Week 4 : Session - 2

Bayesian Detection

So far, we have looked at scenarios where the Hypothesis are deterministic. Now, we look at the system where the Hypothesis are Random, which is of course practical, because each Hypothesis (NULL / ALTERNATIVE) occurs randomly typically with a certain probability.

And this will also leads to the concept of "Minimum P_e Detector". (ie) Minimum Probability of Error decision rule:

Let us consider a Bayesian Binary Hypothesis Testing; where the Hypothesis become random in nature. We denote

$$\left.\begin{aligned} P(H_0) &= \pi_0 \\ P(H_1) &= \pi_1 \end{aligned}\right\} \quad \text{where, } 0 \leq \pi_0, \pi_1 \leq 1 \qquad \pi_0 + \pi_1 = 1.$$

These are called as Prior Probabilities of Hypothesis.

- Decision region for H_1 is $R_1 \subset \mathbb{R}^N$
- Decision region for H_0 is $R_0 = \mathbb{R}^N - R_1$

where, $R_1 \cup R_0 = \mathbb{R}^N$

$R_1 \cap R_0 = \phi$.

So, we are dividing it into two regions such that

if $\bar{y} \in R_1$, then choose H_1 and

if $\bar{y} \in R_0$, then choose H_0.

Also, both the regions are disjoint (ie. their intersection is ϕ) and they are complement of each other (ie together they span the N dimensional space)

⊙ The Probability of error (P_e) is given as

$$P_e = P_r(H_0)\underbrace{P_r(H_1|H_0)}_{P_{FA}} + P_r(H_1)\cdot\underbrace{P_r(H_0|H_1)}_{P_{MD}}$$

$$= \underbrace{\pi_0 \cdot P_r(H_1|H_0)}_{\bar{y} \in R_1 ;\, H_0} + \underbrace{\pi_1 \cdot P_r(H_0|H_1)}_{\bar{y} \in R_0 ;\, H_1}$$

$$= P_r(H_0|H_1)\,\pi_1 + P_r(H_1|H_0)\,\pi_0$$

$$= \underbrace{\pi_1 \int_{R_0} p(\bar{y}|H_1)\,d\bar{y}}_{Pr.\ \bar{y} \in R_0 \text{ given } H_1} + \underbrace{\pi_0 \int_{R_1} p(\bar{y}|H_0)\,d\bar{y}}_{Pr.\ \bar{y} \in R_1 \text{ given } H_0}$$

$$= \pi_1 \int_{R_0} p(\bar{y}|H_1)\,d\bar{y} + \pi_0 \underbrace{\left(1 - \int_{R_0} p(\bar{y}|H_0)\,d\bar{y}\right)}$$

$$\boxed{\begin{array}{l}\because R_1 \text{ and } R_0 \text{ are disjoint, we can write} \\ \displaystyle\int_{R_1} p(\bar{y}|H_0)\,d\bar{y} + \int_{R_0} p(\bar{y}|H_0)\,d\bar{y} = 1\end{array}}$$

$$= \underbrace{\int_{R_0} \left[\pi_1\, p(\bar{y}|H_1) - \pi_0\, p(\bar{y}|H_0)\right] d\bar{y}} + \pi_0$$

Constant

To minimize this quantity, include all $\bar{y}$ in R_0 such that
$\pi_1 p(\bar{y}|H_1) - \pi_0 p(\bar{y}|H_0) \leq 0$.

This is because, when this quantity is $-ve$, the value of the integral decreases, And when this quantity is $+ve$, the value of the integral increases.

So, to minimize the integral, we include all the points $\bar{y}$ such that this quantity is ≤ 0 inside R_0. And at any point where this quantity is ≥ 0 can be out of R_0 (i.e.) R_1.

To minimize, choose R_0 such that

$$\pi_1\, p(\bar{y}|H_1) - \pi_0\, p(\bar{y}|H_0) \leq 0$$

$$\Rightarrow \frac{p(\bar{y}|H_0)}{p(\bar{y}|H_1)} \geq \frac{\pi_1}{\pi_0},$$

(ie) Reduces to LRT with $\tilde{\gamma} = \frac{\pi_1}{\pi_0} = \frac{P_r(H_1)}{P_r(H_0)}$

Further simplification:

Choose H_0 if

$$\pi_0\, p(\bar{y}|H_0) \geq \pi_1\, p(\bar{y}|H_1)$$

$$\Rightarrow \frac{\pi_0\, p(\bar{y}|H_0)}{\pi_0\, p(\bar{y}|H_0) + \pi_1\, p(\bar{y}|H_1)} \geq \frac{\pi_1\, p(\bar{y}|H_1)}{\pi_0\, p(\bar{y}|H_0) + \pi_1\, p(\bar{y}|H_1)}$$

$$\Rightarrow \frac{\cancel{\pi_0\, p(\bar{y}|H_0)}}{\cancel{\pi_0\, p(\bar{y}|H_0)}\left[1 + \frac{\pi_1\, p(\bar{y}|H_1)}{\pi_0\, p(\bar{y}|H_0)}\right]} \geq \frac{\cancel{\pi_1\, p(\bar{y}|H_1)}}{\cancel{\pi_1\, p(\bar{y}|H_1)}\left[\frac{\pi_0\, p(\bar{y}|H_0)}{\pi_1\, p(\bar{y}|H_1)} + 1\right]}$$

$$\Rightarrow \frac{1}{1 + \frac{P_r(H_1)}{P_r(H_0)}\frac{p(\bar{y}|H_1)}{p(\bar{y}|H_0)}} \geq \frac{1}{\frac{P_r(H_0)}{P_r(H_1)}\frac{p(\bar{y}|H_0)}{p(\bar{y}|H_1)} + 1}$$

$$\Rightarrow \frac{P_r(H_0)\cdot p(\bar{y}|H_0)}{P_r(H_0)\, p(\bar{y}|H_0) + P_r(H_1)\, p(\bar{y}|H_1)} \geq \frac{P_r(H_1)\cdot p(\bar{y}|H_1)}{P_r(H_0)\, p(\bar{y}|H_0) + P_r(H_1)\cdot p(\bar{y}|H_1)}$$

$$\Rightarrow P_r(H_0|\bar{y}) \geq P_r(H_1|\bar{y})$$

These are the Aposteriori probabilities. (Baye's Rule)

- So, we choose H_0 if $P_r(H_0|\bar{y}) \geq P_r(H_1|\bar{y})$.

(ie) Choose the Hypothesis with maximum aposteriori probability (MAP).

This is called as MAP decision Rule!

- Otherwise, choose H_1 if $P_r(H_0|\bar{y}) < P_r(H_1|\bar{y})$

- This is the MAP decision rule. Therefore, MAP rule minimizes the Probability of error (P_e).

- Finally, when $\pi_0 = \pi_1 = \frac{1}{2}$

$$\tilde{\gamma} = \frac{\pi_1}{\pi_0} = \frac{1/2}{1/2} = 1.$$

Thus, MAP reduces to ML decision rule!

(ii) When the prior probabilities are equal, MAP becomes the ML rule.

Week 4 : Session - 3.

Consider now, the Signal detection problem

$$H_0 : \bar{y} = \bar{v}$$

$$H_1 : \bar{y} = \bar{s} + \bar{v}$$

Mean, $E\{\bar{v}\} = 0$

Covariance, $E\{\bar{v}\bar{v}^T\} = \sigma^2 I$

WKT, the Test Statistic $\bar{s}^T\bar{y} = \sum_{i=1}^{N-1} y(i)\, s(i)$

Likelihood Ratio Test (LRT)

- choose H_0 if

$$\bar{s}^T\bar{y} = \sum_{i=1}^{N} y(i)\, s(i) \leq \frac{\|\bar{s}\|^2 - 2\sigma^2 \ln \tilde{\gamma}}{2} = \gamma$$

Substitute $\tilde{\gamma} = \frac{\pi_1}{\pi_0}$

$$\Rightarrow \gamma = \frac{\|\bar{s}\|^2 - 2\sigma^2 \ln\left(\frac{\pi_1}{\pi_0}\right)}{2}$$

Thus, choose H_0 if

$$\bar{s}^T\bar{y} \leq \frac{\|\bar{s}\|^2 - 2\sigma^2 \ln\left(\frac{\pi_1}{\pi_0}\right)}{2}$$

This is the MAP rule which minimizes the P_e

⊙ Similarly, choose H_1 if

$$\bar{s}^T \bar{y} = \sum_{i=1}^{N} y(i)\, s(i) > \frac{\|\bar{s}\|^2 - 2\sigma^2 \ln \tilde{\gamma}}{2} = \gamma$$

Substitute $\tilde{\gamma} = \frac{\pi_1}{\pi_0}$

Thus, choose H_1 if

$$\bar{s}^T \bar{y} > \frac{\|\bar{s}\|^2 - 2\sigma^2 \ln\left(\frac{\pi_1}{\pi_0}\right)}{2}$$

⊙ Finally,

$$\boxed{\begin{aligned} &\text{Choose } H_0 \text{ if } \bar{s}^T \bar{y} \le \frac{\|\bar{s}\|^2}{2} - \sigma^2 \ln\left(\frac{\pi_1}{\pi_0}\right) \\ &\text{Choose } H_1 \text{ if } \bar{s}^T \bar{y} > \frac{\|\bar{s}\|^2}{2} - \sigma^2 \ln\left(\frac{\pi_1}{\pi_0}\right) \end{aligned}}$$

⊙ The minimum P_e can now be determined as follows.

Recall,
$$P_{FA} = Q\left(\frac{\gamma}{\sigma \|\bar{s}\|}\right)$$

$$= Q\left(\frac{\frac{\|\bar{s}\|^2}{2} - \sigma^2 \ln\left(\frac{\pi_1}{\pi_0}\right)}{\sigma \|\bar{s}\|}\right)$$

$$= Q\left(\frac{\|\bar{s}\|^2 - 2\sigma^2 \ln\left(\frac{\pi_1}{\pi_0}\right)}{2\sigma \|\bar{s}\|}\right)$$

$$P_D = Q\left(\frac{\gamma - \|\bar{s}\|^2}{\sigma \|\bar{s}\|}\right)$$

$$= Q\left(\frac{\frac{\|\bar{s}\|^2}{2} - \sigma^2 \ln\left(\frac{\pi_1}{\pi_0}\right) - \|\bar{s}\|^2}{\sigma \|\bar{s}\|}\right)$$

$$= Q\left(\frac{-\frac{\|\bar{s}\|^2}{2} - \sigma^2 \ln\left(\frac{\pi_1}{\pi_0}\right)}{\sigma \|\bar{s}\|}\right)$$

$$= Q\left(\frac{-\|\bar{s}\|^2 - 2\sigma^2 \ln\left(\frac{\pi_1}{\pi_0}\right)}{2\sigma \|\bar{s}\|}\right)$$

Therefore, $P_{MD} = 1 - P_D = 1 - Q\left(-\dfrac{\|\bar{s}\|^2 + 2\sigma^2 \ln\left(\frac{\pi_1}{\pi_0}\right)}{2\sigma\|\bar{s}\|}\right)$

$$1 - Q(-x) = Q(x) \longrightarrow \quad = Q\left(\frac{\|\bar{s}\|^2 + 2\sigma^2 \ln\left(\frac{\pi_1}{\pi_0}\right)}{2\sigma\|\bar{s}\|}\right)$$

Therefore, $P_e = \underbrace{\pi_0 P_{FA} + \pi_1 P_{MD}}$

$$= \pi_0\, Q\left(\frac{\|\bar{s}\|^2 - 2\sigma^2 \ln\left(\frac{\pi_1}{\pi_0}\right)}{2\sigma\|\bar{s}\|}\right) + \pi_1\, Q\left(\frac{\|\bar{s}\|^2 + 2\sigma^2 \ln\left(\frac{\pi_1}{\pi_0}\right)}{2\sigma\|\bar{s}\|}\right)$$

$P_r(H_0) P_r(H_1|H_0) + P_r(H_1) \cdot P_r(H_0|H_1)$

$= P_r(H_0) P_{FA} + P_r(H_1) P_{MD}$

We have, $\dfrac{\|\bar{s}\|^2}{\sigma^2} = SNR$

$$\sqrt{\frac{\|\bar{s}\|^2}{\sigma^2}} = \frac{\|\bar{s}\|}{\sigma} = \sqrt{SNR}$$

$$= \pi_0\, Q\left(\frac{1}{2}\frac{\|\bar{s}\|}{\sigma} - \frac{\sigma}{\|\bar{s}\|}\ln\left(\frac{\pi_1}{\pi_0}\right)\right)$$

$$+ \pi_1\, Q\left(\frac{1}{2}\frac{\|\bar{s}\|}{\sigma} + \frac{\sigma}{\|\bar{s}\|}\ln\left(\frac{\pi_1}{\pi_0}\right)\right)$$

$$\boxed{P_e = \pi_0\, Q\left(\frac{1}{2}\sqrt{SNR} - \frac{1}{\sqrt{SNR}}\ln\left(\frac{\pi_1}{\pi_0}\right)\right) + \pi_1\, Q\left(\frac{1}{2}\sqrt{SNR} + \frac{1}{\sqrt{SNR}}\ln\left(\frac{\pi_1}{\pi_0}\right)\right)}$$

$\therefore$ This is the Minimum Probability of Error, P_e.

Simple Example

$SNR = 10\ dB = 10$

$10 \log_{10} SNR = SNR\_dB = 10$

$\log_{10} SNR = 1$

$SNR = 10^1 = 10$

$\pi_1 = 0.8$

$\pi_0 = 1 - \pi_1 = 0.2$

$\frac{\pi_1}{\pi_0} = \frac{0.8}{0.2} = 4$

What is the minimum Probability of error ?

$$P_e = \pi_0\, Q\left(\frac{1}{2}\sqrt{SNR} - \frac{1}{\sqrt{SNR}} \ln\left(\frac{\pi_1}{\pi_0}\right)\right) + \pi_1\, Q\left(\frac{1}{2}\sqrt{SNR} + \frac{1}{\sqrt{SNR}} \ln\left(\frac{\pi_1}{\pi_0}\right)\right)$$

$$= 0.2\, Q\left(\frac{1}{2}\sqrt{10} - \frac{1}{\sqrt{10}} \ln(4)\right) + 0.8\, Q\left(\frac{1}{2}\sqrt{10} + \frac{1}{\sqrt{10}} \ln(4)\right)$$

$$= 0.2\, Q(1.14) + 0.8\, Q(2.01)$$

$$= (0.2 \times 0.127) + (0.8 \times 0.0222)$$

$$= 0.0254 + 0.01776$$

$$= 0.04316$$

This is the minimum value of the probability of error for this signal detection problem on using the MAP detection rule. And what the result says is that, this is the minimum P_e that can be achieved. (ie) no other rule can give a P_e that is lower than this. So, in that sense, the MAP is optimal to minimize the probability of error.