Week 8 : Session 4

11. MIMO Estimation (MMSE/LMMSE perspective)

MIMO system model is given as

$$\underbrace{\begin{bmatrix} y_1(k) \\ y_2(k) \\ \vdots \\ y_r(k) \end{bmatrix}}_{\substack{r\times 1 \\ \text{Output vector}}} = \underbrace{\begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1t} \\ h_{21} & h_{22} & \cdots & h_{2t} \\ \vdots & \vdots & & \vdots \\ h_{r1} & h_{r2} & \cdots & h_{rt} \end{bmatrix}}_{\substack{r\times t \\ \text{Channel Matrix}}} \underbrace{\begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_t(k) \end{bmatrix}}_{\substack{t\times 1 \\ \text{input vector}}} + \underbrace{\begin{bmatrix} v_1(k) \\ v_2(k) \\ \vdots \\ v_r(k) \end{bmatrix}}_{\substack{r\times 1 \\ \text{Noise vector}}}$$

where,

$r \rightarrow$ No. of Receive antennas

$t \rightarrow$ No. of Transmit antennas

MIMO system model can be represented in compact fashion as

$$\bar{y}(k) = H\bar{x}(k) + v(k)$$

Now, consider the transmission of N pilot vectors.

$$\bar{y}(1) = H\,\bar{x}(1) + \bar{v}(1)$$
$$\bar{y}(2) = H\,\bar{x}(2) + \bar{v}(2)$$
$$\vdots$$
$$\bar{y}(N) = H\,\bar{x}(N) + \bar{v}(N)$$

We concatenate them for the purpose of CH EST as

$$\underbrace{\left[\bar{y}(1)\ \bar{y}(2) \cdots \bar{y}(N)\right]}_{r\times N \text{ Output Matrix}} = H \underbrace{\left[\bar{x}(1)\ \bar{x}(2) \cdots \bar{x}(N)\right]}_{t\times N \text{ Pilot Matrix}} + \underbrace{\left[\bar{v}(1)\ \bar{v}(2) \cdots \bar{v}(N)\right]}_{r\times N \text{ Noise Matrix}}$$

This can be represented in compact fashion as

Wide Matrix

$$\underbrace{Y = H X + V}_{\text{MIMO Channel Estimation model}}$$

Pseudo Inverse of $X = X^T(XX^T)^{-1}$ ← if X is Real Matrix

$= X^H(XX^H)^{-1}$ ← if X is Complex Matrix

The ML MIMO channel estimate is

$$\underbrace{\hat{H} = Y X^T (X X^T)^{-1}}_{\text{ML Channel Estimate}}$$

which minimizes the cost function as

$$\min \| Y - HX \|_F^2$$ ← Frobenius Norm.

Frobenius Norm

"Sum of the squares of the magnitudes of all the elements of the matrix."

Vector Norm

Square Root of "Sum of the squares of the magnitudes of all the elements of the matrix"

The LMMSE MIMO channel estimate is

$$\hat{H} = Y X^T \left(X X^T + \frac{1}{SNR} I \right)^{-1}$$

where $SNR = \frac{\sigma_n^2}{\sigma^2}$

As $SNR \to \infty$, LMMSE $\to$ ML

Example:

- Consider the MIMO channel estimation problem with Pilot vectors $\bar{x}(1) = [3 \ -2]^T$, $\bar{x}(2) = [-2 \ 3]^T$, $\bar{x}(3) = [4 \ 2]^T$, $\bar{x}(4) = [2 \ 2]^T$.

What is the Pilot Matrix ?

The Pilot Matrix, $X = \begin{bmatrix} 3 & -2 & 4 & 2 \\ -2 & 3 & 2 & 2 \end{bmatrix}_{t \times N}$

$N = 4$
$t = 2$

- The Output vectors are $\bar{y}(1) = [-2, \ 1 \ \ -3]^T$,

$\bar{y}(2) = [-1 \ \ 3 \ \ 3]^T$,

$\bar{y}(3) = [-1 \ \ -2 \ \ 2]^T$,

$\bar{y}(4) = [-3 \ \ -1 \ \ 1]^T$.

What is the Output Matrix ?

The Output Matrix, $Y = \begin{bmatrix} -2 & -1 & -1 & -3 \\ 1 & 3 & -2 & -1 \\ -3 & 3 & 2 & 1 \end{bmatrix}_{r \times N}$ | $r = 3$, $N = 4$

$\Rightarrow$ The given MIMO System is

$r \times t = 3 \times 2$ MIMO System.

⊙ The LMMSE Channel estimate is given as

$$\hat{H} = YX^T \left(XX^T + \frac{1}{SNR} I \right)^{-1}$$

$SNR = -6\ dB = \frac{1}{4}$.

$$XX^T = \begin{bmatrix} 3 & -2 & 4 & 2 \\ -2 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 3 \\ 4 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 33 & 0 \\ 0 & 21 \end{bmatrix}$$

$$\left(XX^T + \frac{1}{SNR} I \right)^{-1} = \left(\begin{bmatrix} 33 & 0 \\ 0 & 21 \end{bmatrix} + \frac{1}{1/4} I \right)^{-1}$$

$$= \left(\begin{bmatrix} 33 & 0 \\ 0 & 21 \end{bmatrix} + 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 37 & 0 \\ 0 & 25 \end{bmatrix}^{-1}$$

$$YX^T = \begin{bmatrix} -2 & -1 & -1 & -3 \\ 1 & 3 & -2 & -1 \\ -3 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 3 \\ 4 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} -14 & -7 \\ -13 & 1 \\ -5 & 21 \end{bmatrix}$$

Finally, the LMMSE MIMO Channel estimate is

$$\hat{H} = \begin{bmatrix} -14 & -7 \\ -13 & 1 \\ -5 & 21 \end{bmatrix} \begin{bmatrix} 1/37 & 0 \\ 0 & 1/25 \end{bmatrix} = \begin{bmatrix} -\frac{14}{37} & -\frac{7}{25} \\ -\frac{13}{37} & \frac{1}{25} \\ -\frac{5}{37} & \frac{21}{25} \end{bmatrix}$$

MIMO Receivers

How to design the Receiver in MIMO System ?

Consider the MIMO model

$$\bar{y}(k) = H\,\bar{x}(k) + \bar{v}(k)$$

Dropping Time index 'k' for simplicity.

$$\Rightarrow \quad \bar{y} = H\,\bar{x} + \bar{v}$$

Assume, we have estimated the channel and we know H.

Now, how to determine $\bar{x}$ given $\bar{y}$?

(i.e.) what is the estimate of the Transmit vector $\bar{x}$?

(i.e.) $\hat{x} = ?$

① MIMO ZF Receiver

The Least Squares Receiver

$$\underbrace{\min \|\bar{y} - H\bar{x}\|^2}_{\text{Least Squares Problem}}$$

And the Solution for the same is

$$\boxed{\hat{x} = (H^H H)^{-1} H^H \bar{y}}$$

This is termed as the Zero Forcing (ZF) receiver.

where $(H^H H)^{-1} H^H$ is Pseudo-inverse of H.

H is Tall Matrix (i.e.) $r \geq t$.

Example:

consider $\bar{y} = \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix}$, $H = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}_{r \times t}$

$\Rightarrow$ $r = 4$ Receive Antennas, $t = 2$ Transmit Antennas $\Big\}$ $\Rightarrow$ 4×2 MIMO System

What is $\hat{x}$?

The Zero Forcing Estimate can be Calculated as follows.

$$\hat{x} = (H^H H)^{-1} H^H y$$

$$H^H H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix}$$

$$(H^H H)^{-1} = \frac{1}{20} \begin{bmatrix} 30 & -10 \\ -10 & 4 \end{bmatrix}$$

$$(H^H H)^{-1} H^T = \frac{1}{20} \begin{bmatrix} 30 & -10 \\ -10 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1/2 & 0 & -1/2 \\ -3/10 & -1/10 & 1/10 & 3/10 \end{bmatrix}$$

$$(H^H H)^{-1} H^T \vec{y} = \begin{bmatrix} 1 & 1/2 & 0 & -1/2 \\ -3/10 & -1/10 & 1/10 & 3/10 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$$

Therefore, the ZF estimate is

$$\boxed{\hat{x} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}}$$

Output of ZF Receiver

② MIMO LMMSE Receiver.

- Another popular MIMO receiver is the LMMSE receiver.

> **Note:** ML is used when the Unknown quantity is Deterministic.
>
> MMSE / LMMSE is used when the Unknown quantity is Random

We minimize the cost function

$$\min E\left\{\| C^{H}\bar{y} - \bar{x}\|^{2}\right\}$$

The estimate is Linear Transformation of Output

(ie) $\hat{x} = C^{H}\bar{y}$.

Consider the Symbols to be IID, Zero Mean, with Power P.

$$E\{x(i)\} = 0 \;, \quad E\{|x(i)|^{2}\} = P$$

$$\Rightarrow E\{\bar{x}\bar{x}^{H}\} = PI$$

where $P \rightarrow$ Power of Symbols.

The LMMSE Receiver is

$$\boxed{\hat{x} = \left(H^{H}H + \frac{1}{SNR}I\right)^{-1} H^{H}\bar{y}}$$

where $SNR = \frac{P}{N_0}$.

Example :

Consider $\bar{y} = \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix}$, $H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}_{r \times t}$

$\Rightarrow$ $\left.\begin{array}{l} r = 4 \text{ Receive Antennas} \\ t = 2 \text{ Transmit Antennas} \end{array}\right\} \Rightarrow 4 \times 2$ MIMO System.

What is $\hat{x}$ when SNR $= -3$ dB $= \frac{1}{2}$.

The LMMSE estimate can be calculated as follows.

$$\hat{x} = \left(H^H H + \frac{1}{SNR} I\right)^{-1} H^H \bar{y}$$

$$H^H H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = 4I$$

$$H^H H + \frac{1}{SNR} I = 4I + \frac{1}{1/2} I = 6I.$$

$$\left(H^H H + \frac{1}{SNR} I\right)^{-1} H^H = \frac{1}{6} I \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$$\left(H^H H + \frac{1}{SNR} I\right)^{-1} H^H \bar{y} = \frac{1}{6} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Therefore, the LMMSE Estimate is

$$\boxed{\hat{x} = \frac{1}{6} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$ $\leftarrow$ Output of LMMSE Receiver