DETECTION FOR WIRELESS COMMUNICATION

Week 1 : Session 1

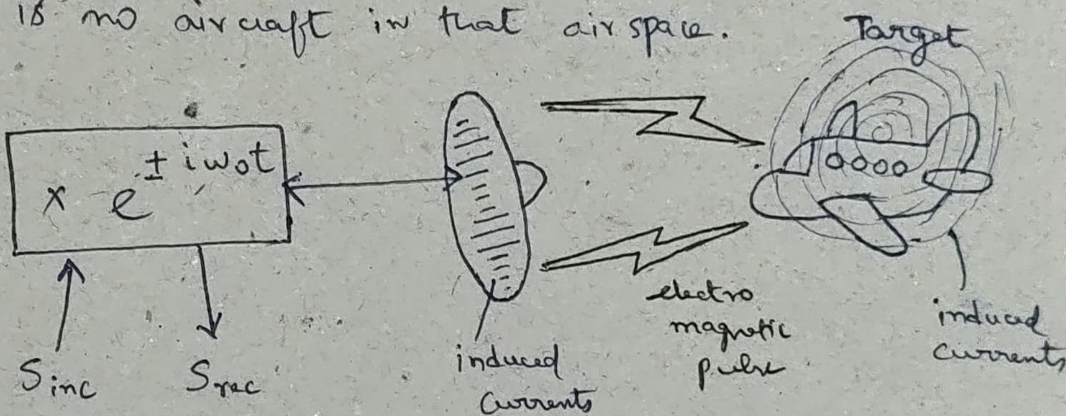
In this module, we'll look into various principles of detection theory for modern Wireless Communication Systems.

Detection :

To detect the presence/absence of a phenomenon (ie. an electrical signal). For instance,

Application 1 : RADAR

- To detect the presence/absence of an aircraft.
- RADAR transmits a pulse and detects the presence of an echo./ reflection
- If the echo is present, that means the aircraft is present. If the echo is absent, then there is no aircraft in that air space.



Application 2 : COMMUNICATION SYSTEMS

- To detect the transmitted symbol belonging to a digital constellation.
- $\underbrace{A+jA,\ A-jA,\ -A+jA,\ -A-jA,}_{\text{4 symbols belonging to QPSK constellation}}$
- $\log_2 4 = 2$ bits/symbol

No. of Symbols

(ie) 2 bits are mapped to a symbol at transmitter, the symbol is transmitted over the channel, At receiver, we detect which symbol has been transmitted and then map it back to the bits.

In communication system, a detector is otherwise called as "DECISION RULE". The decision rule maps the continuous signal 'y' back to one of the 4 QPSK constellation symbols, which in turn again mapped back to bits.

An "optimal Decision Rule" has to be developed, as the received signal would be noisy.

Application 3 : MACHINE LEARNING

- Recognize a face, from a given set of images
- Pattern recognition ML algorithm, maps the images with features such as Eyes, Nose, etc.,

Canonical Detection Problem :

Let us look at the rigorous theoretical underpinnings of the detection problem. (ie) mathematically how do we develope the decision rules? and how do we characterize the performance of the decision rules and improve them?

A simple detection problem can be modeled as "BINARY HYPOTHESIS $\underbrace{\text{TESTING PROBLEM}}$". (ie) There are two Hypothesis.

Testing which of them hypothesis is TRUE

1. NULL Hypothesis (H_0)
2. ALTERNATIVE Hypothesis (H_1)

"choosing / deciding in favor of one of these Hypothesis is known as Binary Hypothesis Testing Problem".

Mathematical Model

1. H_0 → NULL Hypothesis.
   - Signal of interest is ABSENT
   - Only Noise is PRESENT
   - (ie) Output = Noise

$$\begin{aligned} y(1) &= v(1) \\ y(2) &= v(2) \\ &\vdots \\ y(N) &= v(N) \end{aligned}$$

observed samples

IID Gaussian Noise samples with Mean $= 0$, Variance $= \sigma^2$ (ie) $\mathcal{N}(0, \sigma^2)$

In vector form,

$$\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}_{N\times 1} = \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}_{N\times 1}$$

$$\Rightarrow \quad \bar{y} = \bar{v}$$

$\bar{v} \longrightarrow$ $N \times 1$ noise vector

$\bar{y} \longrightarrow$ $N \times 1$ observation vector

$$\Rightarrow \quad \bar{y} = 0 + \bar{v}$$

$0 \longrightarrow$ NULL signal

Since the noise vector contains iid Gaussian noise samples (mean $= 0$, variance $= \sigma^2$), the noise covariance matrix is given by

$$\boxed{E\{\bar{v}\,\bar{v}^T\} = \sigma^2 I}$$

?

$$E\{\bar{v}\,\bar{v}^T\} = E\left\{ \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}_{N\times 1} \begin{bmatrix} v(1) & v(2) & \cdots & v(N) \end{bmatrix}_{1\times N} \right\}$$

$$= E\left\{ \begin{bmatrix} v^2(1) & v(1)v(2) & \cdots \\ v(2)v(1) & v^2(2) & \\ \vdots & & \ddots \end{bmatrix}_{N\times N} \right\}$$

$$E\{v(1)v(2)\} = E\{v(1)\}\,E\{v(2)\} = 0 \cdot 0 = 0$$

$$= \begin{bmatrix} \sigma^2 & 0 & \cdots \\ 0 & \sigma^2 & \\ \vdots & & \ddots \end{bmatrix}_{N\times N}$$

- Diagonal elements will be σ^2 coz Noise variance is σ^2.
- Off-diagonal elements will be 0 coz the noise samples are iid.

$$= \sigma^2 I$$

$N \times N$ identity matrix.

Note :

* Covariance Matrix proportional to I → White Noise
* Covariance Matrix not proportional to I → Colored Noise

② H_1 → ALTERNATIVE Hypothesis.

⊙ Signal of interest is PRESENT

⊙ (ie) Output = Signal + Noise

$$\begin{aligned} y(1) &= s(1) + v(1) \\ y(2) &= s(2) + v(2) \\ &\vdots \\ y(N) &= s(N) + v(N) \end{aligned}$$

observed samples

signal samples

← IID Gaussian Noise samples with Mean = 0, variance = σ^2

(ie) $\mathcal{N}(0, \sigma^2)$

In vector form,

$$\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}_{N\times1} = \begin{bmatrix} s(1) \\ s(2) \\ \vdots \\ s(N) \end{bmatrix}_{N\times1} + \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}_{N\times1}$$

CONSTANT KNOWN SIGNAL

?

$$\Rightarrow \quad \bar{y} = \bar{s} + \bar{v}$$

$\bar{v}$ → $N\times1$ Noise vector

$\bar{s}$ → $N\times1$ signal vector

$\bar{y}$ → $N\times1$ observation vector

Since, $\bar{v}$ is iid Gaussian noise vector,

$$E\{\bar{v}\} = 0$$

$$E\{\bar{v}\bar{v}^T\} = \sigma^2 I$$

For instance, in RADAR, we are trying to detect the reflected signal (i.e. attenuated version of the transmitted signal) which means, the signal is actually known and we are trying to detect is essentially the SIGNAL - is Present or Absent.

Similarly, in communication System, we know that the symbols are modulated on a carrier (say a sinusoid). So we know that the signal is a sinusoid and we are trying to detect if that sinusoid is present or absent.

In the upcoming chapter, we'll study more challenging scenarios where the signal of interest is RANDOM (i.e.) UNKNOWN.

Writing in compact form,

H_0 :

$$\bar{y} = \bar{v}$$

$$\bar{y} \sim \mathcal{N}(0, \sigma^2 I)$$

H_1 :

$$\bar{y} = \bar{s} + \bar{v} \qquad (\bar{y} \text{ is shifted by } \bar{s})$$

$$\bar{y} \sim \mathcal{N}(\bar{s}, \sigma^2 I)$$

Mean, $E\{\bar{y}\} = E\{\bar{s} + \bar{v}\}$

$$= \bar{s} + \cancel{E\{\bar{v}\}} \rightarrow 0$$

$$= \bar{s}$$

N-dimensional / Multidimensional Gaussian

Hypothesis Testing : Given $\bar{y}$, how to choose between H_0 and H_1 ?

Recall, Gaussian Random vector $\bar{v}$ contains iid samples $(v(1), v(2), \ldots, v(N))$ with Mean $=0$ and Variance $=\sigma^2$.

$$E\{v(i)\} = 0$$

$$E\{v^2(i)\} = \sigma^2$$

$$E\{v(i) \cdot v(j)\} = 0 \quad , \quad i \neq j.$$

What is the Joint PDF ?

Since the noise samples are independent, the Joint PDF will be the product of the Marginal PDFs, where each marginal PDF is Gaussian.

For instance, $v(i) \sim \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-v^2(i)}{2\sigma^2}}$

Joint PDF of noise samples, $f_{\bar{v}}(\bar{v})$ is

$$f_{\bar{v}}(\bar{v}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-v^2(1)}{2\sigma^2}} \times \ldots \times \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-v^2(N)}{2\sigma^2}}$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{N/2} e^{-\frac{\sum_{i=1}^{N} v^2(i)}{2\sigma^2}}$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{N/2} e^{-\frac{\|\bar{v}\|^2}{2\sigma^2}}$$

Multivariate Gaussian PDF
iid Gaussian RVs
with mean $=0$.
Variance $= \sigma^2 I$

where, $\|\bar{v}\|^2 = v_1^2 + v_2^2 + \ldots + v_N^2$.

Multidimensional PDF of the Gaussian noise vector.

Week 1 : Session 2

Likelihood of Hypothesis

The PDF represents the likelihood of Hypothesis. Larger the value of PDF, greater is the likelihood of the Hypothesis.

For instance, if we have two hypothesis (H_A and H_B), Under H_A, the outcomes has higher probability and at H_B, the outcomes has lower probability, then naturally there is higher likelihood that the underlying hypothesis is H_A.

So, this is essentially the intuition / thinking behind the likelihood of Hypothesis. (ie) the PDF of $\bar{y}$ represents the likelihood of the corresponding hypothesis.

So, the likelihood of Hypothesis answers the question, "How likely is the Hypothesis ?". The answer is basically somehow proportional to the PDF.

① NULL Hypothesis (H_0)

$$y(i) = v(i)$$

$v(i) \rightarrow \mathcal{N}(0, \sigma^2)$

$y(i) \rightarrow \mathcal{N}(0, \sigma^2)$

Likelihood of Null Hypothesis H_0

$$\boxed{p(\bar{y}; H_0) = \left(\frac{1}{2\pi\sigma^2}\right)^{N/2} e^{-\frac{\sum_{i=1}^{N} y^2(i)}{2\sigma^2}}}$$

② ALTERNATIVE Hypothesis (H_1)

$$y(i) = s(i) + v(i)$$

$$y(i) \sim \mathcal{N}(s(i), \sigma^2)$$

$y(i)$ is shifted by mean $s(i)$.

Likelihood of Alternative Hypothesis H_1

$$p(\bar{y}; H_1) = \left(\frac{1}{2\pi\sigma^2}\right)^{N/2} e^{-\frac{\sum_{i=1}^{N}(y(i)-s(i))^2}{2\sigma^2}}$$

Which Hypothesis to choose?

LOGIC: Choose Hypothesis with higher Likelihood!

(ie) Choose H_0 if

$$p(\bar{y}; H_0) \geq \tilde{\gamma} \times p(\bar{y}; H_1)$$

$$\Rightarrow \frac{p(\bar{y}; H_0)}{p(\bar{y}; H_1)} \geq \tilde{\gamma}$$

$\tilde{\gamma}$ → Arbitrary Threshold

$\frac{p(\bar{y}; H_0)}{p(\bar{y}; H_1)}$ → Likelihood Ratio

And hence the name Likelihood Ratio Test (LRT).

(ie) we choose H_0 if the ratio of "Likelihood of H_0" to "Likelihood of H_1" exceeds the arbitrary threshold $\tilde{\gamma}$.

Special case:

If $\tilde{\gamma} = 1$, then it becomes Maximum Likelihood (ML) detection / Maximum Likelihood (ML) decision rule.

(ie) choose H_0 if

$$p(\bar{y}; H_0) \geq p(\bar{y}; H_1)$$

$$\Rightarrow \frac{p(\bar{y}; H_0)}{p(\bar{y}; H_1)} \geq 1$$

Simplify LRT.

(ii) Choose H_0 if

$$\frac{p(\bar{y};H_0)}{p(\bar{y};H_1)} \geq \tilde{\gamma}$$

$$\Rightarrow \frac{\cancel{\left(\frac{1}{2\pi\sigma^2}\right)^{N/2}} e^{-\frac{\sum_{i=1}^{N} y^2(i)}{2\sigma^2}}}{\cancel{\left(\frac{1}{2\pi\sigma^2}\right)^{N/2}} e^{-\frac{\sum_{i=1}^{N}(y(i)-s(i))^2}{2\sigma^2}}} \geq \tilde{\gamma}$$

$$\boxed{\begin{array}{l}\log e^{x} = x \\ \log(A/B) = \log A - \log B\end{array}}$$

Taking $\ln$ on both sides

$$\Rightarrow \frac{-\sum_{i=1}^{N} y^2(i)}{2\sigma^2} - \frac{-\sum_{i=1}^{N}(y(i)-s(i))^2}{2\sigma^2} \geq \ln\tilde{\gamma}$$

$$\Rightarrow \sum_{i=1}^{N}(y(i)-s(i))^2 - \sum_{i=1}^{N} y^2(i) \geq 2\sigma^2 . \ln\tilde{\gamma}$$

$$\Rightarrow \sum_{i=1}^{N}\left(\cancel{y^2(i)} + s^2(i) - 2y(i)s(i)\right) - \cancel{\sum_{i=1}^{N} y^2(i)} \geq 2\sigma^2 . \ln\tilde{\gamma}$$

$$\Rightarrow \sum_{i=1}^{N} y(i)s(i) \leq \frac{\sum_{i=1}^{N} s^2(i) - 2\sigma^2\ln\tilde{\gamma}}{2}$$

$$\Rightarrow \sum_{i=1}^{N} y(i)s(i) \leq \frac{\|\bar{s}\|^2 - 2\sigma^2\ln\tilde{\gamma}}{2}$$

$$\Rightarrow \sum_{i=1}^{N} y(i)s(i) \leq \gamma$$

where, $\boxed{\gamma = f(\tilde{\gamma}) = \frac{\|\bar{s}\|^2 - 2\sigma^2\ln\tilde{\gamma}}{2}}$

In vector form,

$$\Rightarrow \bar{s}^T\bar{y} \leq \gamma$$

where, $\bar{s} = \begin{bmatrix} s(1) \\ \vdots \\ s(N) \end{bmatrix}$, $\bar{y} = \begin{bmatrix} y(1) \\ \vdots \\ y(N) \end{bmatrix}$

This Inner Product is called as "MATCHED FILTER".

otherwise, Choose H_1 if,

$$\sum_{i=1}^{N} y(i)\, s(i) > \gamma$$

$$\Rightarrow \bar{s}^T \bar{y} > \gamma$$

Test Statistic

$$\boxed{\begin{array}{l} \text{The final LRT is given as} \\ \odot \text{ Choose } H_0 \text{ if } \bar{s}^T\bar{y} \le \gamma \\ \odot \text{ Choose } H_1 \text{ if } \bar{s}^T\bar{y} > \gamma \end{array}}$$

Threshold

(ii) $\bar{s}^T \bar{y} = \sum_{i=1}^{N} y(i)\, s(i)$ → Test statistic

→ Maximizes SNR in White Gaussian Noise

→ Matched Filter.

Recall, in communication system, for AWGN detection, the matched filter is optimum, because it maximizes the SNR.

(ii) At the receiver, first we have a Matched Filter, which is matched to the pulse shape of the transmitter.

Week 1 : Session 3.

ML Detector

① Maximum Likelihood is a special case of LRT (i.e) by setting $\tilde{\gamma} = 1$.

We have,

$$\gamma = \frac{\|\bar{s}\|^2 - 2\sigma^2 \ln \tilde{\gamma}}{2}$$

$$= \frac{\|\bar{s}\|^2 - 2\sigma^2 (0)}{2}$$

$$\gamma = \frac{\|\bar{s}\|^2}{2}$$

Performance of the detector (LRT)

Just because of the fact that we have come up with a decision rule doesn't mean that it is always going to give us the correct decision. Some of the decisions may be incorrect.

For instance, the signal might be present. But the decision might be that the signal is Absent. So we need to characterise the rate / frequency / probability of which the decisions are accurate. Also the probability with which the decisions are in error

Characterizing the performance $\Rightarrow$ Probability of incorrect decision $\Rightarrow$ Probability of error

Let's discuss two things to characterize the performance.

① Probability of false alarm : P_{FA}

② Probability of detection : P_D

① Probability of False alarm (P_{FA}). — Performance (LRT)

Probability that the test falsely detects the presence of signal under Ho.

FALSE ALARM — SIGNAL ABSENT, but decision is SIGNAL PRESENT

This is happening because fundamentally the signal is embedded in noise. (ie) what we observe at the receiver is not a clean signal. Hence, depending on the nature / level of noise, the decision can be swayed either way.

So, if there is no noise, then naturally the decision would have been 100% accurate. But the problem is, In any practical system, we have noise and because of the noise, the decisions are going to be in error with a certain probability, that is essentially what we need to find. (ie) P_{FA}.

What is P_{FA} ?

Given underlying Hypothesis Ho, what is the probability that the decision is H_1.

$$\boxed{\text{Lower is the } P_{FA}\text{, Better is the Test.!}}$$

ideally P_{FA} should be very Low ≈ 0

When does FA occur ?

FALSE ALARM occur when, Under Ho (SIGNAL ABSENT)

(ie) $y(i) = v(i)$, $\boxed{\sum_{i=1}^{N} y(i)\, s(i)} > \gamma$

Test Statistic.

$$\Rightarrow \bar{s}^T \bar{y} > \gamma$$

$$\Rightarrow \sum_{i=1}^{N} v(i) \cdot s(i) > \gamma$$

$$\Rightarrow \bar{s}^T \bar{v} > \gamma$$

Probability of this, would be P_{FA}

Note: Under Ho (NULL Hypothesis / NO signal), Output = Noise.

$$y(i) = v(i) \sim \mathcal{N}(0, \sigma^2).$$

WKT, Test statistic

$$\sum_{i=1}^{N} y(i)\, s(i) \qquad \text{where } y(i) \text{ are Gaussian!}$$

$$\Rightarrow \underbrace{y(1)\,s(1) + y(2)\,s(2) + \dots + y(N)\,s(N)}_{\text{Linear Combination of Gaussian RV's.}}$$

Therefore, Test statistic is GAUSSIAN..

What is the Mean and Variance of Test statistic ?

Mean :

$$E\left\{\sum_{i=1}^{N} y(i)\,s(i)\right\} = E\left\{\sum_{i=1}^{N} v(i)\,s(i)\right\} \qquad \because \text{Under } H_0,\ y(i) = v(i)$$

$$= \sum_{i=1}^{N} s(i)\, \cancel{E\{v(i)\}}^{\,0}$$

$$= 0$$

Variance :

$E\{(X-\mu)^2\}$

$$E\left\{\left(\sum_{i=1}^{N} y(i)\,s(i) - 0\right)^2\right\} = E\left\{\left(\sum_{i=1}^{N} v(i)\,s(i)\right)^2\right\}$$

$$= E\left\{\sum_{i=1}^{N} s^2(i)\,v^2(i) + \sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N} v(i)\,v(j)\,s(i)\,s(j)\right\}$$

$$E\{v(i)\,v(j)\} = E\{v(i)\} \cdot E\{v(j)\} = 0 \cdot 0 = 0$$

$$= \sum_{i=1}^{N} E\{v^2(i)\}\, s^2(i) + \sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N} \cancel{E\{v(i)\,v(j)\}}^{\,0}\, s(i)\,s(j)$$

$$= \sum_{i=1}^{N} \sigma^2\, s^2(i)$$

$$= \sigma^2 \sum_{i=1}^{N} s^2(i)$$

$$= \sigma^2 \|s\|^2$$

Standard deviation :

$$SD = \sqrt{\text{variance}} = \sqrt{\sigma^2 \|\bar{s}\|^2} = \sigma \|\bar{s}\|$$

Therefore, Under H_0, the Test statistic is Gaussian with mean $= 0$ and variance $= \sigma^2 \|\bar{s}\|^2$

(ie) $\sum_{i=1}^{N} y(i)\, s(i) \sim \mathcal{N}(0, \sigma^2 \|\bar{s}\|^2)$

Therefore, Probability of False Alarm,

$$P_{FA} = P_r\left(\sum_{i=1}^{N} y(i)\, s(i) > \gamma \right)$$

$$= \underbrace{P_r\left(\mathcal{N}(0, \sigma^2 \|\bar{s}\|^2) > \gamma \right)}$$

What is the probability that the Gaussian RV with mean $= 0$, variance $= \sigma^2 \|\bar{s}\|^2$ is $> \gamma$?

$$= \underbrace{P_r\left(\frac{\mathcal{N}(0, \sigma^2 \|\bar{s}\|^2) - 0}{\sigma \|\bar{s}\|} > \frac{\gamma - 0}{\sigma \|\bar{s}\|} \right)}$$

Dividing both sides by SD and Subtract with Mean

Property

In a Gaussian RV, if we
- Subtract it by the Mean and
- Divide it by the SD

Then it becomes : Standard Gaussian RV with Mean $= 0$, Variance $= 1$

(ie) $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1)$

$$= P_r\left(\mathcal{N}(0,1) > \frac{\gamma}{\sigma \|\bar{s}\|} \right)$$

$$= Q\left(\frac{\gamma}{\sigma \|\bar{s}\|} \right) = P_{FA}.$$

What is $Q(\cdot)$?

- This is the Gaussian Q function.
- $Q(\cdot) \rightarrow$ Complementary Cumulative Distribution function (CCDF) of the Standard Gaussian RV (Mean $= 0$, Variance $= 1$).

PDF of Standard Gaussian RV

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad ; \mu = 0, \sigma^2 = 1$$

$$\text{CDF} = P_r(X \leq x)$$

$$\text{CCDF} = P_r(X \geq x) = 1 - \text{CDF} = 1 - P_r(X \leq x).$$

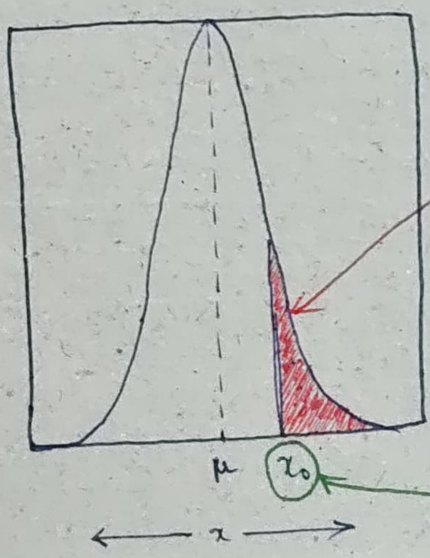
$Q(x)$ is defined as the CCDF of the Standard Gaussian RV ($\mu = 0, \sigma^2 = 1$).

$$\text{(i.e.) } Q(x) = P(X \geq x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \, dt$$

Therefore,

$$Q\left(\frac{\gamma}{\sigma \|\bar{a}\|}\right) = \int_{\frac{\gamma}{\|\bar{a}\|}}^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt$$

$$= P_r\left(\mathcal{N}(0,1) \geq \frac{\gamma}{\|\bar{a}\|}\right)$$



Area under PDF of Standard Gaussian RV.
(aka) Tail Probability of Standard Gaussian RV.
(aka) Gaussian Q function

$\frac{\gamma}{\|\bar{a}\|}$

Week 2 : Session 1

② **Probability of Detection (P_D) - Performance (LRT)**

Probability that the test correctly detects the presence of signal under H1.

For instance, in the Blood test report from the Lab, the disease is correctly diagnozed. Similarly, in RADAR, it correctly detects the presence of Aircraft.

DETECTION - Signal is Present, And the decision is also SIGNAL is Present.

What is P_D ?

Given underlying Hypothesis H1, what is the probability that the decision is H1.

When does detection occur?

DETECTION occurs when, under H_1 (SIGNAL PRESENT)

(ie) $y(i) = s(i) + v(i)$,

$$\boxed{\sum_{i=1}^{N} y(i)\, s(i)} > \gamma$$

Test statistic

$$\Rightarrow \bar{s}^T \bar{y} > \gamma$$

$$\Rightarrow \sum_{i=1}^{N} \left(s(i) + v(i) \right) \cdot s(i) > \gamma$$

$$\Rightarrow \underbrace{\bar{s}^T (\bar{s} + \bar{v}) > \gamma}$$

Probability of this, would be P_D.

Higher is the P_D, Better is the Test !

Ideally P_D should be very High ≈ 1

Note: Under H_1 (ALTERNATIVE Hypothesis / SIGNAL PRESENT),

output = Constant known Signal + Noise.

$$y(i) = s(i) + v(i)$$

where, $v(i) \sim \mathcal{N}(0, \sigma^2)$

Since, $s(i)$ shifts the mean, $y(i) \sim \mathcal{N}(s(i), \sigma^2)$

Thus, Both $v(i)$ and $y(i)$ are Gaussian.

Mean:

$$E\left\{\sum_{i=1}^{N} y(i)\, s(i)\right\} = E\left\{\sum_{i=1}^{N} \left(s(i) + v(i)\right) s(i)\right\}$$

$$= \sum_{i=1}^{N} s^2(i) + s(i)\, \cancel{E\{v(i)\}}^{\to 0}$$

$$= \sum_{i=1}^{N} s^2(i) + 0$$

$$= \|\bar{s}\|^2$$

Variance:

$E\left\{(x-\mu)^2\right\}$

$$E\left[\left(\sum_{i=1}^{N} y(i)\, s(i) - \sum_{i=1}^{N} s^2(i)\right)^2\right] = E\left\{\left(\sum_{i=1}^{N} s(i)\left[y(i) - s(i)\right]\right)^2\right\}$$

$$= E\left\{\left(\sum_{i=1}^{N} s(i)\left[\cancel{s(i)} + v(i) - \cancel{s(i)}\right]\right)^2\right\}$$

$$= E\left\{\left(\sum_{i=1}^{N} s(i)\, v(i)\right)^2\right\}$$

$$= \sum_{i=1}^{N} s^2(i)\, E\left\{v^2(i)\right\}$$

$$= \|\bar{s}\|^2 \sigma^2$$

$$= \sigma^2 \|\bar{s}\|^2$$

Standard deviation :

$$SD = \sqrt{\text{variance}} = \sqrt{\sigma^2 \|\bar{s}\|^2} = \sigma \|\bar{s}\|$$

$$\boxed{\begin{array}{l}\text{Therefore, under } H_1 \text{, the Test statistic is} \\ \text{Gaussian with mean} = \|\bar{s}\|^2 \text{ and variance} = \sigma^2 \|\bar{s}\|^2. \\ \text{(ii) } \sum_{i=1}^{N} y(i)\, s(i) \sim \mathcal{N}\left(\|\bar{s}\|^2, \sigma^2 \|\bar{s}\|^2\right)\end{array}}$$

Therefore, Probability of Detection,

$$P_D = P_r\left(\sum_{i=1}^{N} y(i)\, s(i) > \gamma\right)$$

$$= P_r\left(\mathcal{N}\left(\|\bar{s}\|^2, \sigma^2\|\bar{s}\|^2\right) > \gamma\right)$$

$$= \underbrace{P_r\left(\frac{\mathcal{N}\left(\|\bar{s}\|^2, \sigma^2\|\bar{s}\|^2\right) - \|\bar{s}\|^2}{\sigma\|\bar{s}\|} > \frac{\gamma - \|\bar{s}\|^2}{\sigma\|\bar{s}\|}\right)}_{\text{Divide both sides by SD, And Subtract with Mean}}$$

Standard Gaussian RV.

$$= P_r\left(\mathcal{N}(0,1) > \frac{\gamma - \|\bar{s}\|^2}{\sigma\|\bar{s}\|}\right)$$

$$= Q\left(\frac{\gamma - \|\bar{s}\|^2}{\sigma\|\bar{s}\|}\right)$$

$\because Q(\cdot)$ is the CCDF of Standard Gaussian RV. $(\mu = 0, \sigma^2 = 1)$

$$= P_D.$$

$P_r(\text{decision} = H_1 \mid H_0)$

$P_r(\text{decision} = H_1 \mid H_1)$

$$\boxed{\begin{array}{l}\text{Therefore, } P_{FA} \text{ and } P_D \text{ are given as} \\ \left.\begin{array}{l} P_{FA} = Q\left(\dfrac{\gamma}{\sigma\|\bar{s}\|}\right) \\ P_D = Q\left(\dfrac{\gamma - \|\bar{s}\|^2}{\sigma\|\bar{s}\|}\right)\end{array}\right\} \begin{array}{l} P_{FA} \text{ and } P_D \\ \text{together characterize} \\ \text{the performance} \\ \text{of the Test !}\end{array}\end{array}}$$

What is the role of γ?

Closer Examination:

$$H_1 : \bar{y}^T\bar{s} \sim \mathcal{N}\left(\|\bar{s}\|^2, \sigma^2\|\bar{s}\|^2\right)$$

$$H_0 : \bar{y}^T\bar{s} \sim \mathcal{N}\left(0, \sigma^2\|\bar{s}\|^2\right)$$

Observe that, both the cases are Gaussian with same variances $\sigma^2\|\bar{s}\|^2$, and the mean is different (mean of one is shifted w.r.t other)

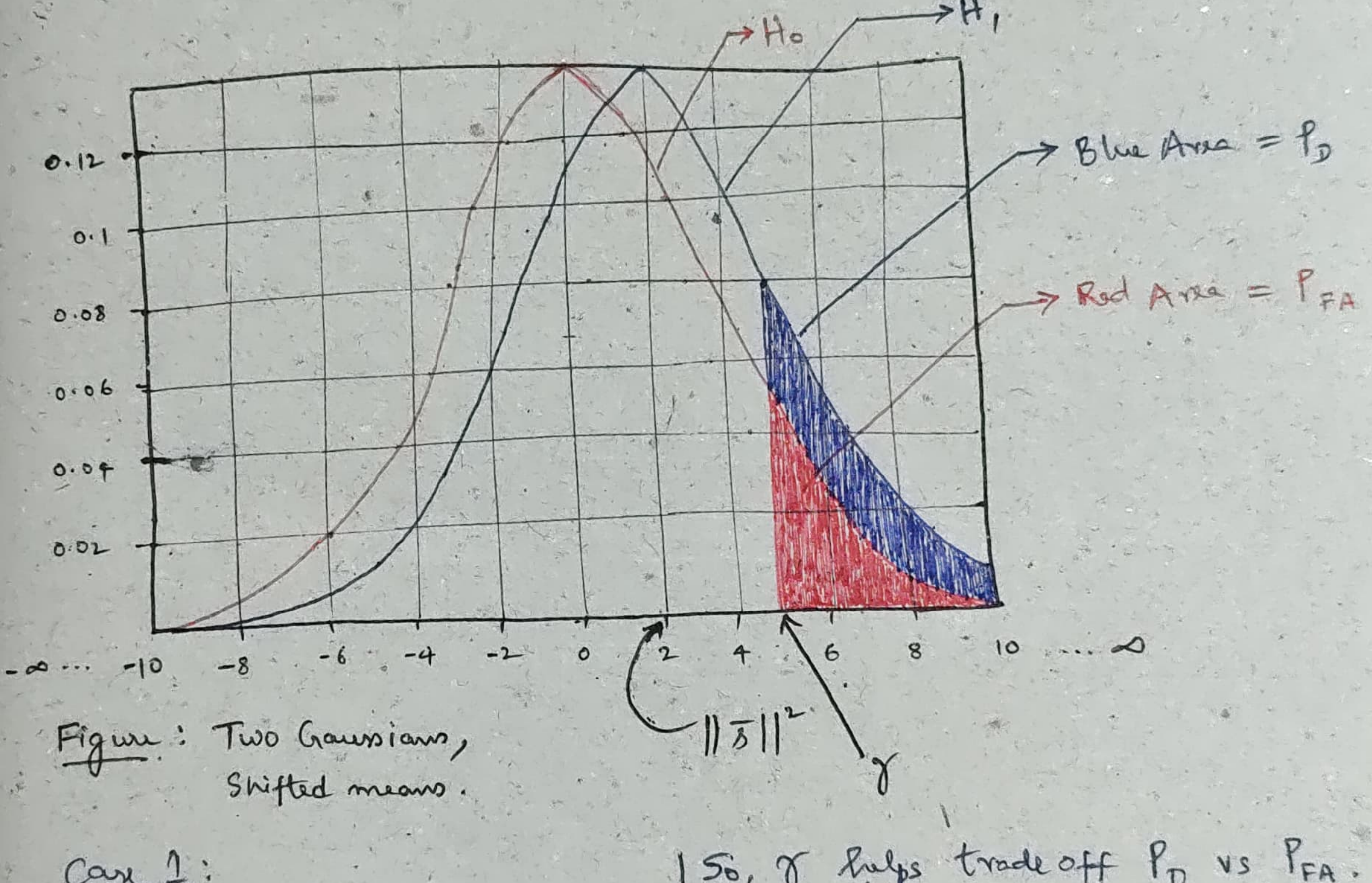


Figure: Two Gaussians, Shifted means.

Case 1:

As $\gamma \to \infty$,

$P_D = 0$ (P_D Very Poor!)

$P_{FA} = 0$ (P_{FA} Very Good!)

Case 2:

As $\gamma \to -\infty$,

$P_D = 1$ (P_D Very Good!)

$P_{FA} = 1$ (P_{FA} Very Poor!)

So, γ helps trade off P_D vs P_{FA}. (i) γ helps to control the performance of the test, which is not possible in ML rule coz $\gamma = 1$ (fixed) in ML rule. (ii) P_D and P_{FA} are fixed in ML rule.

The advantage offered by LRT over ML is that, LRT has the parameter γ, which can be tuned to achieve desirable P_D and P_{FA}.

Sometimes, we might want very high value of P_D by compromizing P_{FA}. In this case, γ has to be pushed to the left. Alternatively, sometimes we might want very less value of P_{FA} by compromizing P_D. In this case, γ has to be pushed to the right. Hence γ helps trade off P_D vs P_{FA}, which is not possible in ML detector.

$$\gamma \text{ high} \Rightarrow P_D \text{ low} \ \& \ P_{FA} \text{ low}$$

$$\gamma \text{ low} \Rightarrow P_D \text{ high} \ \& \ P_{FA} \text{ high}$$

Which γ to choose? That depends on what are the values of P_D and P_{FA} we want.

Receiver Operating characteristic (ROC)

ROC is P_D as a function of P_{FA}.

We have,
$$P_{FA} = Q\left(\frac{\gamma}{\sigma \|\bar{s}\|}\right)$$

$$\Rightarrow \gamma = \sigma \|\bar{s}\| \, Q^{-1}(P_{FA})$$

Substitute γ in P_D.

we have,
$$P_D = Q\left(\frac{\gamma - \|\bar{s}\|^2}{\sigma \|\bar{s}\|}\right)$$

$$= Q\left(\frac{\sigma \|\bar{s}\| \, Q^{-1}(P_{FA}) - \|\bar{s}\|^2}{\sigma \|\bar{s}\|}\right)$$

$$= Q\left(Q^{-1}(P_{FA}) - \frac{\|\bar{s}\|}{\sigma}\right)$$

$$= Q\left(Q^{-1}(P_{FA}) - \sqrt{\frac{\|\bar{s}\|^2}{\sigma^2}}\right)$$

ROC $\rightarrow$

$$\boxed{P_D = Q\left(Q^{-1}(P_{FA}) - \sqrt{SNR}\right) \quad \text{where } SNR = \frac{\|\bar{s}\|^2}{\sigma^2}}$$

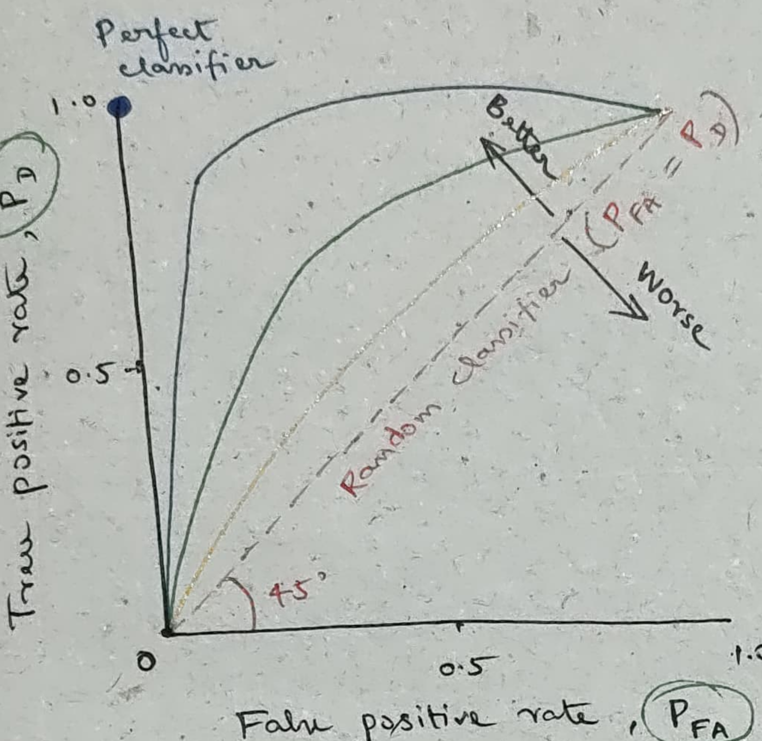


Figure : ROC Curve

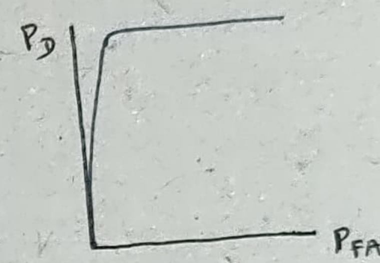


Figure : Ideal ROC

- ROC will always lie above the 45° line
- For given P_{FA}, we want maximum P_D. (ie) Ideally we want $P_{FA} = 0$ and $P_D = 1$. So, the moment P_{FA} is shifted to right, P_D has to shoot up to 1. This is called Ideal ROC.

Performance of the detector (ML) Week 2 : Session 2

WKT, For ML detection, $\tilde{\gamma} = 1$.

Recall, ML simply chooses the Hypothesis which has maximum likelihood.

(i) choose H_0 if $p(\bar{y}; H_0) \geq p(\bar{y}; H_1)$

$$\Rightarrow \frac{p(\bar{y}; H_0)}{p(\bar{y}; H_1)} \geq 1. \quad \leftarrow \tilde{\gamma} = 1$$

We have,

$$\gamma = \frac{\|\bar{s}\|^2 - 2\sigma^2 \ln \tilde{\gamma}}{2}$$

$$= \frac{\|\bar{s}\|^2}{2}$$

(ii) if we set $\gamma = \frac{\|\bar{s}\|^2}{2}$, then LRT reduces to ML decision rule.

Therefore, for ML, the P_{FA} and P_D are given as

$$P_{FA} = Q\left(\frac{\gamma}{\sigma\|\bar{s}\|}\right) = Q\left(\frac{\|\bar{s}\|^2/2}{\sigma\|\bar{s}\|}\right) = Q\left(\frac{\|\bar{s}\|}{2\sigma}\right)$$

$$P_D = Q\left(\frac{\gamma - \|\bar{s}\|^2}{\sigma\|\bar{s}\|}\right) = Q\left(\frac{\|\bar{s}\|^2/2 - \|\bar{s}\|^2}{\sigma\|\bar{s}\|}\right) = Q\left(\frac{-\|\bar{s}\|}{2\sigma}\right)$$

Probability of misdetection (P_{MD}) ?

$P_D \rightarrow$ Probability of detecting the signal under $\underbrace{\text{Hypothesis } H_1}_{\text{SIGNAL PRESENT}}$.

$P_{MD} \rightarrow$ Probability of missing detecting the signal under $\underbrace{\text{Hypothesis } H_1}_{\text{SIGNAL PRESENT}}$.

$\boxed{1 - Q(-x) = Q(x)}$

Therefore,

$$P_{MD} = 1 - P_D = 1 - Q\left(\frac{-\|\bar{s}\|}{2\sigma}\right) = Q\left(\frac{\|\bar{s}\|}{2\sigma}\right)$$

Here, both P_{FA} and P_{MD} are $Q\left(\frac{\|\bar{s}\|}{2\sigma}\right)$. which are naturally going to be Symmetric, coz when we go to Maximum Likelihood, everything is Symmetric. (i.e) Threshold is in the middle (i.e) $\frac{\|\bar{s}\|^2}{2}$.

In terms of SNR,

$$P_{FA} = P_{MD} = Q\left(\frac{\|\bar{s}\|}{2\sigma}\right) = Q\left(\sqrt{\frac{\|\bar{s}\|^2}{4\sigma^2}}\right) = Q\left(\frac{\sqrt{SNR}}{2}\right)$$

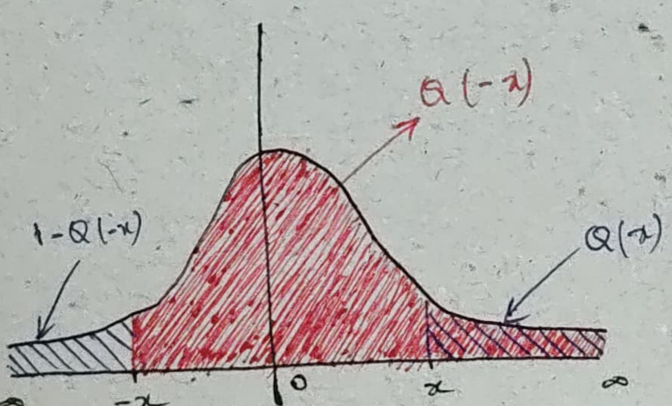
Let us consider a simple example to illustrate the performance of ML detector. Consider a scenario where

$$P_r(H_0) = P_r(H_1) = \frac{1}{2}$$

This is typically the case that we have many times in practice, which is called EQUIPROBABLE. (i.e) Both Hypothesis are Equiprobable.

For instance, when the constellation such as BPSK / ASK / FSK typically have both the symbols +1/-1 or 0/1 is equiprobable.

Symmetry property of Standard Normal Gaussian RV



$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-t^2/2}\, dt$$

$$Q(-x) = \frac{1}{\sqrt{2\pi}} \int_{-x}^{\infty} e^{-t^2/2}\, dt$$

$$\underbrace{1 - Q(-x)}_{P(X \leq -x)} = \underbrace{Q(x)}_{P(X \geq x)}$$

Therefore, the Probability of error (P_e) is given as

$$P_e = P_r(H_0) \cdot P_r(H_1 | H_0) + P_r(H_1) \cdot P_r(H_0 | H_1)$$

$$= P_r(H_0) \cdot P_{FA} + P_r(H_1) \cdot P_{MD}$$

$$= \frac{1}{2} P_{FA} + \frac{1}{2} P_{MD} = \frac{1}{2} Q\left(\frac{\|\bar{s}\|}{2\sigma}\right) + \frac{1}{2} Q\left(\frac{\|\bar{s}\|}{2\sigma}\right)$$

$$\boxed{P_e = Q\left(\frac{\|\bar{s}\|}{2\sigma}\right)}$$

Overall Probability of erroneous decision.

Example

Consider ASK system, where we have constellation $\{0, A\}$.

No. of symbols $= 2$;

No. of bits / symbol $= \log_2 2 = 1$;

Signal $= A$; No. of sample, $N = 1$;

Hypothesis,

$H_0 : y = v$

$H_1 : y = A + v$

Typically, in Comm. System $\sigma^2 = \frac{N_0}{2}$

Therefore, $P_e = Q\left(\frac{\|\bar{s}\|}{2\sigma}\right) = Q\left(\frac{A}{2\sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{A^2}{2N_0}}\right)$

- Each symbol carries one bit
- Let E_b be the energy per bit (i.e. Average Energy / bit)
- Let each symbol be equiprobable

(i.e.) $P_r(0) = P_r(A) = \frac{1}{2}$

$P_r(H_0)$ $P_r(H_1)$

$$E_b = \underbrace{P_r(H_0) \times 0}_{\text{Energy for } H_0} + \underbrace{P_r(H_1) \times A^2}_{\text{Energy for } H_1}$$

$$= \frac{1}{2} \times 0 + \frac{1}{2} \times A^2$$

$$= \frac{A^2}{2}$$

$\Rightarrow A^2 = 2 E_b$

$\Rightarrow A = \sqrt{2 E_b}$

Therefore, $P_e = Q\left(\sqrt{\frac{A^2}{2N_0}}\right) = Q\left(\sqrt{\frac{2 E_b}{2 N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$

This is termed as Bit Error Rate for ASK. (Probability that bit is in error).

For BPSK, $BER = Q\left(\sqrt{\frac{2 E_b}{N_0}}\right)$, which gives a 3 dB better performance for the same E_b.