Week 3 : Session 1

Generalized ML Detection

In Binary Hypothesis testing problem, under Hypothesis H_0, let us consider the model

H_0 :

$$
\begin{aligned}
y(1) &= s_0(1) + v(1) \\
y(2) &= s_0(2) + v(2) \\
&\vdots \\
y(N) &= s_0(N) + v(N)
\end{aligned}
$$

← iid Gaussian noise samples
mean = 0
Variance = σ^2

$s_0(i)$ - Signal corresponding to H_0.

Unlike the previous case where we have the SIGNAL ABSENT under H_0 (NULL Hypothesis), here we have the SIGNAL PRESENT. It is still called H_0 (NULL Hypothesis) except now it is a general scenario.

We have a signal corresponding to Hypothesis H_0, and a different signal corresponding to Hypothesis H_1. And how do we differentiate between these two signals. This is the generalized version of the ML Detection Problem.

In vector form,

$$
H_0 : \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} = \begin{bmatrix} s_0(1) \\ s_0(2) \\ \vdots \\ s_0(N) \end{bmatrix} + \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}
$$

$$
\Rightarrow \boxed{\bar{y} = \bar{s}_0 + \bar{v} \quad ; H_0}
$$

→ Signal for H_0

Similarly, under Hypothesis H_1, we have a different signal.

$\underline{H_1}$:

$$y(1) = s_1(1) + v(1)$$
$$y(2) = s_1(2) + v(2)$$
$$\vdots$$
$$y(N) = s_1(N) + v(N)$$

In vector form;

$$H_1 : \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} = \begin{bmatrix} s_1(1) \\ s_1(2) \\ \vdots \\ s_1(N) \end{bmatrix} + \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}$$

$$\Rightarrow \boxed{\bar{y} = \bar{s}_1 + \bar{v} \;;\; H_1}$$

Signal for H_1.

- Noise samples in $\bar{v}$ (ie) $v(1), v(2), \ldots, v(N)$ are iid Gaussian with mean $= 0$, variance $= \sigma^2$
- $\bar{s}_0$, $\bar{s}_1$ are known signals

Signal corresponding to Hypothesis H_1

Signal corresponding to Hypothesis H_0

In compact form,

$$H_0 : \bar{y} = \bar{s}_0 + \bar{v}$$
$$H_1 : \bar{y} = \bar{s}_1 + \bar{v}$$

This is still Binary Hypothesis Testing Problem.
What is the LRT ?

We define $\tilde{y} = \bar{y} - \bar{s}_0$

Thus, $H_0 : \bar{y} = \bar{s}_0 + \bar{v}$

$\Rightarrow \bar{y} - \bar{s}_0 = \bar{v}$

$\Rightarrow \tilde{y} = \bar{v} \; ; H_0$

Similarly,

$H_1 : \bar{y} = \bar{s}_1 + \bar{v}$

$\Rightarrow \bar{y} - \bar{s}_0 = \bar{s}_1 - \bar{s}_0 + \bar{v}$

$\Rightarrow \tilde{y} = \bar{s} + \bar{v} \; ; H_1$

where $\bar{s} = \bar{s}_1 - \bar{s}_0$.

Hence reduced to original signal detection problem:

$H_0 : \tilde{y} = \bar{v}$

$H_1 : \tilde{y} = \bar{s} + \bar{v}$

with $\bar{y} \rightarrow \tilde{y}$ and $\bar{s} \rightarrow \bar{s}_1 - \bar{s}_0$.

We know the LRT for this. (i.e)

- choose H_0 if

$$\bar{s}^T \tilde{y} \le \gamma$$

$$(\bar{s}_1 - \bar{s}_0)^T \tilde{y} \le \gamma$$

$$\underbrace{(\bar{s}_1 - \bar{s}_0)^T (\bar{y} - \bar{s}_0)} \le \gamma$$

- choose H_1 if

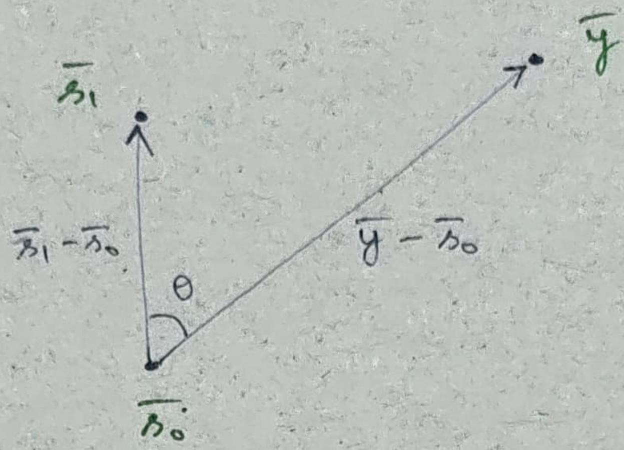
$$\bar{s}^T \tilde{y} > \gamma$$

$$(\bar{s}_1 - \bar{s}_0)^T \tilde{y} > \gamma.$$

$$\underbrace{(\bar{s}_1 - \bar{s}_0)^T (\bar{y} - \bar{s}_0)} > \gamma$$

TEST STATISTIC for "Generalized Signal Detection Problem".

Pictorial intuition



Test statistic = Dot / Inner product b/w these 2 vectors

(ie) $(\bar{s}_1 - \bar{s}_0)^T (\bar{y} - \bar{s}_0)$

$= |\bar{s}_1 - \bar{s}_0| \, |\bar{y} - \bar{s}_0| \cos\theta$

In original signal detection problem, we have a single signal and we are detecting either its presence or absence. Whereas, in generalized signal detection problem, we have two signals ($\bar{s}_0$ and $\bar{s}_1$) and we are detecting either its $\bar{s}_0$ or $\bar{s}_1$ occuring. Hence its a general version, still a Binary Hypothesis Testing Problem.

Therefore, P_{FA} and P_D are given as

$$P_{FA} = Q\left(\frac{\gamma}{\sigma \|\bar{s}\|}\right) = Q\left(\frac{\gamma}{\sigma \|\bar{s}_1 - \bar{s}_0\|}\right)$$

$$P_D = Q\left(\frac{\gamma - \|\bar{s}\|^2}{\sigma \|\bar{s}\|}\right) = Q\left(\frac{\gamma - \|\bar{s}_1 - \bar{s}_0\|^2}{\sigma \|\bar{s}_1 - \bar{s}_0\|}\right)$$

For Maximum Likelihood (ML), Set

$$\gamma = \frac{\|\bar{s}\|^2}{2} = \frac{\|\bar{s}_1 - \bar{s}_0\|^2}{2}$$

Thus, choose H_0 if $(\bar{s}_1 - \bar{s}_0)^T \tilde{y} \leq \frac{\|\bar{s}_1 - \bar{s}_0\|^2}{2}$

choose H_1 if $(\bar{s}_1 - \bar{s}_0)^T \tilde{y} > \frac{\|\bar{s}_1 - \bar{s}_0\|^2}{2}$

where $\tilde{y} = \bar{y} - \bar{s}_0$

And, the P_{FA} and P_D are given as

$$P_{FA} = Q\left(\frac{\|\bar{s}\|}{2\sigma}\right) = Q\left(\frac{\|\bar{s}_1 - \bar{s}_0\|}{2\sigma}\right)$$

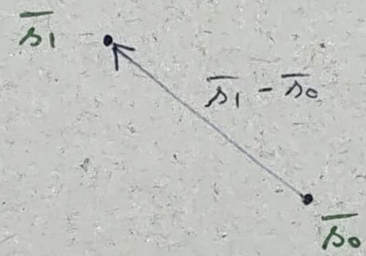
$$P_{MD} = Q\left(\frac{\|\bar{s}\|}{2\sigma}\right) = Q\left(\frac{\|\bar{s}_1 - \bar{s}_0\|}{2\sigma}\right)$$

Probability of Miss detection, $P_{MD} = 1 - P_D$.

And, P_e is given as

$$P_e = Q\left(\frac{\|\bar{s}\|}{2\sigma}\right) = Q\left(\frac{\|\bar{s}_1 - \bar{s}_0\|}{2\sigma}\right) = Q\left(\frac{d}{2\sigma}\right)$$

distance b/w two signals.



$\|\bar{s}_1 - \bar{s}_0\|$ is the distance b/w $\bar{s}_0$ and $\bar{s}_1$. So, to minimize P_e, we maximize the distance b/w the signals.

Thus, the Probability of error depends only on the distance between these two signals $\bar{s}_0$ and $\bar{s}_1$.

Therefore, P_e is given as

$$P_e = P_r(H_0) \cdot P_r(H_1 | H_0) + P_r(H_1) \cdot P_r(H_0 | H_1)$$

$$= \frac{1}{2} P_{FA} + \frac{1}{2} P_{MD}$$

$$= Q\left(\frac{\|\bar{s}\|}{2\sigma}\right) = Q\left(\frac{\|\bar{s}_1 - \bar{s}_0\|}{2\sigma}\right) = Q\left(\frac{d}{2\sigma}\right)$$

where, d is the distance b/w the points $\bar{s}_1, \bar{s}_0$.

$$d = \|\bar{s}_1 - \bar{s}_0\|$$

Example:

consider BPSK system, where we have constellation $\{-A, A\}$

No. of symbols $= 2$;

No. of bits/symbol $= \log_2 2 = 1$;

Signal $= A$; No. of sample, $N = 1$;

Hypothesis,

$$H_0 : y = -A + v$$
$$H_1 : y = A + v$$

Typically, in comm. System $\sigma^2 = \frac{N_0}{2}$

where $s_0 = -A$, $s_1 = A$.

Therefore, $P_e = Q\left(\frac{\|\bar{s}_1 - \bar{s}_0\|}{2\sigma}\right) = Q\left(\frac{\|A - (-A)\|}{2\sqrt{N_0/2}}\right)$

$$= Q\left(\frac{2A}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{2A^2}{N_0}}\right)$$

- Each symbol carries one bit
- Let E_b be the Energy per bit
- Let each symbol be equiprobable.

(i.e) $P_r(-A) = P_r(A) = \frac{1}{2}$

$P_r(H_0)$ $P_r(H_1)$

$$E_b = \underbrace{P_r(H_0) \times A^2}_{\text{Energy for } H_0} + \underbrace{P_r(H_1) \times (-A)^2}_{\text{Energy for } H_1}$$

$$= \frac{1}{2}(A^2) + \frac{1}{2}\cdot(-A)^2$$

$$= A^2$$

$\Rightarrow A = \sqrt{E_b}$

$\Rightarrow A^2 = E_b$

Therefore, $P_e = Q\left(\sqrt{\frac{2A^2}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$

This is termed as Bit Error Rate for BPSK. (Probability that bit is in error)

ASK VS BPSK

BER of ASK	BER of BPSK
$Q\left(\sqrt{\frac{E_b}{N_0}}\right)$	$Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$

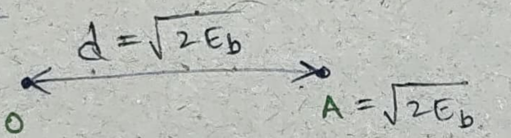
Inference:
For same E_b, BER of BPSK is LOWER than BER of ASK..

Why?

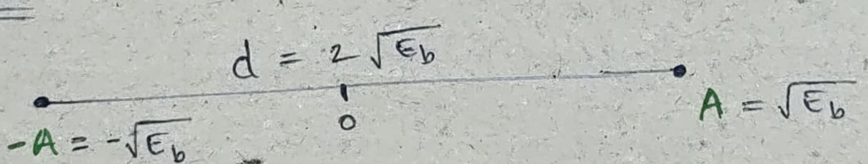
- Infact, BPSK is 3 dB MORE EFFICIENT than ASK, since BPSK needs half the E_b to achieve same BER as ASK. (ie) 3dB less!
- For instance, if E_b of ASK is 8 dB, then E_b for BPSK to achieve the same BER would be 3 dB lower. (ie) E_b of BPSK is 5dB.

Intuition (ASK vs. BPSK Distance Properties).

ASK:



BPSK:

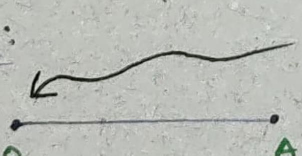


For same E_b, distance of BPSK $>$ distance of ASK.

This happens because BPSK is ANTIPODAL. (ie) centred around zero.

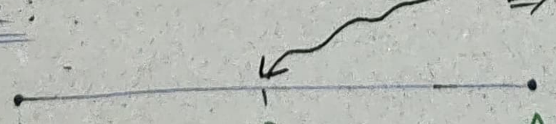
ASK:

Not Symmetric about 0
$\Rightarrow$ Energy is higher



BPSK:

Symmetric about 0
$\Rightarrow$ Energy is minimum



Therefore, constellation has to be symmetric about 0 to minimize P_e.