Week 6 : Session 2

Detection of Random Signals

So far, we've considered the signals $\bar{s}$, $\bar{s}_0$, $\bar{s}_1$ as deterministic signals. Now, how to perform detection when the signals are Random (ie) $\bar{s}$ is an unknown signal, its statistical properties are known, then how to perform detection?

Consider a random signal detection problem,

- Under Hypothesis H_0, the signal is absent, only noise is present which are iid Gaussian noise samples with mean = 0, variance = σ^2

$$\begin{aligned} y(1) &= v(1) \\ y(2) &= v(2) \\ &\vdots \\ y(N) &= v(N) \end{aligned}$$

iid Gaussian noise samples

$$\Rightarrow \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} = \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}$$

(In vector form)

$$\Rightarrow \bar{y} = \bar{v}$$

where, $\bar{v}$ = $N \times 1$ noise vector

- Under Hypothesis H_1, the signal is present, which are Random in nature. Thus the Signals are iid Gaussian with mean = 0, variance = σ^2. And of course, the signals $s(i)$ are independent of noise $v(j)$. $E\{s(i) \cdot v(j)\} = 0 \ \forall \ i, j$.

$$\begin{aligned} y(1) &= s(1) + v(1) \\ y(2) &= s(2) + v(2) \\ &\vdots \\ y(N) &= s(N) + v(N) \end{aligned}$$

iid Gaussian signal samples

$$\Rightarrow \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} = \begin{bmatrix} s(1) \\ s(2) \\ \vdots \\ s(N) \end{bmatrix} + \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix} \quad \text{(In vector form)}$$

$$\Rightarrow \bar{y} = \bar{s} + \bar{v}$$

where, $\bar{s}$ = $N \times 1$ Gaussian Signal vector

$\bar{v}$ = $N \times 1$ Gaussian Noise vector

$E\{\bar{s}\} = 0 \longrightarrow$ Mean

$E\{\bar{s}\bar{s}^T\} = \sigma_s^2 I \longrightarrow$ Signal covariance Matrix

$E\{\bar{v}\} = 0 \longrightarrow$ Mean

$E\{\bar{v}\bar{v}^T\} = \sigma^2 I \longrightarrow$ Noise covariance Matrix

---

$s(i) \rightarrow$ iid Gaussian with mean $= 0$, Variance $= \sigma_s^2$

$v(i) \rightarrow$ iid Gaussian with mean $= 0$, Variance $= \sigma^2$

The Likelihoods are as follows :

- Under Hypothesis H_0, we have $y(i) = v(i)$ (i.e) Only Gaussian Noise present

The likelihood under NULL Hypothesis H_0 is

$$p(\bar{y}; H_0) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-y^2(1)}{2\sigma^2}} \times \cdots \times \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-y^2(N)}{2\sigma^2}}$$

$$\Rightarrow p(\bar{y}; H_0) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}} e^{\frac{-\|\bar{y}\|^2}{2\sigma^2}}$$

- Under Hypothesis H_1, we have $y(i) = s(i) + v(i)$.

(i.e) $y(i) \rightarrow$ Sum of Gaussians $s(i)$ and $v(i)$

$\Rightarrow y(i) \rightarrow$ Gaussian.

Thus, $E\{y(i)\} = E\{s(i)\} + E\{v(i)\} = 0 + 0 = 0$.

$$E\{y^2(i)\} = E\{(s(i) + v(i))^2\}$$

$$= E\{s^2(i)\} + E\{v^2(i)\} + \underbrace{2E\{\cancel{s(i) \cdot v(i)}\}}_{0}$$

$$= \sigma_s^2 + \sigma^2 + 0$$

Thus, $y(i) \rightarrow$ iid Gaussian with mean $= 0$, Variance $= \sigma^2 + \sigma_s^2$

$$y(i) \sim \mathcal{N}(0, \sigma_s^2 + \sigma^2)$$

The likelihood under ALTERNATIVE Hypothesis H_1 is

$$p(\bar{y}\,;H_1) = \frac{1}{\sqrt{2\pi(\sigma^2+\sigma_0^2)}}\, e^{\frac{-y^2(1)}{2(\sigma^2+\sigma_0^2)}} \times \cdots\cdots$$

$$\cdots\cdots \times \frac{1}{\sqrt{2\pi(\sigma^2+\sigma_0^2)}}\, e^{\frac{-y^2(N)}{2(\sigma^2+\sigma_0^2)}}$$

$$\Rightarrow p(\bar{y}\,;H_1) = \left(\frac{1}{2\pi(\sigma^2+\sigma_0^2)}\right)^{\frac{N}{2}} e^{\frac{-\|\bar{y}\|^2}{2(\sigma^2+\sigma_0^2)}}$$

Likelihood Ratio Test (LRT)

$\log AB = \log A + \log B$

$\log \frac{A}{B} = \log A - \log B$

$\log x^n = n \log x$

$\log(e^{-x}) = -x \log e = -x$

⊙ Choose H_0 if

$$\frac{p(\bar{y}\,;H_0)}{p(\bar{y}\,;H_1)} \geq \tilde{\gamma}$$

Threshold

$$\Rightarrow \frac{\left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}} e^{-\frac{1}{2}\frac{\|\bar{y}\|^2}{\sigma^2}}}{\left(\frac{1}{2\pi(\sigma^2+\sigma_0^2)}\right)^{\frac{N}{2}} e^{-\frac{1}{2}\frac{\|\bar{y}\|^2}{\sigma^2+\sigma_0^2}}} \geq \tilde{\gamma}$$

Take natural log on both sides.

$$\Rightarrow \ln\left(\left(\frac{1}{\sigma^2}\right)^{\frac{N}{2}} \cdot e^{-\frac{1}{2}\frac{\|\bar{y}\|^2}{\sigma^2}}\right) - \ln\left(\left(\frac{1}{\sigma^2+\sigma_0^2}\right)^{\frac{N}{2}} \cdot e^{-\frac{1}{2}\frac{\|\bar{y}\|^2}{\sigma^2+\sigma_0^2}}\right) \geq \ln\tilde{\gamma}$$

$$\Rightarrow \ln\left(\frac{1}{\sigma^2}\right)^{\frac{N}{2}} + \ln e^{-\frac{1}{2}\frac{\|\bar{y}\|^2}{\sigma^2}} - \ln\left(\frac{1}{\sigma^2+\sigma_0^2}\right)^{\frac{N}{2}} - \ln e^{-\frac{1}{2}\frac{\|\bar{y}\|^2}{\sigma^2+\sigma_0^2}} \geq \ln\tilde{\gamma}$$

$$\Rightarrow \frac{N}{2}\ln\left(\frac{1}{\sigma^2}\right) + \left(-\frac{1}{2}\frac{\|\bar{y}\|^2}{\sigma^2}\right) - \frac{N}{2}\ln\left(\frac{1}{\sigma^2+\sigma_0^2}\right) - \left(-\frac{1}{2}\frac{\|\bar{y}\|^2}{\sigma^2+\sigma_0^2}\right) \geq \ln\tilde{\gamma}$$

$$\Rightarrow \frac{N}{2}\left[\ln\left(\frac{1}{\sigma^2}\right) - \ln\left(\frac{1}{\sigma^2+\sigma_0^2}\right)\right] + \|\bar{y}\|^2\left(\frac{1}{2(\sigma^2+\sigma_0^2)} - \frac{1}{2\sigma^2}\right) \geq \ln\tilde{\gamma}$$

$$\Rightarrow \frac{N}{2}\ln\left(\frac{1/\sigma^2}{1/\sigma^2+\sigma_0^2}\right) + \|\bar{y}\|^2\left(\frac{1}{2(\sigma^2+\sigma_0^2)} - \frac{1}{2\sigma^2}\right) \geq \ln\tilde{\gamma}$$

$$\Rightarrow \frac{N}{2}\ln\left(\frac{\sigma^2+\sigma_0^2}{\sigma^2}\right) - \ln\tilde{\gamma} \geq \|\bar{y}\|^2\left(\frac{1}{2\sigma^2} - \frac{1}{2(\sigma^2+\sigma_0^2)}\right)$$

$$\Rightarrow \|\bar{y}\|^2 \leq \frac{\frac{N}{2}\ln\left(\frac{\sigma^2+\sigma_0^2}{\sigma^2}\right) - \ln\tilde{\gamma}}{\left(\frac{1}{2\sigma^2} - \frac{1}{2(\sigma^2+\sigma_0^2)}\right)} = \gamma$$

Thus, $\boxed{\begin{array}{ll} \text{Choose } H_0 \text{ if} & \|\bar{y}\|^2 \leq \gamma \\ \text{Choose } H_1 \text{ if} & \|\bar{y}\|^2 > \gamma \end{array}}$

where, $\|\bar{y}\|^2 = \underbrace{|y(1)|^2 + \dots + |y(N)|^2}_{\text{Energy of Signal}}$

As the Energy is being compared with the threshold, this is basically known as the **ENERGY DETECTOR**, which is very useful when the signal is unknown and random.

The deterministic signal detection / classification can be used when the signal is known, but when the signals are unknown, we use the Energy detector which is very simple / popular / low complexity / Efficient.

Unlike the Matched filter, which is a coherent detector (performs $\bar{s}^T\bar{y}$ or $\bar{s}^H\bar{y}$), the Energy detector is a non coherent detector (only concerned with the Energy, ignoring phase).

Coherent Detector $\longrightarrow$ Matching Both the Amplitude (Energy) and the Phase

Non-coherent Detector $\longrightarrow$ Only concerned with the Amplitude (Energy) and ignoring the Phase.

Q The P_{FA} is given as follows.

$$\Pr\left(\|\bar{y}\|^2 \geq \gamma \,;\, H_0\right)$$

$$\Rightarrow \Pr\left(|y(1)|^2 + \dots + |y(N)|^2 \geq \gamma \,;\, H_0\right)$$

Observe that, under H_0, each $y(i) = v(i) \sim \mathcal{N}(0, \sigma^2)$

$$\Rightarrow y^2(1) + \dots + y^2(N) \geq \gamma$$

Divide both sides by σ^2

$$\Rightarrow \frac{y^2(1)}{\sigma^2} + \dots + \frac{y^2(N)}{\sigma^2} \geq \frac{\gamma}{\sigma^2}$$

$$\Rightarrow \left(\frac{y(1)}{\sigma}\right)^2 + \dots + \left(\frac{y(N)}{\sigma}\right)^2 \geq \frac{\gamma}{\sigma^2}$$

$$\Rightarrow \left(\frac{v(1)}{\sigma}\right)^2 + \dots + \left(\frac{v(N)}{\sigma}\right)^2 \geq \frac{\gamma}{\sigma^2}$$

observe that, each $\frac{y(i)}{\sigma}$ will be a Standard Gaussian random variable of mean = 0, variance = 1.

$$\frac{y(i)}{\sigma} = \frac{v(i)}{\sigma} \sim \mathcal{N}(0,1)$$

This is because, recall, from Gaussian, when we subtract the mean and divided by standard deviation, we get Standard Gaussian Random variable with mean = 0, variance = 1.

Thus, $\left(\frac{y(1)}{\sigma}\right)^2 + \cdots\cdots + \left(\frac{y(N)}{\sigma}\right)^2 \geq \frac{\gamma}{\sigma^2}$

$$\underbrace{\left(\frac{v(1)}{\sigma}\right)^2 + \cdots\cdots + \left(\frac{v(N)}{\sigma}\right)^2}_{\text{Sum of squares of N standard Zero mean, Unit variance iid Gaussian Random variables.}} \geq \frac{\gamma}{\sigma^2}$$

This is what we called as Chi-Squared Random variable with N degrees of freedom $\left(\chi^2_N\right)$.

Infact, we can call this here as "Central Chi-squared RV" with N degrees of freedom coz each of the standard Gaussian RV has zero mean.

Suppose, if the Gaussian RV has non-zero mean, then it is called as "Non-Central chi-squared RV"

The PDF of the central chi-squared RV with N degrees of freedom is given as

$$f_x(x) = \frac{1}{2^{N/2}\,\Gamma\left(\frac{N}{2}\right)}\, x^{\frac{N}{2}-1}\, e^{-\frac{1}{2}x}, \quad x \geq 0.$$

where,

$$\Gamma\left(\frac{N}{2}\right) = \int_0^\infty t^{\frac{N}{2}-1}\, e^{-t}\, dt.$$

For a positive number m, $\Gamma m = (m-1)!$

Week 6 : Session 3

P_{FA} is given as (Under H_0, Probability decision = H_1)

$$P_{FA} = P_r\left(\underbrace{\left(\frac{v(1)}{\sigma}\right)^2 + \dots + \left(\frac{v(N)}{\sigma}\right)^2}_{\chi_N^2} > \frac{\gamma}{\sigma^2}\right)$$

$$\Rightarrow P_{FA} = Q_{\chi_N^2}\left(\frac{\gamma}{\sigma^2}\right)$$

Complementary Cumulative Distribution Function

where, $Q_{\chi_N^2}(\cdot) \rightarrow$ CCDF of the χ_N^2 RV with N degrees of freedom.

$$Q_{\chi_N^2}(x) = \int_x^\infty \frac{1}{2^{N/2}\Gamma(\frac{N}{2})} e^{-\frac{1}{2}t} \cdot t^{\frac{N}{2}-1} dt$$

Set $t = 2u \Rightarrow dt = 2\,du$.

$$Q_{\chi_N^2}(x) = \frac{1}{2^{N/2}\Gamma(\frac{N}{2})} \int_{x/2}^\infty e^{-u} (2u)^{\frac{N}{2}-1} 2\,du$$

$$= \frac{1}{\Gamma(\frac{N}{2})} \underbrace{\int_{x/2}^\infty e^{-u} \cdot u^{\frac{N}{2}-1} du}$$

This is known as Upper incomplete Gamma Function, $\Gamma(\frac{N}{2}, \frac{x}{2})$.

$$= \frac{\Gamma(\frac{N}{2}, \frac{x}{2})}{\Gamma(\frac{N}{2})}$$

Upper Incomplete Gamma function

$$\Gamma(m,x) = \int_x^\infty e^{-t} t^{m-1} dt$$

Lower Incomplete Gamma function

$$\Gamma(m,x) = \int_0^x e^{-t} t^{m-1} dt$$

$$\Rightarrow P_{FA} = Q_{\chi_N^2}\left(\frac{\gamma}{\sigma^2}\right)$$

$$\boxed{P_{FA} = \frac{\Gamma\left(\frac{N}{2}, \frac{\gamma}{2\sigma^2}\right)}{\Gamma\left(\frac{N}{2}\right)}}$$

⊙ The P_D is given as follows.

$$P_r\left(\|\bar{y}\|^2 \geq \gamma \,;\, H_1\right)$$

$$\Rightarrow P_r\left(|y(1)|^2 + \dots + |y(N)|^2 \geq \gamma \,;\, H_1\right)$$

Observe that, under H_1, each $y(i) = s(i) + v(i) \sim \mathcal{N}(0, \sigma^2 + \sigma_s^2)$

$$\Rightarrow y^2(1) + \dots + y^2(N) \geq \gamma$$

Divide both sides by $\underline{\underline{\sigma^2 + \sigma_s^2}}$

$$\Rightarrow \frac{y^2(1)}{\sigma^2+\sigma_s^2} + \dots + \frac{y^2(N)}{\sigma^2+\sigma_s^2} \geq \frac{\gamma}{\sigma^2+\sigma_s^2}$$

$$\Rightarrow \underbrace{\left(\frac{y(1)}{\sqrt{\sigma^2+\sigma_s^2}}\right)^2 + \dots + \left(\frac{y(N)}{\sqrt{\sigma^2+\sigma_s^2}}\right)^2}_{\substack{\text{Sum of squares of } N \text{ standard} \\ \text{zero mean, unit variance iid} \\ \text{Gaussian Random Variables.}}} \geq \frac{\gamma}{\sigma^2+\sigma_s^2}$$

$\Rightarrow \chi^2_N$

Observe that, each $\frac{y(i)}{\sqrt{\sigma^2+\sigma_s^2}}$ will be a standard Gaussian random variable of mean $= 0$, variance $= 1$.

$$\frac{y(i)}{\sqrt{\sigma^2+\sigma_s^2}} \sim \mathcal{N}(0,1).$$

P_D is given as (under H_1, Probability decision $= H_1$)

$$P_D = P_r\left(\underbrace{\left(\frac{y(1)}{\sqrt{\sigma^2+\sigma_s^2}}\right)^2 + \dots + \left(\frac{y(N)}{\sqrt{\sigma^2+\sigma_s^2}}\right)^2}_{\chi^2_N} \geq \frac{\gamma}{\sigma^2+\sigma_s^2}\right)$$

$$\Rightarrow P_D = Q_{\chi^2_N}\left(\frac{\gamma}{\sigma^2+\sigma_s^2}\right) = \frac{\Gamma\left(\frac{N}{2}, \frac{\gamma}{2(\sigma^2+\sigma_s^2)}\right)}{\Gamma\left(\frac{N}{2}\right)}$$

$$\boxed{P_D = \frac{\Gamma\left(\frac{N}{2}, \frac{\gamma}{2(\sigma^2+\sigma_s^2)}\right)}{\Gamma\left(\frac{N}{2}\right)}}$$

⊙ Receiver Operating Characteristic (ROC)

We have, $P_{FA} = Q_{\chi_N^2}\left(\frac{\gamma}{\sigma^2}\right)$; $P_D = Q_{\chi_N^2}\left(\frac{\gamma}{\sigma^2+\sigma_s^2}\right)$

ROC = ?

$$\gamma = \sigma^2 Q^{-1}_{\chi_N^2}(P_{FA})$$

$$\Rightarrow \boxed{P_D = Q_{\chi_N^2}\left(\frac{\sigma^2 Q^{-1}_{\chi_N^2}(P_{FA})}{\sigma^2+\sigma_s^2}\right)} \leftarrow \text{ROC}$$

Example : $N = 2$.

(i) PDF of the central Chi-squared RV with N degrees of freedom

$$f_x(x) = \frac{1}{2^{N/2}\,\Gamma\left(\frac{N}{2}\right)} x^{\frac{N}{2}-1} \cdot e^{-\frac{1}{2}x}$$

$$= \frac{1}{2^1\,\Gamma(1)} \cdot x^0 \cdot e^{-\frac{1}{2}x}$$

$$= \frac{1}{2} e^{-\frac{1}{2}x}$$ where $x \geq 0$

$\Gamma(m) = (m-1)!$

$\Gamma(1) = (1-1)!$

$= 0! = 1$

$\Gamma(1) = \int_0^\infty e^{-t}\,dt = 1$

(ii) CCDF of the central chi-squared RV with N degrees of freedom.

$$Q_{\chi_N^2}(x) = \int_x^\infty \frac{1}{2^{N/2}\,\Gamma\left(\frac{N}{2}\right)} e^{-\frac{1}{2}t} \cdot t^{\frac{N}{2}-1}\,dt$$

$$Q_{\chi_2^2}(x) = \int_x^\infty \frac{1}{2^1\,\Gamma(1)} e^{-\frac{1}{2}t} \cdot t^0\,dt$$

$$= \int_x^\infty \frac{1}{2} e^{-\frac{1}{2}t}\,dt$$

$$= -e^{-\frac{1}{2}t}\Big|_x^\infty$$

$$= e^{-\frac{1}{2}x}$$

(iii) P_{FA}, P_D, ROC

- $\underbrace{P_{FA} = Q_{\chi_2^2}\left(\frac{\gamma}{\sigma^2}\right) = e^{-\frac{\gamma}{2\sigma^2}}}$

$$\gamma = \underbrace{-2\sigma^2 \ln(P_{FA})}$$

- $P_D = Q_{\chi_2^2}\left(\frac{\gamma}{\sigma^2+\sigma_s^2}\right) = e^{-\frac{\gamma}{2(\sigma^2+\sigma_s^2)}}$

$$\Rightarrow P_D = e^{-\frac{1}{2(\sigma^2+\sigma_s^2)}\left(-2\sigma^2 \ln(P_{FA})\right)}$$

$$= e^{\frac{\sigma^2}{\sigma^2+\sigma_s^2}\cdot \ln(P_{FA})}$$

$$= e^{\ln(P_{FA})^{\frac{\sigma^2}{\sigma^2+\sigma_s^2}}}$$

$$\boxed{P_D = P_{FA}^{\frac{\sigma^2}{\sigma^2+\sigma_s^2}}} \rightarrow \text{Receiver Operating Characteristic (ROC)}$$

Week 7 : Session 1

Chi - Squared Approximation

Let us develop an approximation for a Chi-squared RV with N degrees of freedom. WKT, this can be written as

$$u = \underbrace{u_1^2 + u_2^2 + \dots + u_N^2}$$

where each u_i's is a Standard Gaussian with zero mean and unit variance (i.e) $u_i \sim \mathcal{N}(0,1)$.

Sum of the squares of N iid standard Normal RVs, which is called Central Chi-Squared RV.

u_i's are iid Standard Normal RVs.

u_i, u_i^2 → Both are iid RVs

u_i → Gaussian

u_i^2 → Not Gaussian

⊙ What happens when $N \to \infty$?

Whenever we take the Sum of N iid Random variables, we use the Central Limit theorem.

$$u = u_1^2 + u_2^2 + \dots + u_N^2$$

As $N \to \infty$, u becomes Gaussian because of the Central Limit theorem.

(ie) $u \sim \mathcal{N}$ (mean, variance).

$E\{u\}$ = Mean = ?

$E\{(u-\bar{u})^2\}$ = Variance = ?

$$E\{u\} = E\{u_1^2 + u_2^2 + \dots + u_N^2\}$$

$$= \sum_{i=1}^{N} E\{u_i^2\}$$

$u_i \sim \mathcal{N}(0,1)$

$\Rightarrow E\{u_i^2\} = 1$

$$= \sum_{i=1}^{N} (1)$$

$$= N$$ ← Mean of u

$$E\{(u-\bar{u})^2\} = E\{(u - E\{u\})^2\}$$

$$= E\{u^2\} - (E\{u\})^2$$

$$= E\{u^2\} - N^2$$

$$E\{u^2\} = E\left\{\left(\sum_{i=1}^{N} u_i^2\right)^2\right\}$$

$$= E\left\{\sum_{i=1}^{N} u_i^4 + 2\sum_{i=1}^{N}\sum_{j=1}^{i-1} u_i^2 u_j^2\right\}$$

$$= \sum_{i=1}^{N} E\{u_i^4\} + 2\sum_{i=1}^{N}\sum_{j=1}^{i-1} E\{u_i^2 u_j^2\}$$

For a Gaussian RV,

$E\{u_i^4\} = 3\sigma^4 = 3$

Since u_i, u_j are independent, $i \neq j$

$E\{u_i^2 u_j^2\} = E\{u_i^2\} \cdot E\{u_j^2\}$

$= 1 \times 1 = 1$

$$= \sum_{i=1}^{N} (3) + 2 \sum_{i=1}^{N}\sum_{j=1}^{i-1} (1)$$

Total No. of combinations ($i \neq j$)

$= {}^N C_2 = \frac{N(N-1)}{2}$

$$= 3N + \cancel{2} \cdot \frac{N(N-1)}{\cancel{2}}$$

$$= 3N + N^2 - N$$

$$= N^2 + 2N.$$

$$\therefore E\{(u-\bar{u})^2\} = N^2 + 2N - N^2$$

$$= 2N$$ ← Variance of u

Therefore, as $N \to \infty$, the chi-squared RV with N degrees of freedom approaches Normal distribution with mean $= N$, variance $= 2N$.

$$\boxed{\chi^2_N \sim \mathcal{N}(N, 2N) \quad \text{as } N \to \infty}$$

⊙ When $N \to \infty$, P_{FA} is given as

$$P_{FA} = Q_{\chi^2_N}\left(\frac{\tilde{\gamma}}{\sigma^2}\right) = P_r\left(\chi^2_N \geq \frac{\tilde{\gamma}}{\sigma^2}\right)$$

$$= P_r\left(\mathcal{N}(N, 2N) \geq \frac{\tilde{\gamma}}{\sigma^2}\right)$$

$$\boxed{P_{FA} = Q\left(\frac{\frac{\tilde{\gamma}}{\sigma^2} - N}{\sqrt{2N}}\right)}$$

⊙ When $N \to \infty$, P_D is given as

$$P_D = Q_{\chi^2_N}\left(\frac{\tilde{\gamma}}{\sigma^2 + \sigma_0^2}\right) = P_r\left(\chi^2_N \geq \frac{\tilde{\gamma}}{\sigma^2 + \sigma_0^2}\right)$$

$$= P_r\left(\mathcal{N}(N, 2N) \geq \frac{\tilde{\gamma}}{\sigma^2 + \sigma_0^2}\right)$$

$$\boxed{P_D = Q\left(\frac{\frac{\tilde{\gamma}}{\sigma^2 + \sigma_0^2} - N}{\sqrt{2N}}\right)}$$

⊙ When $N \to \infty$, the Receiver Operating Characteristic (ROC) is

$$P_{FA} = Q\left(\frac{\frac{\tilde{\gamma}}{\sigma^2} - N}{\sqrt{2N}}\right)$$

$$\Rightarrow \tilde{\gamma} = \left(\sqrt{2N}\, Q^{-1}(P_{FA}) + N\right)\sigma^2$$

$$P_D = Q\left(\frac{\frac{\tilde{\gamma}}{\sigma^2 + \sigma_0^2} - N}{\sqrt{2N}}\right) = Q\left(\frac{\frac{\left(\sqrt{2N}\, Q^{-1}(P_{FA}) + N\right)\sigma^2}{\sigma^2 + \sigma_0^2} - N}{\sqrt{2N}}\right)$$

$$= Q\left(\frac{\frac{\sigma^2}{\sigma^2 + \sigma_0^2}\left(\sqrt{2N}\, Q^{-1}(P_{FA}) + N\right) - N}{\sqrt{2N}}\right)$$

$$= Q\left(\frac{\frac{\sigma^2}{\sigma^2 + \sigma_0^2}\left(\cancel{\sqrt{2N}}\, Q^{-1}(P_{FA})\right)}{\cancel{\sqrt{2N}}} + \frac{\frac{\sigma^2}{\sigma^2 + \sigma_0^2}(N)}{\sqrt{2N}} - \frac{N}{\sqrt{2N}}\right)$$

$$= Q\left(\frac{\sigma^2}{\sigma^2 + \sigma_0^2}\, Q^{-1}(P_{FA}) + \frac{\sqrt{N}\cancel{\sqrt{N}}}{\sqrt{2\cancel{N}}}\left(\frac{\sigma^2}{\sigma^2 + \sigma_0^2} - 1\right)\right)$$

$$= Q\left(\frac{\sigma^2}{\sigma^2+\sigma_s^2}\,Q^{-1}(P_{FA}) + \sqrt{\frac{N}{2}}\left(\frac{\cancel{\sigma^2}-\cancel{\sigma^2}-\sigma_s^2}{\sigma^2+\sigma_s^2}\right)\right)$$

$$\boxed{P_D = Q\left(\frac{\sigma^2}{\sigma^2+\sigma_s^2}\,Q^{-1}(P_{FA}) - \sqrt{\frac{N}{2}}\;\frac{\sigma_s^2}{\sigma^2+\sigma_s^2}\right)}$$