Week 8 : Session 3

MIMO Spectrum Sensing

The model for $r \times t$ MIMO system can be stated as follows.

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_r \end{bmatrix}_{r\times 1} = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1t} \\ h_{21} & & & \\ \vdots & & & \\ h_{r1} & \cdots & & h_{rt} \end{bmatrix}_{r\times t} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_t \end{bmatrix}_{t\times 1} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_r \end{bmatrix}_{r\times 1}$$

$$\Rightarrow \bar{y} = H \bar{x} + \bar{v}$$

where,

$\bar{y} \rightarrow r \times 1$ Receive vector

$\bar{x} \rightarrow t \times 1$ Transmit vector

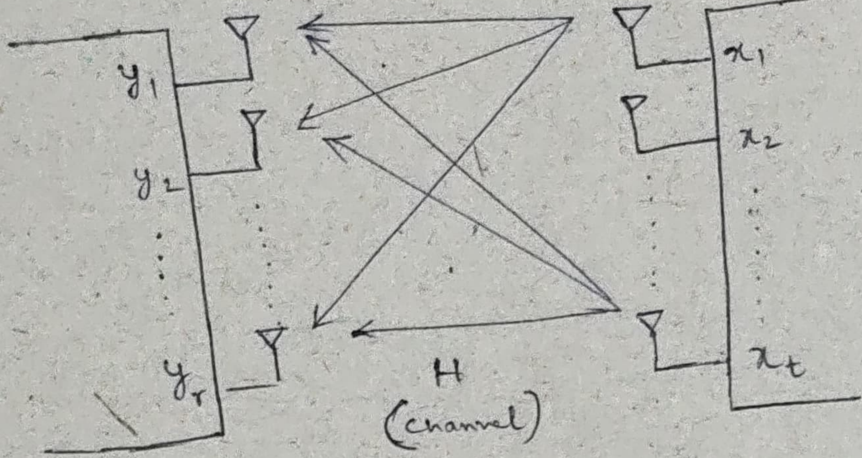
$\bar{v} \rightarrow r \times 1$ Noise vector

$H \rightarrow r \times t$ channel Matrix

$h_{ij} \rightarrow$ Channel coefficient between i^{th} Rx. Antenna and j^{th} Tx. Antenna

$r \rightarrow$ No. of Receive antennas

$t \rightarrow$ No. of Transmit antennas



We assume, the channel coefficients Symmetric complex Gaussian with zero mean and unit variance.

(ie) $h_{ij} \sim \mathcal{CN}(0,1)$.

$$\Rightarrow E\{h_{ij}\} = 0$$

$$E\{|h_{ij}|^2\} = 1$$

Since these are symmetric complex Gaussian, the real part and imaginary part are independent, and have variance $= \frac{1}{2}$ (half of the total variance). Thus, h_{ij} is also known as Rayleigh Fading channel coefficient.

(ie) $|h_{ij}| = a_{ij} \sim$ Rayleigh RV.

For MIMO Spectrum sensing, One can use the Test Statistic

$$\|\bar{y}\|^2 = \underbrace{|y_1|^2 + |y_2|^2 + \cdots + |y_r|^2}_{\text{ENERGY of the output.}}$$

Now, compare the Test statistic $\|\bar{y}\|^2$ with a suitable threshold γ to get the Energy Detector (ED).

$$\left.\begin{array}{ll} \text{Choose } H_0 : & \|\bar{y}\|^2 \leq \gamma \\ \text{Choose } H_1 : & \|\bar{y}\|^2 > \gamma \end{array}\right\} \text{ENERGY DETECTOR}$$

where, $H_0 \rightarrow$ Primary user absent

$H_1 \rightarrow$ Primary user present

Let us now characterize the performance of MIMO spectrum sensing algorithm.

⊙ consider Hypothesis H_0

- The Probability of false alarm (P_{FA}) is

$$P_{FA} = P_r\left(\|\bar{y}\|^2 > \gamma ; H_0\right).$$

$$\bar{y} \to \bar{v}$$

$$y_k = v_k$$

k^{th} receive antenna

$$= \underbrace{v_{k,I}}_{\text{Inphase component}} + j\, \underbrace{v_{k,Q}}_{\text{Quadrature component}}$$

$$\underbrace{v_{k,I}, v_{k,Q}}_{\text{iid Gaussian with mean} = 0,\ \text{Var} = \sigma^2/2} \sim \mathcal{N}\left(0, \frac{\sigma^2}{2}\right)$$

Thus,

$$\|\bar{y}\|^2 > \gamma$$

$$\Rightarrow \|\bar{v}\|^2 > \gamma$$

r = No. of receive antennas in MIMO system

$$\Rightarrow \sum_{k=1}^{r} |v_{k,I}|^2 + |v_{k,Q}|^2 > \gamma$$

$$\Rightarrow \underbrace{\sum_{k=1}^{r} \left|\frac{v_{k,I} - 0}{\sigma/\sqrt{2}}\right|^2 + \left|\frac{v_{k,Q} - 0}{\sigma/\sqrt{2}}\right|^2}_{\text{Sum of squares of } 2r \text{ iid Standard Normal RVs with mean} = 0,\ \text{Var} = 1.} > \frac{\gamma}{\sigma^2/2}$$

This is called Central chi-squared RV with $2r$ degrees of freedom.

$$\Rightarrow P_{FA} = P_r\left(\|\bar{v}\|^2 > \gamma ; H_0\right)$$

$$\Rightarrow P_{FA} = P_r\left(\frac{\|\bar{v}\|^2}{\sigma^2/2} > \frac{\gamma}{\sigma^2/2} ; H_0\right)$$

$$\Rightarrow \boxed{P_{FA} = Q_{\chi^2_{2r}}\left(\frac{\gamma}{\sigma^2/2}\right)}$$

CCDF of Central chi-squared RV with $2r$ degrees of freedom.

⊙ Consider the Hypothesis H_1

- The Probability of detection (P_D) is

$$P_D = P_r\left(\|y\|^2 > \gamma \; ; \; H_1\right)$$

Consider the BPSK symbols x_i with power P,

where $x_i \in \{\sqrt{P}, -\sqrt{P}\}$

Thus, $E\{x_i\} = 0$

$|x_i|^2 = P$

Now, we can write, for each k,

$$y_k = \sum_{l=1}^{t} h_{kl}\, x_l + v_k$$

where, $x_l \rightarrow$ symbol transmitted on l^{th} Transmit antenna

$h_{k,l}, v_k \rightarrow$ Complex Gaussian

$y_k \rightarrow$ Linear combination of Symmetric complex Gaussian RV's.

Thus, $y_k \rightarrow$ Zero mean Circularly Symmetric complex gaussian.

Now, $E\{y_k\} = 0$, $E\{v_k\} = 0$, $E\{h_{kl}\} = 0$.

$$E\{|y_k|^2\} = \sum_{l=1}^{t} \underbrace{E\{|h_{kl}|^2\}}_{1} \underbrace{|x_l|^2}_{P} + \underbrace{E\{|v_k|^2\}}_{\sigma^2}$$

$$= tP + \sigma^2$$

Therefore, $y_k \sim \mathcal{CN}\left(0, \frac{tP+\sigma^2}{2}\right)$

$E\{h_{kl} \cdot v_k^*\} = 0$ (cross term)

$E\{h_{kl} \cdot h_{kp}^*\} = 0$ (when $l \neq p$, coz there are iid channel coeffs between different antennas)

$E\{|h_{kl}|^2\} = 1$ (coz $h_{kl} \sim \mathcal{CN}(0,1)$)

$E\{|v_k|^2\} = \sigma^2$

So, $\underbrace{y_k = y_{k,I}}_{\text{Inphase component}} + j \underbrace{y_{k,Q}}_{\text{Quadrature component}}$

where, $\underbrace{y_{k,I}, y_{k,Q}}_{\text{iid zero mean Gaussian}} \sim \mathcal{N}\left(0, \frac{tP+\sigma^2}{2}\right)$

Then, $\|\bar{y}\|^2 > \gamma$

$$\Rightarrow \sum_{k=1}^{r} |y_{k,I}|^2 + |y_{k,Q}|^2 > \gamma$$

$$\Rightarrow \underbrace{\sum_{k=1}^{r} \left|\frac{y_{k,I}}{\sqrt{\frac{tP+\sigma^2}{2}}}\right|^2 + \left|\frac{y_{k,Q}}{\sqrt{\frac{tP+\sigma^2}{2}}}\right|^2}_{\text{sum of squares of } 2r \text{ iid zero mean standard Gaussian RV.}} > \frac{\gamma}{\frac{tP+\sigma^2}{2}}$$

This is called as Central chi-squared RV with $2r$ degrees of freedom

$$\Rightarrow P_D = P_r\left(\|\bar{y}\|^2 > \gamma \,;\, H_1\right)$$

$$\Rightarrow P_D = P_r\left(\frac{\|\bar{y}\|^2}{(tP+\sigma^2)/2} > \frac{\gamma}{(tP+\sigma^2)/2} \,;\, H_1\right)$$

$$\Rightarrow \boxed{P_D = Q_{\chi^2_{2r}}\left(\frac{\gamma}{(tP+\sigma^2)/2}\right)}$$

CCDF of central chi-squared RV with $2r$ degrees of freedom.

- Receiver operating characteristic (ROC)

$$P_{FA} = Q_{\chi^2_{2r}}\left(\frac{\gamma}{\sigma^2/2}\right)$$

$$\Rightarrow \gamma = \frac{\sigma^2}{2} Q^{-1}_{\chi^2_{2r}}(P_{FA})$$

$$P_D = Q_{\chi^2_{2r}}\left(\frac{\frac{\sigma^2}{2} Q^{-1}_{\chi^2_{2r}}(P_{FA})}{(tP+\sigma^2)/2}\right)$$

$$P_D = Q_{\chi^2_{2r}}\left(\frac{\sigma^2}{tP+\sigma^2} Q^{-1}_{\chi^2_{2r}}(P_{FA})\right)$$

WKT, $SNR = \frac{P}{\sigma^2}$

where $P = |x_i|^2$

$\sigma^2 = E\{|v_k|^2\}$

$$\Rightarrow P_D = Q_{\chi^2_{2r}}\left(\frac{1}{\frac{tP}{\sigma^2}+1} Q^{-1}_{\chi^2_{2r}}(P_{FA})\right)$$

$$\Rightarrow \boxed{P_D = Q_{\chi^2_{2r}}\left(\frac{Q^{-1}_{\chi^2_{2r}}(P_{FA})}{(t \times SNR)+1}\right)}$$

ROC

P_D in terms of P_{FA}.