Week 3 : Session 2

Detection with Multiple Symbols

In this session, we'll look at the Multiple Hypothesis Testing Problem and the classic example for Multiple Hypothesis Testing is the detection of multiple symbols in a constellation. Another example could be "Face detection and Recognition".
(ii) Mapping a set of features to one possible face from a set of faces. We'll have a Face database, making a decision about a face and mapping it to one of the faces from a DB.

So, in order to do that, let us first start with the Binary Hypothesis Testing Problem and simplify that detection rule, so that we can then get to a Multiple Hypothesis testing problem.

Recall, the Binary Hypothesis testing problem is

$$H_0 : \bar{y} = \bar{s}_0 + \bar{v}$$

$$H_1 : \bar{y} = \bar{s}_1 + \bar{v}$$

where, $\bar{s}_0 \rightarrow$ signal for H_0

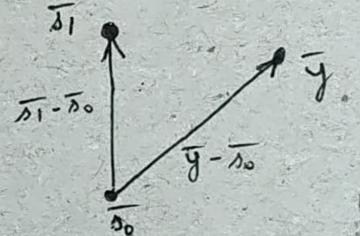
$\bar{s}_1 \rightarrow$ signal for H_1

And, we seen that the optimal detector is

- Choose H_0 if

$$\bar{s}^T \tilde{y} \le \gamma$$

$$\Rightarrow \underbrace{(\bar{s}_1 - \bar{s}_0)^T (\bar{y} - \bar{s}_0)}_{\text{Inner Product}} \le \gamma$$



- Else, choose H_1

In particular for the ML detector, we've seen that

$$\gamma = \frac{\|\bar{s}\|^2}{2} = \frac{\|\bar{s}_1 - \bar{s}_0\|^2}{2}$$

Therefore, choose H_0 if

$$(\bar{s}_1 - \bar{s}_0)^T \underbrace{(\bar{y} - \bar{s}_0)}_{\tilde{y}} \le \frac{\|\bar{s}_1 - \bar{s}_0\|^2}{2}$$

This can be further simplified as follows. $\|\bar{u}\|^2 = \bar{u}^T\bar{u}$

$$\Rightarrow 2(\bar{s}_1 - \bar{s}_0)^T(\bar{y} - \bar{s}_0) \leq \|\bar{s}_1 - \bar{s}_0\|^2$$

$$\Rightarrow 2(\bar{s}_1^T - \bar{s}_0^T)(\bar{y} - \bar{s}_0) \leq (\bar{s}_1 - \bar{s}_0)^T(\bar{s}_1 - \bar{s}_0)$$

$$\Rightarrow \left.\begin{array}{l} 2\bar{s}_1^T\bar{y} - 2\bar{s}_0^T\bar{y} \\ \cancel{-2\bar{s}_1^T\bar{s}_0} + 2\|\bar{s}_0\|^2 \end{array}\right\} \leq \|\bar{s}_1\|^2 + \|\bar{s}_0\|^2 \cancel{-2\bar{s}_1^T\bar{s}_0}$$

$$\Rightarrow \|\bar{s}_0\|^2 - 2\bar{s}_0^T\bar{y} \leq \|\bar{s}_1\|^2 - 2\bar{s}_1^T\bar{y}$$

$$\Rightarrow \|\bar{s}_0\|^2 + \underline{\|\bar{y}\|^2} - 2\bar{s}_0^T\bar{y} \leq \|\bar{s}_1\|^2 + \underline{\|\bar{y}\|^2} - 2\bar{s}_1^T\bar{y}$$

$$\Rightarrow \|\bar{y} - \bar{s}_0\|^2 \leq \|\bar{y} - \bar{s}_1\|^2$$

$$\Rightarrow \|\bar{y} - \bar{s}_0\| \leq \|\bar{y} - \bar{s}_1\|$$

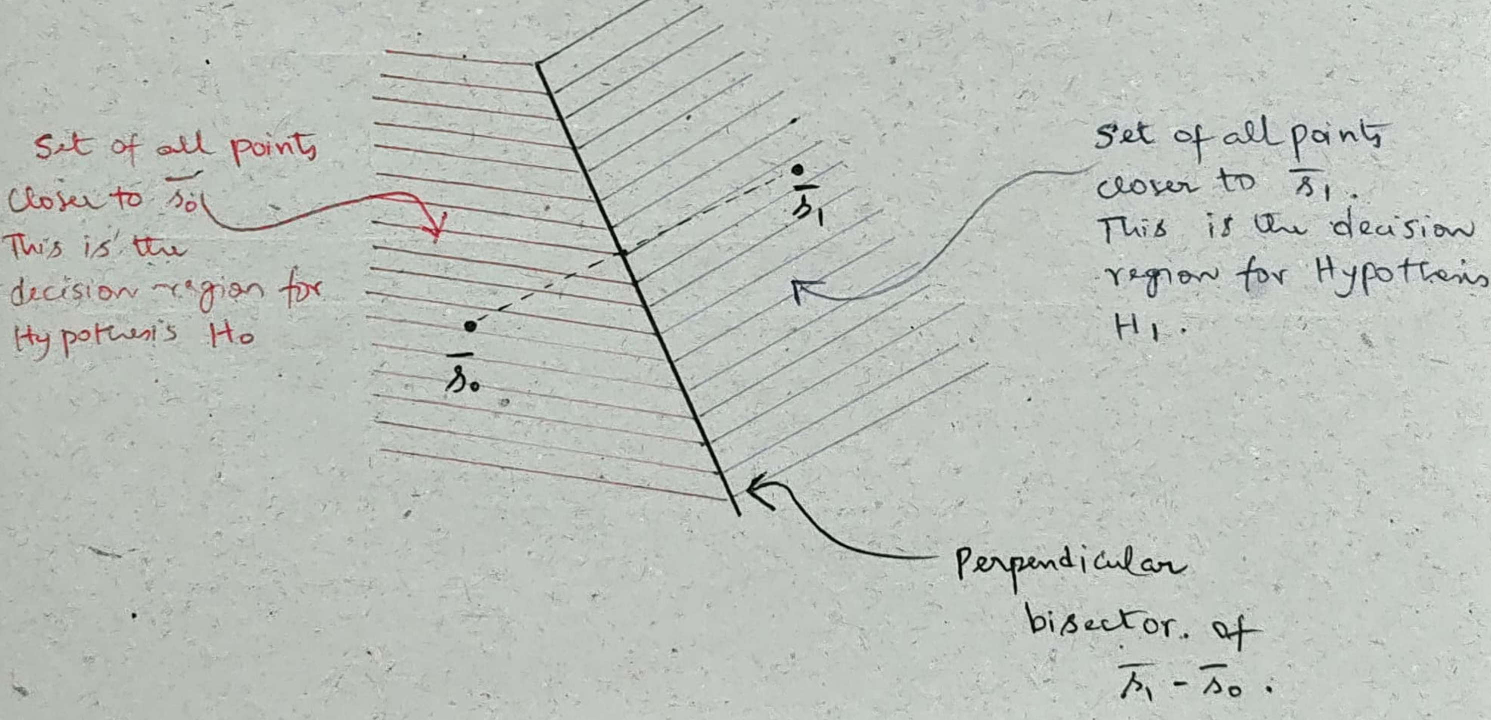
Therefore, for ML detector, what we have is,

Choose H_0 if,

$$\|\bar{y} - \bar{s}_0\| \leq \|\bar{y} - \bar{s}_1\|$$

This is known as Minimum distance decoder. (ie)

Choose the Hypothesis corresponding to the signal which is closest to the output vector. (ie) if $\bar{s}_0$ is closest to $\bar{y}$, then choose Hypothesis H_0. And if $\bar{s}_1$ is closer to $\bar{y}$, then choose Hypothesis H_1.



So, when the points are closer to $\bar{s}_0$, we choose H_0.
And, when the points are closer to $\bar{s}_1$, we choose H_1.

Summary :

We have two signal $\bar{s}_0$ and $\bar{s}_1$ in N dimensional space. And if we look at the $\perp^r$ bisector (i.e) Hyperplane that divides it into two regions. The region of points closer to $\bar{s}_0$ is the decision region corresponding to hypothesis H_0. And the set of points closer to $\bar{s}_1$ is the region corresponding to Hypothesis H_1.

The points on the hyperplane which are equidistant to both Hypothesis signal $\bar{s}_0$ and $\bar{s}_1$; we can put them in any Hypothesis. But, by convension, we are putting them in Hypothesis H_0. We can as well put them in Hypothesis H_1 which will not affect the performance of the test. We can arbitrarily choose Hypotheses H_0 or H_1..

Thus,

- choose H_0 if $\|\bar{y}-\bar{s}_0\| \leq \|\bar{y}-\bar{s}_1\|$.
- choose H_1 if $\|\bar{y}-\bar{s}_1\| < \|\bar{y}-\bar{s}_0\|$.
- choose the closest signal !

This is called Nearest Neighbor decision rule/ Nearest Neighbor decoder/ NN decoder.

Now, we can readily extend this to a Multiple signal decoding problem / Multiple Hypothesis Testing Problem.

Consider now a multiple hypothesis testing problem

$$\begin{aligned} H_0 &: \bar{y} = \bar{s}_0 + \bar{v} \\ H_1 &: \bar{y} = \bar{s}_1 + \bar{v} \\ &\vdots \\ H_{M-1} &: \bar{y} = \bar{s}_{M-1} + \bar{v} \end{aligned}$$

M-ary Hypothesis Testing Problem.

M signals

Now, how do we choose the decision rule? The decision rule is the Nearest Neighbor decoder for this multiple Hypothesis testing problem.

(ii) Choose H_i corresponding to signal $\bar{s}_i$ which is closest to the output vector $\bar{y}$.

Choose H_i such that

$$i = \arg\min_{i} \|\bar{y} - \bar{s}_i\|$$

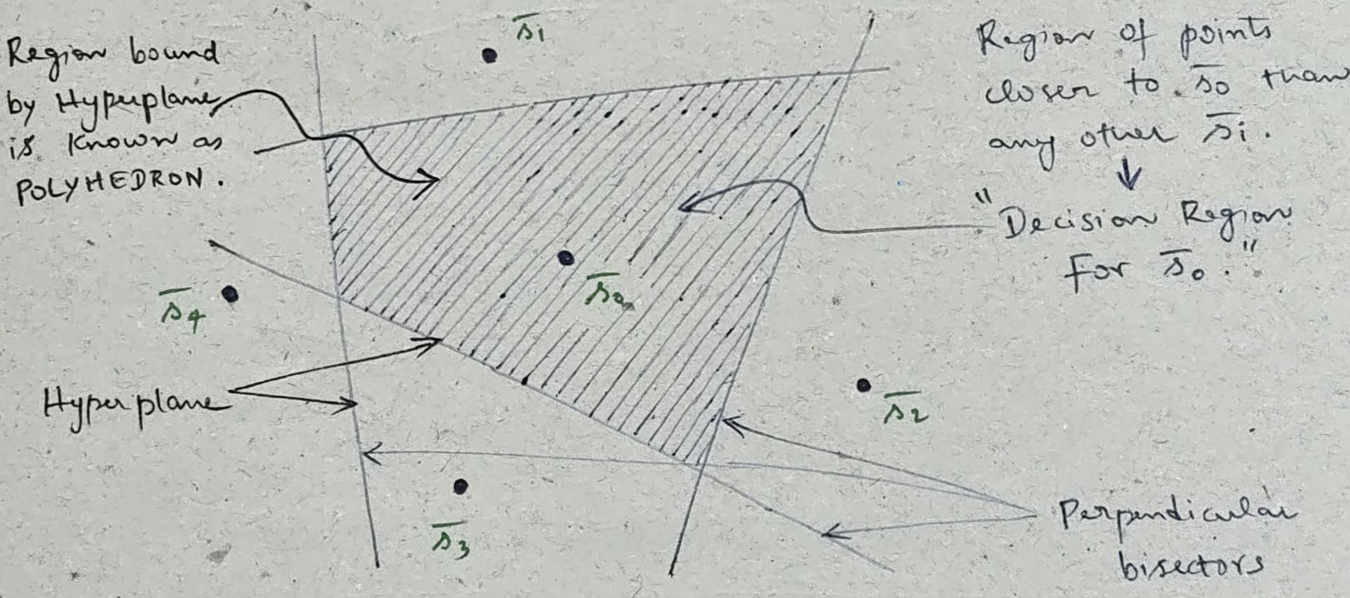


Figure of Decision Region

Draw the perpendicular bisectors for every pair of signals, which forms the region of intersection. The points that are closer to $\bar{s}_0$ than any other constellation points, is the decision region corresponding to $\bar{s}_0$, which is the region of points that will be mapped to $\bar{s}_0$ in our NN decision rule.

The decision region for each hypothesis is a POLYHEDRON.

For Minimum distance decision rule (ie) For an M-ary Hypothesis testing problem, what is the Probability of Error (Pe) of symbol decoding ?

We start with the notion of Confusion Probability $i \to j$, which is basically the probability that $\bar{s}_i$ will be confused for $\bar{s}_j$.

$$P_{\bar{s}_i \to \bar{s}_j} = Q\left(\frac{\|\bar{s}_i - \bar{s}_j\|}{2\sigma}\right)$$

Now, the Probability of error for symbol i is the Union of confused events.

$$P_{e,i} = P\left(\bigcup_{j \neq i} \bar{s}_i \to \bar{s}_j\right)$$

$$\Rightarrow P_{e,i} \leq \sum_{j \neq i} P_{\bar{s}_i \to \bar{s}_j}$$

$$= \sum_{j \neq i} Q\left(\frac{\|\bar{s}_i - \bar{s}_j\|}{2\sigma}\right)$$

WKT, Union bound in Probability,

$P(A \cup B \cup C \ldots)$

$\leq$

$P(A) + P(B) + P(C) + \ldots$

Sum of confusion probabilities which acts as upper bound for $P_{e,i}$. This is called as "UNION BOUND".

At high SNR especially, this UNION BOUND, where we simply upper bounded by the sum of probabilities of the events, provides a very good approximation to the probability of error.

Generally, $P(A \cup B) = P(A) + P(B) + P(A \cap B)$
But here, we are ignoring the probabilities of intersection of confusion events.

This is done in order to overcome the complexity.

Probability of Error (for M dimensional constellation)

$$P_e = \sum_i P_i \cdot P_{e,i}$$

where, $P_i = \frac{1}{M}$ (i.e. in an M-ary Hypothesis Testing probability, if the symbols are equiprobable then $P_i = 1/M$.)

Therefore,

$$P_e = \sum_i \frac{1}{M} P_{e,i}$$

$$= \frac{1}{M} \sum_i \underbrace{\sum_{j \neq i} Q\left(\frac{\|\bar{s}_i - \bar{s}_j\|}{2\sigma}\right)}$$

This expression is dominated by $\bar{s}_j$ which are closest to $\bar{s}_i$. So, choose only those $\bar{s}_j$ which are closest to $\bar{s}_i$. And we call them as Nearest neighbors.

$$\boxed{P_e = \frac{1}{M} \sum_i N^i_{min} \, Q\left(\frac{d^i_{min}}{2\sigma}\right)}$$

where, N^i_{min} → Number of nearest neighbors to $\bar{s}_i$

d^i_{min} → Distance of nearest neighbors to $\bar{s}_i$

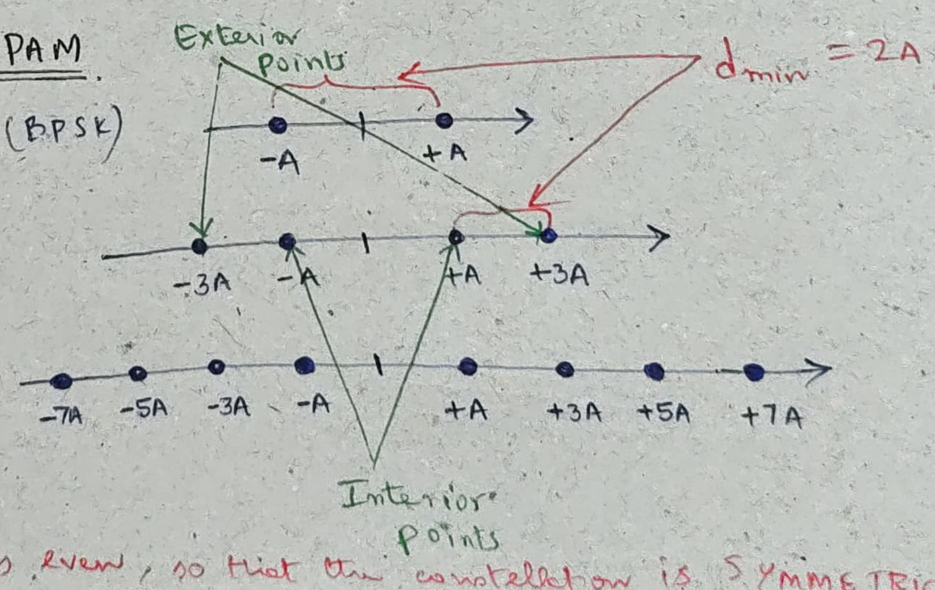
even & power of 2
integer bits/sym

M-ary PAM

① 2-PAM (BPSK)

② 4-PAM

③ 8-PAM



M is always even, so that the constellation is SYMMETRIC.

Example : 8-ary PAM.

Recall; 8-ary PAM has 8 constellation points.

$-7A, -5A, -3A, -A, +A, +3A, +5A, +7A$

In general, the i^{th} constellation point of M-ary PAM is given by

$$\boxed{s_i = \big(2i - (M-1)\big) A, \quad \text{where } i = 0, 1, \ldots, M-1}$$

And, the distance between the neighbor constellation points, $d_{min} = 2A$.

This is a multiple Hypothesis Testing problem (M-ary Hypothesis Testing Problem).

$H_0 : y = s_0 + v$

$H_1 : y = s_1 + v$

$\vdots$

$H_{M-1} : y = s_{M-1} + v$

For any interior point,

- No. of nearest neighbors, $N^i_{min} = 2$
- Distance of nearest neighbors, $d^i_{min} = 2A$

Probability of error, for the interior points

$$P_{e,i} = N^i_{min}\, Q\left(\frac{d^i_{min}}{2\sigma}\right)$$

$$= 2\, Q\left(\frac{2A}{2\sigma}\right)$$

$$= 2\, Q\left(\frac{A}{\sigma}\right)$$

For any boundary points, (Ex. $-7A$, $+7A$)

- No. of nearest neighbors, $N^i_{min} = 1$.
- Distance of nearest neighbors, $d^i_{min} = 2A$

Probability of error, for the boundary points

$$P_{e,i} = N^i_{min}\, Q\left(\frac{d^i_{min}}{2\sigma}\right)$$

$$= 1 \cdot Q\left(\frac{2A}{2\sigma}\right)$$

$$= \underline{\underline{Q\left(\frac{A}{\sigma}\right)}}$$

Therefore, the overall probability of error,

$$P_e = \frac{1}{M} \sum_i N^i_{min}\, Q\left(\frac{d^i_{min}}{2\sigma}\right)$$

For $M-2$ interior points

For 2 points on the boundary

$$= \frac{1}{M}(M-2)\cdot 2\, Q\left(\frac{A}{\sigma}\right) + \frac{1}{M}\, 2\, Q\left(\frac{A}{\sigma}\right)$$

$$= \frac{1}{M}(2M-4+2)\, Q\left(\frac{A}{\sigma}\right) = \frac{2M-2}{M}\, Q\left(\frac{A}{\sigma}\right)$$

$$= 2\left(1-\frac{1}{M}\right) Q\left(\frac{A}{\sigma}\right)$$

Express A in terms of Average Symbol power E_s.

– PTO

Week 4 : Session 1

In the previous session, we were looking at the PAM constellation, where the i^{th} constellation point of M-ary PAM is given by

$$S_i = (2i - (M-1))A, \text{ where } i = 0, 1, \ldots, M-1.$$

The Average symbol power,

FORMULAE :

① Sum of squares of n natural numbers.
$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

② Sum of n natural numbers
$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}.$$

$$E_s = \sum_i P(s_i)\,|s_i|^2$$

$$= \frac{1}{M}\sum_i |s_i|^2$$

$$= \frac{1}{M}\sum_{i=0}^{M-1}(2i-(M-1))^2 A^2$$

$$= \frac{1}{M}\sum_{i=0}^{M-1}\left(4i^2 + (M-1)^2 - 4i(M-1)\right)A^2$$

$$= \frac{A^2}{M}\left[4\sum_{i=0}^{M-1} i^2 + (M-1)^2\sum_{i=0}^{M-1}(1) - 4(M-1)\sum_{i=0}^{M-1} i\right]$$

$$1^2 + 2^2 + \ldots + (M-1)^2 = \frac{(M-1)(M-1+1)(2(M-1)+1)}{6} = \frac{(M-1)M(2M-2+1)}{6} = \frac{(M-1)M(2M-1)}{6}$$

$$\underbrace{1+1+\ldots+1}_{M \text{ times}} = M$$

$$1 + 2 + \ldots + (M-1) = \frac{(M-1)(M-1+1)}{2} = \frac{(M-1)M}{2}$$

$$= \frac{A^2}{M}\left[4\frac{(M-1)M(2M-1)}{6} + (M-1)^2 M - 4(M-1)\frac{(M-1)M}{2}\right]$$

$$= \frac{A^2}{M}\left[\frac{2}{3}(M-1)M(2M-1) + (M-1)^2 M - 2(M-1)^2 M\right]$$

$$= \frac{A^2}{M}\left[\frac{2}{3}(M-1)M(2M-1) - (M-1)^2 M\right]$$

$$= \frac{A^2}{M} M(M-1)\left[\frac{2}{3}(2M-1) - (M-1)\right]$$

$$= A^2(M-1)\left[\frac{4M-2-(3M-3)}{3}\right]$$

$$= \frac{A^2}{3}(M-1)(M+1) = \frac{A^2}{3}(M^2+M-M-1)$$

$$\boxed{E_s = \frac{A^2}{3}(M^2-1)} \Rightarrow \boxed{A = \sqrt{\frac{3E_s}{M^2-1}}}$$

Therefore, P_e reduces to

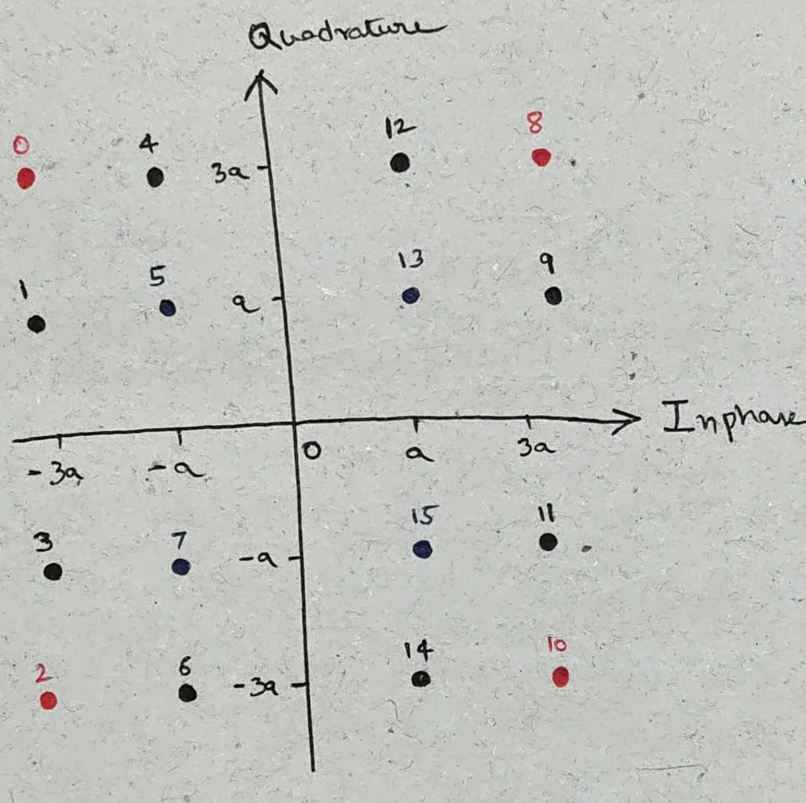
$$P_e = 2\left(1-\frac{1}{M}\right) Q\left(\frac{A}{\sigma}\right) = 2\left(1-\frac{1}{M}\right) Q\left(\sqrt{\frac{3\,E_s}{(M^2-1)\,\underbrace{N_0/2}_{\sigma^2 = N_0/2}}}\right)$$

$$\boxed{P_e = 2\left(1-\frac{1}{M}\right) Q\left(\sqrt{\frac{6\,E_s}{(M^2-1)\,N_0}}\right)}$$

This is the Probability of Symbol error for M-ary PAM. (a.k.a) Symbol Error Rate (SER)

QAM Constellation

Let us now consider a QAM constellation. (M-ary QAM)



- We have complex signal in QAM.
  - Inphase → Real part
  - Quadrature → Imaginary part
- We can think of as
  - PAM constellation in Inphase
  - PAM constellation in Quadrature

(i.e) PAM Inphase + j PAM Quadrature

In this QAM constellation, we see 4-PAM in Inphase and 4-PAM in Quadrature. Hence $4 \times 4 = 16$ QAM. Also QAM is a SQUARE constellation, symmetric about zero, which maximizes the distance for a given symbol power. So, QAM is Energy efficient constellation.

M-ary QAM constellation can be modeled as

$$\boxed{x_I + j\, x_Q}$$

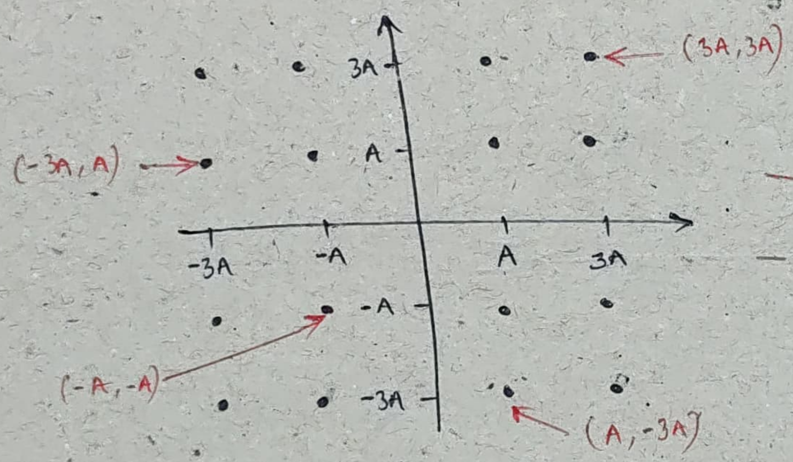
where $x_I, x_Q \in \sqrt{M}$ PAM.

Example: $M = 16$ QAM;

$x_I \in 4$ PAM (i.e.) $\{-3A, -A, A, 3A\}$

$x_Q \in 4$ PAM (i.e.) $\{-3A, -A, A, 3A\}$

Total No. of Symbols in M-QAM $= \sqrt{M} \times \sqrt{M}$



SQUARE CONSTELLATION

- E_s is the Total QAM average symbol power
- Power of x_I, x_Q is $\frac{E_s}{2}$

(i.e.) x_I and x_Q is symmetric, Each has Half power!

- SER of in-phase and Quadrature constellation
  - Recall, inphase and Quadrature are $\sqrt{M}$ PAM. Therefore, the SER can be derived by substituting M by $\sqrt{M}$ in the $P_{e,PAM}$:

(i) $$P_{e,PAM} = 2\left(1-\frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{6(E_s/2)}{(\sqrt{M}^2-1)N_0}}\right)$$

$$= 2\left(1-\frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3E_s}{(M-1)N_0}}\right)$$

- Probability of symbol error for QAM (ii) SER of QAM is given by

$$P_{e,QAM} = 1 - P_r(\text{Both } x_I, x_Q \text{ NOT in error})$$

$P_r(x_I \text{ not in error}) = 1 - P_{e,PAM}$

$P_r(x_Q \text{ not in error}) = 1 - P_{e,PAM}$

$P_r(\text{Both } x_I, x_Q \text{ NOT in error}) = (1-P_{e,PAM})^2$

$\because x_I, x_Q$ are Independent events.

$$\boxed{P_{e,QAM} = 1-(1-P_{e,PAM})^2}$$

$$= 1-(1+P_{e,PAM}^2 - 2P_{e,PAM})$$

$$= 2P_{e,PAM} - P_{e,PAM}^2$$

$$\approx 2P_{e,PAM}$$

($\because$ At High SNR, $P_{e,PAM}$ is very small (10^{-4} or 10^{-5}). Hence, we can ignore the square of $P_{e,PAM}$)

$$= 2\times 2\left(1-\frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3E_s}{(M-1)N_0}}\right)$$

$$\boxed{P_{e,QAM} = 4\left(1-\frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3E_s}{(M-1)N_0}}\right)}$$