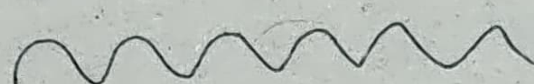
Week 7 : Session 2

Generalized Likelihood Ratio Test (GLRT)

As the name implies, GLRT generalizes the LRT technique, when we have unknown parameters in the detection problem. (ie) GLRT is for Detection of Signals, when certain parameters of the signal are Unknown.

Example : Signal detection when Carrier frequency is UNKNOWN.



$$f_c \in [B_L, B_H] \quad , \quad f_c \text{ is Unknown.}$$

This occurs frequently in practice.

To illustrate the GLRT, consider the Binary Hypothesis Testing problem (Modified Signal Detection Problem)

- Under NULL Hypothesis H_0 : $\bar{y} = \bar{v}$
- Under ALTERNATIVE Hypothesis H_1 : $\bar{y} = A\bar{s} + \bar{v}$

where, $A \rightarrow$ Unknown parameter (Scaling factor)
Ex. Carrier Amplitude, Channel coefficient; ...

$\bar{s} \rightarrow$ Known Signal vector

$$\bar{s} = \begin{bmatrix} s(1) \\ s(2) \\ \vdots \\ s(N) \end{bmatrix}$$

$\bar{v} \rightarrow$ Noise vector (Gaussian iid mean $= 0$, var $= \sigma^2$)

$$\bar{v} = \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}$$

The LRT for this problem is,

Likelihood corresponding to H_1 with unknown parameter A

Choose H_1 if

$$\frac{p(\bar{y}\,;\,A, H_1)}{p(\bar{y}\,;\,H_0)} > \tilde{\gamma}$$

Likelihood corresponding to H_0

$$\Rightarrow \frac{\cancel{\left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}}}\, e^{-\frac{\|\bar{y}-A\bar{s}\|^2}{2\sigma^2}}}{\cancel{\left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}}}\, e^{-\frac{\|\bar{y}\|^2}{2\sigma^2}}} > \tilde{\gamma}$$

Taking ln on both sides.

$$\Rightarrow \left(\frac{-\|\bar{y}-A\bar{s}\|^2}{2\sigma^2}\right) - \left(\frac{-\|\bar{y}\|^2}{2\sigma^2}\right) > \ln \tilde{\gamma}$$

$$\Rightarrow \|\bar{y}\|^2 - \|\bar{y}-A\bar{s}\|^2 > 2\sigma^2 \ln \tilde{\gamma}$$

$$\Rightarrow \cancel{\|\bar{y}\|^2} - \left(\cancel{\|\bar{y}\|^2} + A^2\|\bar{s}\|^2 - 2A\bar{s}^T\bar{y}\right) > 2\sigma^2 \ln\tilde{\gamma}$$

$$\Rightarrow -A^2\|\bar{s}\|^2 + 2A\bar{s}^T\bar{y} > 2\sigma^2 \ln\tilde{\gamma}$$

$$\Rightarrow 2A\bar{s}^T\bar{y} - A^2\|\bar{s}\|^2 > 2\sigma^2\ln\tilde{\gamma}$$

Since A is unknown scaling factor, what value of A to choose ?

Choose the value of unknown parameter 'A', such that it maximizes the likelihood $\hat{A}$ corresponding to the unknown parameter 'A'. (ie) choose the Maximum Likelihood (ML) Estimate of A.

$$\boxed{\text{(ie) } \hat{A} = \arg\max\, p(\bar{y}\,;\,A, H_1)}$$

Thus, the LRT is modified as,

Choose H_1 if

$$\frac{p(\bar{y}\,;\,\hat{A}_{ML}, H_1)}{p(\bar{y}\,;\,H_0)} > \tilde{\gamma}$$

This is termed as Generalized Likelihood Ratio Test (GLRT)

The ML estimate of A can be found as follows.

$$\hat{A} = \arg\max \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}} e^{\frac{-\|\bar{y}-A\bar{s}\|^2}{2\sigma^2}}$$

ignore constants, Maximize Exponent $\equiv$ Minimize

$$= \arg\min \frac{\|\bar{y}-A\bar{s}\|^2}{\cancel{2\sigma^2}}$$

$$= \arg\min \|\bar{y}-A\bar{s}\|^2$$

This is known as Least squares Problem. The Solution for this is,

$$\hat{A} = \underbrace{(\bar{s}^T\bar{s})^{-1}\bar{s}^T}\bar{y}$$

Pseudo-inverse of $\bar{s}$

$$= \frac{\bar{s}^T\bar{y}}{(\bar{s}^T\bar{s})}$$

$$\boxed{\hat{A} = \frac{\bar{s}^T\bar{y}}{\|\bar{s}\|^2}}$$

This is the ML estimate of A. Substitute the same in modified LRT.

$$\Rightarrow 2\hat{A}\bar{s}^T\bar{y} - \hat{A}^2\|\bar{s}\|^2 > 2\sigma^2 \ln\tilde{\gamma}.$$

$$\Rightarrow 2 \cdot \frac{\bar{s}^T\bar{y}}{\|\bar{s}\|^2}\cdot\bar{s}^T\bar{y} - \frac{(\bar{s}^T\bar{y})^2}{\|\bar{s}\|^{\cancel{4}\,2}}\cancel{\|\bar{s}\|^2} > 2\sigma^2\ln\tilde{\gamma}$$

$$\Rightarrow 2\frac{(\bar{s}^T\bar{y})^2}{\|\bar{s}\|^2} - \frac{(\bar{s}^T\bar{y})^2}{\|\bar{s}\|^2} > 2\sigma^2\ln\tilde{\gamma}$$

$$\Rightarrow \frac{(\bar{s}^T\bar{y})^2}{\|\bar{s}\|^2} > 2\sigma^2\ln\tilde{\gamma}$$

$$\Rightarrow (\bar{s}^T\bar{y})^2 > 2\sigma^2\ln\tilde{\gamma}\,\|\bar{s}\|^2$$

$$\Rightarrow (\bar{s}^T\bar{y}) \overset{?}{>} \sqrt{2\sigma^2\ln\tilde{\gamma}\,\|\bar{s}\|^2}$$

$$\Rightarrow \bar{s}^T\bar{y} > \|\bar{s}\|\sqrt{2\sigma^2\ln\tilde{\gamma}}$$

This ambiguity arises, as the sign of the scaling factor is NOT known.

(ie) A $<$ +ve / −ve ?

$$\Rightarrow \bar{s}^T\bar{y} > \gamma \quad \text{(or)} \quad \bar{s}^T\bar{y} < -\gamma$$

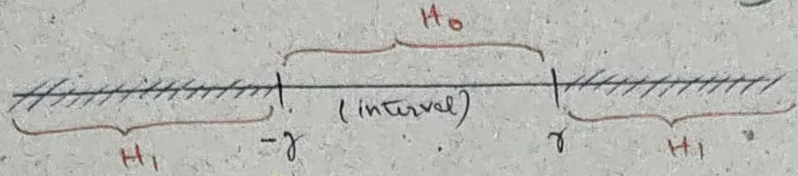
where, $\gamma = \|\bar{s}\|\sqrt{2\sigma^2\ln\tilde{\gamma}}$

Also, if A is complex, then the phase is UNKNOWN.

Therefore, the GLRT is,

- Choose H_1 if $|\bar{s}^T\bar{y}| > \gamma$
  where $\bar{s}^T\bar{y} > \gamma$ (or) $\bar{s}^T\bar{y} < -\gamma$
- Choose H_0 if $-\gamma < \bar{s}^T\bar{y} < \gamma$

} Generalized Matched Filter



Generalized Matched filter

- $|\bar{s}^T\bar{y}|$
- $(\bar{s}^T\bar{y})^2 \rightarrow$ Energy output of Matched Filter
  "Matched Filter + Energy Detector"
- $|\bar{s}^H\bar{y}|^2 \rightarrow$ For complex signals.

Week 7 : Session 3 :

Now, let us characterize the performance of GLRT.

- P_{FA} can be found as follows.

$P_{FA} \rightarrow$ Under H_0, Probability decision is Signal Present.

$$P_{FA} = P_r\left(\bar{s}^T\bar{y} > \gamma \text{ (or) } \bar{s}^T\bar{y} < -\gamma \,;\, H_0\right)$$

WKT, under H_0 : $\bar{y} = \bar{v}$

where $\bar{v}$ - contains iid Gaussian noise samples with mean $= 0$, variance $= \sigma^2$.

(ie) $\bar{v} \sim \mathcal{N}(0, \sigma^2 I)$.

$\bar{s}^T\bar{y} = \bar{s}^T\bar{v}$

Mean : $E\{\bar{s}^T\bar{v}\} = \bar{s}^T E[\bar{v}] = \bar{s}^T . 0 = 0$.

Variance : $E\{(\bar{s}^T\bar{v})^2\} = E\{\bar{s}^T\bar{v} . \bar{v}^T\bar{s}\}$

$= \bar{s}^T E[\bar{v}\bar{v}^T]\,\bar{s}$

$= \bar{s}^T . \sigma^2 I . \bar{s}$

$= \sigma^2 \|\bar{s}\|^2$

Thus,

$$\bar{s}^T\bar{v} \sim \mathcal{N}(0, \sigma^2\|\bar{s}\|^2)$$

$$\Rightarrow P_{FA} = P_r\left(\bar{s}^T\bar{v} > \gamma \ \text{(or)}\ \bar{s}^T\bar{v} < -\gamma\right)$$

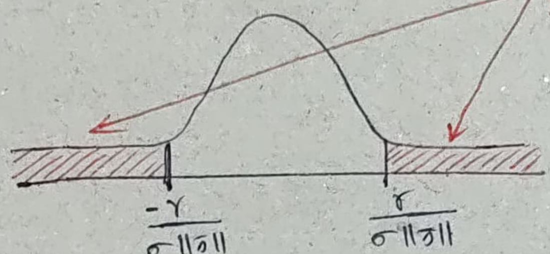
$$\Rightarrow P_{FA} = P_r\left(\mathcal{N}(0,\sigma^2\|\bar{s}\|^2) > \gamma \ \text{(or)}\ \mathcal{N}(0,\sigma^2\|\bar{s}\|^2) < -\gamma\right)$$

$$\Rightarrow P_{FA} = P_r\left(\underbrace{\frac{\mathcal{N}(0,\sigma^2\|\bar{s}\|^2) - 0}{\sigma\|\bar{s}\|}}_{\mathcal{N}(0,1)} > \frac{\gamma - 0}{\sigma\|\bar{s}\|} \ \text{(or)}\ \underbrace{\frac{\mathcal{N}(0,\sigma^2\|\bar{s}\|^2) - 0}{\sigma\|\bar{s}\|}}_{\mathcal{N}(0,1)} < -\frac{\gamma - 0}{\sigma\|\bar{s}\|}\right)$$

Standard Gaussian Random variable

$$\Rightarrow P_{FA} = P_r\left(\mathcal{N}(0,1) > \frac{\gamma}{\sigma\|\bar{s}\|} \ \text{(or)}\ \mathcal{N}(0,1) < \frac{-\gamma}{\sigma\|\bar{s}\|}\right)$$

Both these probabilities are equal



$$\Rightarrow \boxed{P_{FA} = 2\underbrace{Q\left(\frac{\gamma}{\sigma\|\bar{s}\|}\right)}}$$

CCDF of Standard Normal

- P_D can be found as follows.

$P_D \rightarrow$ Under H_1, Probability decision is signal present.

$$P_D = P_r\left(\bar{s}^T\bar{y} > \gamma \ \text{(or)}\ \bar{s}^T\bar{y} < -\gamma \ ;\ H_1\right)$$

WKT, under H_1 : $\bar{y} = A\bar{s} + \bar{v}$

$$\bar{s}^T\bar{y} = \bar{s}^T(A\bar{s} + \bar{v})$$

$$\bar{s}^T\bar{y} = \underbrace{A\|\bar{s}\|^2}_{\text{Mean}} + \bar{s}^T\bar{v}$$

where, $\bar{s}^T\bar{v} \sim \mathcal{N}(0, \sigma^2\|\bar{s}\|^2)$

$$\bar{s}^T\bar{y} \sim \mathcal{N}(A\|\bar{s}\|^2, \sigma^2\|\bar{s}\|^2)$$

$$\Rightarrow P_D = P_r\left(\mathcal{N}(A\|\bar{s}\|^2,\sigma^2\|\bar{s}\|^2) > \gamma \ \text{(or)}\ \mathcal{N}(A\|\bar{s}\|^2,\sigma^2\|\bar{s}\|^2) < -\gamma\right)$$

$$\Rightarrow P_D = P_r\left(\underbrace{\frac{\mathcal{N}(A\|\bar{s}\|^2,\sigma^2\|\bar{s}\|^2) - A\|\bar{s}\|^2}{\sigma\|\bar{s}\|}}_{\mathcal{N}(0,1)} > \frac{\gamma - A\|\bar{s}\|^2}{\sigma\|\bar{s}\|} \ \text{(or)}\ \underbrace{\frac{\mathcal{N}(A\|\bar{s}\|^2,\sigma^2\|\bar{s}\|^2) - A\|\bar{s}\|^2}{\sigma\|\bar{s}\|}}_{\mathcal{N}(0,1)} < \frac{-\gamma - A\|\bar{s}\|^2}{\sigma\|\bar{s}\|}\right)$$

$$\Rightarrow P_D = P_r\left(\mathcal{N}(0,1) > \frac{\gamma - A\|\bar{s}\|^2}{\sigma\|\bar{s}\|} \text{ (or) } \mathcal{N}(0,1) < \frac{-\gamma - A\|\bar{s}\|^2}{\sigma\|\bar{s}\|} \right)$$

$$\Rightarrow P_D = Q\left(\frac{\gamma - A\|\bar{s}\|^2}{\sigma\|\bar{s}\|}\right) + \left[1 - Q\left(\frac{-\gamma - A\|\bar{s}\|^2}{\sigma\|\bar{s}\|}\right)\right]$$

$$\Rightarrow \boxed{P_D = Q\left(\frac{\gamma - A\|\bar{s}\|^2}{\sigma\|\bar{s}\|}\right) + Q\left(\frac{\gamma + A\|\bar{s}\|^2}{\sigma\|\bar{s}\|}\right)}$$

- Receiver Operating Characteristic (ROC) ← P_D as a function of P_{FA}

WKT,
$$P_{FA} = 2\,Q\left(\frac{\gamma}{\sigma\|\bar{s}\|}\right)$$

$$\Rightarrow \gamma = \sigma\|\bar{s}\|\,Q^{-1}\left(\frac{P_{FA}}{2}\right)$$

Thus,
$$P_D = Q\left(\frac{\sigma\|\bar{s}\|\,Q^{-1}\left(\frac{P_{FA}}{2}\right) - A\|\bar{s}\|^2}{\sigma\|\bar{s}\|}\right) + Q\left(\frac{\sigma\|\bar{s}\|\,Q^{-1}\left(\frac{P_{FA}}{2}\right) + A\|\bar{s}\|^2}{\sigma\|\bar{s}\|}\right)$$

$$\Rightarrow \boxed{P_D = Q\left(Q^{-1}\left(\frac{P_{FA}}{2}\right) - \frac{A\|\bar{s}\|}{\sigma}\right) + Q\left(Q^{-1}\left(\frac{P_{FA}}{2}\right) + \frac{A\|\bar{s}\|}{\sigma}\right)}$$

This is the ROC of GLRT with unknown scaling factor.