Week 1 : Session 3.

2. CHANNEL ESTIMATION

In the previous chapter, we've looked at Maximum Likelihood framework, simple problem of estimation of an unknown parameter from noisy measurements. In this chapter, we'll look at a problem that is of fundamental importance in wireless communication which is called as CHANNEL ESTIMATION problem.

The wireless communication system can be modeled as

$$y = hx + v$$

where,

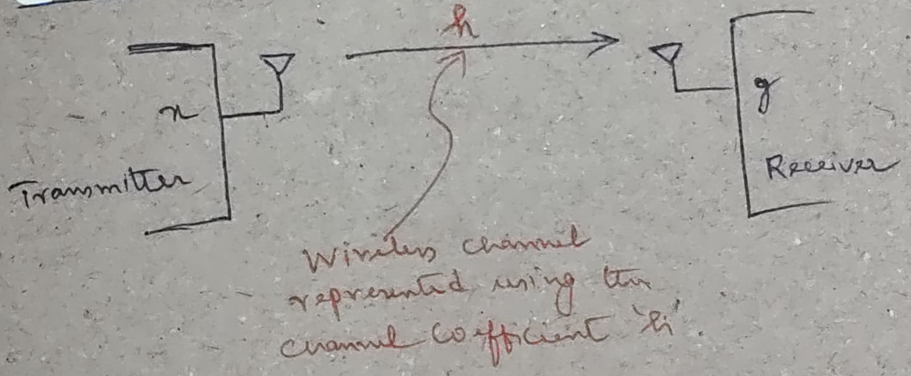
$x \rightarrow$ Transmit / input symbol

$y \rightarrow$ Received / output symbol

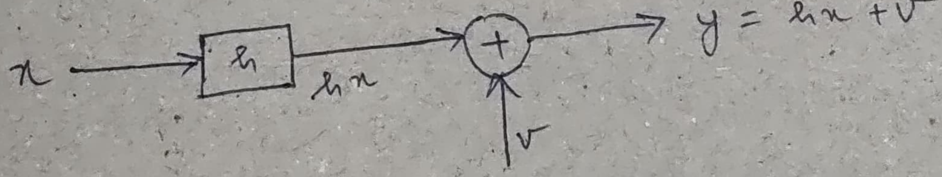
$n \rightarrow$ Noise

$h \rightarrow$ Unknown Fading channel coefficient, which is necessary for decoding at Receiver; has to be estimated.

SISO Channel Schematic



Wireless channel represented using the channel coefficient 'h'.



The channel coefficient h is unknown. Estimating this is termed as CHANNEL ESTIMATION. This is a very important problem of key significance in any wireless communication system, because without this, decoding is not possible at the receiver.

How to perform channel Estimation?

In order to estimate the channel, we transmit known symbol or fixed symbols from the transmitter. These fixed symbols are termed as PILOT Symbols. Pilot symbols are purely for the purpose of Channel Estimation.

$$\underbrace{x(1),\ x(2),\ \ldots,\ x(N)}_{\text{Training symbols / Pilots}}$$

Pilot symbols are Predetermined / Known / Fixed symbols which do not carry any information.

The input-output model for the transmission of Pilot Symbol is given as

PILOTS

$$\begin{aligned} y(1) &= h\,x(1) + v(1) \\ y(2) &= h\,x(2) + v(2) \\ &\ \vdots \\ y(N) &= h\,x(N) + v(N) \end{aligned}$$

where,

$$\bar{y} = \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}$$ is the $N \times 1$ output vector.

$$\bar{x} = \begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(N) \end{bmatrix}$$ is the $N \times 1$ Pilot vector

$$\underbrace{x(1), x(2), \ldots, x(N)}_{\text{Pilot symbols}} \qquad \underbrace{y(1), y(2), \ldots, y(N)}_{\text{Output symbols}}$$

Now, consider the k^{th} observation,

$$y(k) = h\,x(k) + v(k)$$

where, $x(k)$ is the k^{th} pilot

$y(k)$ is the k^{th} output

Since $v(k)$ is Gaussian with mean $= 0$, var $= \sigma^2$ (ie) $v(k) \sim \mathcal{N}(0, \sigma^2)$

$y(k)$ will be the shifted version of $v(k)$ by h times.

(ie) $y(k)$ will also be Gaussian with mean $= h\,x(k)$ and variance $= \sigma^2$. (ie) $y(k) \sim \mathcal{N}(h\,x(k), \sigma^2)$

The PDF of $y(k)$ is given as follows

$$f_{Y(k)}(y(k)) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(y(k) - h\,x(k))^2}{2\sigma^2}}$$

The Noise samples are iid, the Joint PDF of observations is obtained by the product of the individual PDFs.

$$f_{\bar{Y}}(\bar{y}) = f_{Y(1)}(y(1)) \times \cdots\cdots \times f_{Y(N)}(y(N))$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2\sigma^2}(y(1) - h\,x(1))^2} \times \cdots\cdots$$

$$\cdots\cdots \times \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2\sigma^2}(y(N) - h\,x(N))^2}$$

$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N e^{-\frac{1}{2\sigma^2}\sum_{k=1}^{N}(y(k) - h\,x(k))^2}$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{N}{2}} e^{-\frac{1}{2\sigma^2}\sum_{k=1}^{N}(y(k) - h\,x(k))^2}$$

As the Joint PDF can be viewed as a function of the unknown parameter (h), it becomes the Likelihood, which is basically a measure of "How well the parameter 'h' is able to explain the observation vector $\bar{y}$". (ie) The value of h corresponding to which the PDF of the observation vector $\bar{y}$ is maximum, that is the value of the parameter h which has maximum likelihood.

So, to compute the estimate of 'h', we maximize the likelihood. This is MLE.

The likelihood function w.r.t h is given by

$$p(\bar{y}; h) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N e^{-\frac{1}{2\sigma^2}\sum_{k=1}^{N}(y(k) - h\,x(k))^2}$$

The estimate of h is obtained by maximizing the likelihood.

$$\hat{h} = \max \; p(\bar{y}; h)$$

constant

$$\max \; \underbrace{\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N}_{\text{constant}} e^{-\overset{\text{constant}}{\boxed{\frac{1}{2\sigma^2}}}\sum_{k=1}^{N} \left(y(k) - h\,x(k)\right)^2}$$

due to '−' in the exponent

$$\equiv \underline{\underline{\min}} \; \underbrace{\sum_{k=1}^{N} \left(y(k) - h\,x(k)\right)^2}_{\text{Equivalent cost function}}$$

Minimizing the equivalent cost function can be achieved by taking derivative w.r.t h and set equal to 0.

$$\frac{d}{dh} \sum_{k=1}^{N} \left(y(k) - h\,x(k)\right)^2 = 0$$

$$\Rightarrow \sum_{k=1}^{N} \cancel{2}\left(y(k) - h\,x(k)\right)\left(\cancel{-}x(k)\right) = 0$$

$$\Rightarrow \sum_{k=1}^{N} y(k)\cdot x(k) = \sum_{k=1}^{N} h\,x^2(k)$$

$$\Rightarrow \hat{h} = \frac{\overset{\text{inner product, } \bar{x}^T\bar{y}}{\boxed{\sum_{k=1}^{N} y(k)\cdot x(k)}}}{\boxed{\sum_{k=1}^{N} x^2(k)}} \leftarrow \|\bar{x}\|^2 \text{, assuming all quantities are REAL.}$$

$$\Rightarrow \boxed{\bar{h} = \frac{\bar{x}^T\bar{y}}{\|\bar{x}\|^2} = \frac{\bar{x}^T\bar{y}}{\bar{x}^T\bar{x}}}$$

This is the <u>channel Estimate</u> / <u>ML Estimate of channel.</u>

For complex quantities ($\bar{x}, \bar{y}$ are COMPLEX), the channel estimate is given as

$$\boxed{\hat{h} = \frac{\sum_{k=1}^{N} x^*(k)\cdot y(k)}{\sum_{k=1}^{N} |x(k)|^2} = \frac{\bar{x}^H\bar{y}}{\|\bar{x}\|^2} = \frac{\bar{x}^H\bar{y}}{\bar{x}^H\bar{x}}}$$

Recall, in any Communication system, inputs/outputs are always complex (i.e) complex Baseband representation in which signals are represented as 'Inphase' + j 'Quadrature'. For the sake of simplicity, we consider these quantities to be REAL.

Week 2 : Session 1

Properties of MLE (Pilot based channel Estimation)

We now explore the properties of the ML estimate.

$$\text{MLE} \Rightarrow \hat{h} = \frac{\sum_{k=1}^{N} x(k)\cdot \underbrace{y(k)}_{\text{Gaussian}}}{\sum_{k=1}^{N} x^2(k)}$$

$$= \sum_{k=1}^{N} \frac{x(k)}{\|\bar{x}\|^2}\cdot y(k)$$

$h \rightarrow$ deterministic unknown quantity

$\hat{h} \rightarrow$ random quantity.

⊙ What is the distribution of $\hat{h}$?

Since $y(k)$ is Gaussian, $\hat{h}$ is a linear combination of Gaussian RVs. Hence, $\hat{h}$ is also Gaussian RV.

⊙ What is the mean of $\hat{h}$?

$$E\{\hat{h}\} = E\left\{\frac{\sum_{k=1}^{N} x(k)\cdot y(k)}{\sum_{k=1}^{N} x^2(k)}\right\}$$

$$= \frac{\sum_{k=1}^{N} x(k)\cdot E\{y(k)\}}{\sum_{k=1}^{N} x^2(k)}$$

$$= \frac{\sum_{k=1}^{N} x(k)\cdot E\{h\, x(k) + v(k)\}}{\sum_{k=1}^{N} x^2(k)}$$

$$= \frac{\sum_{k=1}^{N} x(k)\cdot h\cdot x(k)\cdot \cancelto{0}{E\{v(k)\}}}{\sum_{k=1}^{N} x^2(k)}$$

$$= \frac{\cancel{\sum_{k=1}^{N} x^2(k)}\cdot h}{\cancel{\sum_{k=1}^{N} x^2(k)}}$$

$$\boxed{E\{\hat{h}\} = h} \leftarrow$$

Here, the estimate of the unknown parameter '$\hat{h}$' coincides with the True parameter 'h'. Such an Estimate is termed as UNBIASED ESTIMATE.

⊙ What is the Mean Square Error (MSE) / Variance of $\hat{h}$?

$$E\{(\hat{h}-\underset{\text{Mean}}{h})^2\} = E\left\{\left(\frac{\sum_{k=1}^{N} x(k)\,y(k)}{\sum_{k=1}^{N} x^2(k)} - h\right)^2\right\}$$

$$= E\left\{\left(\frac{\sum_{k=1}^{N} x(k)\,y(k) - h\sum_{k=1}^{N} x^2(k)}{\sum_{k=1}^{N} x^2(k)}\right)^2\right\}$$

$$= \frac{E\left\{\left(\sum_{k=1}^{N} x(k)\left[y(k) - h\,x(k)\right]\right)^2\right\}}{\|\bar{x}\|^4}$$

$$= \frac{E\left[\left(\sum_{k=1}^{N} x(k).v(k)\right)^2\right]}{\|\bar{x}\|^4}$$

Since $v(i)$ is iid,

$E\{v(k).v(l)\} = \sigma^2\delta(k-l)$

$$= \frac{E\left\{\left(\sum_{k=1}^{N} x(k).v(k)\right)\left(\sum_{l=1}^{N} x(l).v(l)\right)\right\}}{\|\bar{x}\|^4}$$

$$= \frac{\sum_{k=1}^{N}\sum_{l=1}^{N} x(k).x(l).E\{v(k).v(l)\}}{\|\bar{x}\|^4}$$

Because of $\delta(k-l)$, only the terms where $k = l$ will survive

$$= \frac{\sum_{k=1}^{N}\sum_{l=1}^{N} \sigma^2 x(k)\,x(l).\delta(k-l)}{\|\bar{x}\|^4}$$

$$= \frac{\sigma^2\sum_{k=1}^{N} x^2(k)}{\|\bar{x}\|^4}$$

$$= \frac{\sigma^2\|\bar{x}\|^2}{\|\bar{x}\|^4}$$

$$\boxed{E\{(\hat{h}-h)^2\} = \frac{\sigma^2}{\|\bar{x}\|^2}} \longleftarrow \text{MSE / Variance}$$

observe, MSE decreases as $\frac{1}{\|\bar{x}\|^2}$

MSE $= \frac{\sigma^2}{\|\bar{x}\|^2} \propto \frac{1}{\|\bar{x}\|^2}$, where $\|\bar{x}\|^2$ is Energy of PILOT.

(i) if we transmit pilots with very high energy, the channel Estimation error is going to be lower.

Therefore, $\hat{h}$ is Gaussian with mean $= h$, Variance $= \frac{\sigma^2}{\|\bar{x}\|^2}$

$$\boxed{\hat{h} \sim \mathcal{N}\left(h, \frac{\sigma^2}{\|\bar{x}\|^2}\right)}$$

Example :

Consider the input/Pilot vector $\bar{x} = [1 \;\; -1 \;\; 1 \;\; -1]^T$,

Output vector $\bar{y} = [2 \;\; -3 \;\; -2 \;\; 1]^T$.

(i) What is the Maximum likelihood estimate $\hat{h}$ of the unknown channel coefficient h?

$$\hat{h} \hat{=} \frac{\bar{x}^T \bar{y}}{\|\bar{x}\|^2} = \frac{[1 \;\; -1 \;\; 1 \;\; -1]\begin{bmatrix} 2 \\ -3 \\ -2 \\ 1 \end{bmatrix}}{4}$$

$$\|\bar{x}\| = \sqrt{(1)^2 + (-1)^2 + (1)^2 + (-1)^2}$$

$$= \sqrt{4} \qquad = \frac{2 + 3 - 2 - 1}{4} = \frac{2}{4} = \frac{1}{2}$$

$$\|\bar{x}\|^2 = 4$$

$$\boxed{\hat{h} = \frac{1}{2}}$$

(ii) Given the iid Gaussian noise with zero mean, Var $= \sigma^2 = 2$, What is the variance of the ML estimate?

$$\text{Variance} = \text{MSE} = \frac{\sigma^2}{\|\bar{x}\|^2} = \frac{2}{4} = \frac{1}{2}$$

At the Receiver, the received symbol y is given as

$$y = hx + v.$$

The receiver has to estimate the transmitted/information symbol x

$$\hat{x} = \frac{y}{h} \quad (\text{Assuming ideal case}, v = 0)$$

Here, 'h' is unknown. So, we estimate the unknown channel coefficient ($\hat{h}$) first, and then estimate the transmitted/information symbol x.

$$\hat{x} = \frac{y}{\hat{h}}.$$

This process is known as EQUALIZATION.

So, naturally, if the channel estimation $\hat{h}$ is good, the equalization performance is good. If the $\hat{h}$ is poor, then the equalization performance is also poor. Therefore, accurate estimation of the channel is very important in a wireless communication system.