_1 such that P_D is maximum for given $P_{FA} = \alpha$.

Let optimal detector choose

- $H_1 : \bar{y} \in R_1 \subset R^{N\times 1}$,
- $H_0 : \bar{y} \in R_0 = R^{N\times 1} - R_1$

$R_0 = \mathbb{R}^N - R_1$

$R_0 \cap R_1 = \phi$

$R_0 \cup R_1 = \mathbb{R}^N$

Therefore,

Probability decision : H_1
Given H_1
$\bar{y} \in R_1$

$$P_D = P_r(\bar{y} \in R_1 \,;\, H_1)$$

$$= \int_{R_1} p(\bar{y}\,;\,H_1)\, d\bar{y}$$

integrate PDF of $\bar{y}$ corresponding to H_1 over R_1

Probability that the signal is correctly detected when the underlying Hypothesis H_1. So, we have to look at the PDF corresponding to H_1, integrate it over the region R_1.

Probability decision : H_1
Given H_0
$\bar{y} \in R_1$

$$P_{FA} = P_r(\bar{y} \in R_1 \,;\, H_0)$$

$$= \int_{R_1} p(\bar{y}\,;\,H_0)\, d\bar{y}$$

integrate PDF of $\bar{y}$ corresponding to H_0 over R_1

How to solve the Constrained Optimization problem?

We use Lagrange Multiplier (λ). (ie) whenever we have "Minimize $f(x)$ subject to the condition $g(x)=0$", then we take Lagrangian.

$$\boxed{f(x) + \lambda\, g(x)}$$

Therefore, "Maximize P_D subject to $\underbrace{P_{FA} = \alpha}$"

$\nwarrow \alpha - P_{FA} = 0$

Lagrangian is

$$P_D + \lambda(\alpha - P_{FA})$$

$$= \int_{R_1} p(\bar{y}; H_1)\, d\bar{y} + \lambda\left(\alpha - \int_{R_1} p(\bar{y}; H_0)\, d\bar{y}\right)$$

$$= \int_{R_1} p(\bar{y}; H_1)\, d\bar{y} + \lambda\alpha - \lambda \int_{R_1} p(\bar{y}; H_0)\, d\bar{y}$$

Fixed for R_1

$$= \int_{R_1} \left[p(\bar{y}; H_1) - \lambda p(\bar{y}; H_0) \right] d\bar{y} + \lambda\alpha$$

How to choose R_1 to maximize integral?

Put all points $\bar{y}$ in R_1 such that

$$\underbrace{p(\bar{y}; H_1) > \lambda\, p(\bar{y}; H_0)}$$

LRT

So, to maximize P_D, include all points $\bar{y}$ in R_1 such that $\underbrace{p(\bar{y}; H_1) > \lambda\, p(\bar{y}; H_0)}$.

Choose H_1 if $\bar{y} \in R_1$

Choose H_1 if

$$\frac{p(\bar{y}; H_0)}{p(\bar{y}; H_1)} < \frac{1}{\lambda}$$

And choose H_0 if

$$p(\bar{y}; H_1) \leq \lambda\, p(\bar{y}; H_0)$$

$$\Rightarrow \underbrace{\frac{p(\bar{y}; H_0)}{p(\bar{y}; H_1)} \geq \frac{1}{\lambda} = \tilde{\gamma}}$$

LRT

Thus, the NP criterion clearly justifies that LRT is the OPTIMAL TEST, in the sense that LRT maximizes P_D for the given value of P_{FA}.

How to choose λ?

Use the constraint $P_{FA} = \alpha$. (i.e) choose λ such that $P_{FA} = \alpha$.

Constant Signal Detection.

Given $P_{FA} = \alpha$

$$P_{FA} = Q\left(\frac{\gamma}{\sigma \|\bar{s}\|}\right) = \alpha$$

$$\Rightarrow \boxed{\gamma = \sigma \|\bar{s}\| Q^{-1}(\alpha)}$$

(ii) If we set this value of γ in LRT, we get decision rule that maximizes P_D for $P_{FA} = \alpha$

$$\text{Choose } H_0 \text{ if } \bar{s}^T \bar{y} \leq \gamma = \sigma \|\bar{s}\| Q^{-1}(\alpha)$$
$$\text{choose } H_1 \text{ if } \bar{s}^T \bar{y} > \gamma = \sigma \|\bar{s}\| Q^{-1}(\alpha)$$

Thus, the NP criterion explicitly proves that the LRT is the optimal test, which maximizes the P_D for a given value of P_{FA}.