Week 5 : Session 3.

7. Orthogonal Frequency Division Multiplexing (OFDM).

→ OFDM is one of the most prominent wireless technology / Multicarrier modulation technique / waveform that is used in cellular as well as wifi standards.

→ OFDM is one of the most extensively used wireless technologies.

→ OFDM is used in 4G LTE, 5G NR, wifi (802.11n/ac/ax/...)

→ OFDM is widely employed in most of the modern cellular and Wi-Fi Systems.

→ OFDM enables Ultra High Data Rates. For instance,

4G → 150 - 200 Mbps

5G → 1 Gbps

To understand OFDM, let us first understand the motivation for OFDM.

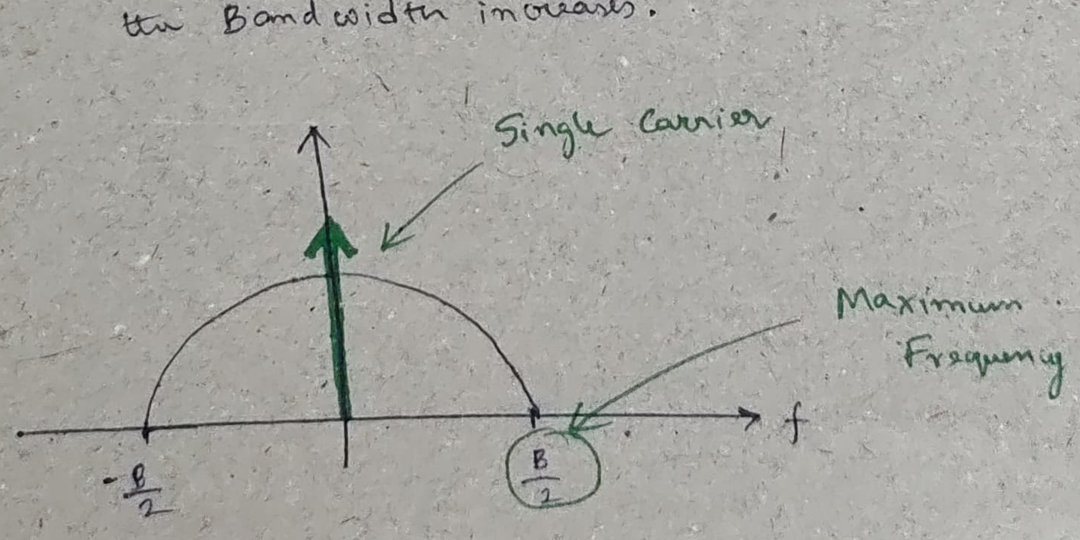
Single carrier Modulation. (conventional / orthodox / Traditional)

⊙ Consider bandwidth $\frac{B}{2}$ and single carrier

⊙ symbol duration $\propto \frac{1}{B}$

As Bandwidth increases, symbol duration decreases.

(i.e) We are transmitting symbols at a much higher rate, as the Bandwidth increases.



⊙ For instance, Bandwidth, $B = 10\,MHz$

Symbol duration, $T_{sym} = \frac{1}{B} = \frac{1}{10\,MHz} = 0.1\,\mu s$.

The symbol duration above is extremely small !!

This leads to Inter symbol Interference (ISI) / Frequency selective Distortion.

⊙ In a wireless communication system, when the Symbol duration becomes much smaller than the Delay Spread, ISI occurs.

Delay spread $\approx 2 - 3\,\mu s$.

Symbol duration $= 0.1\,\mu s$.

Here, the Symbol duration is significantly smaller than the Delay Spread. Thus ISI occurs.

Multi carrier Modulation

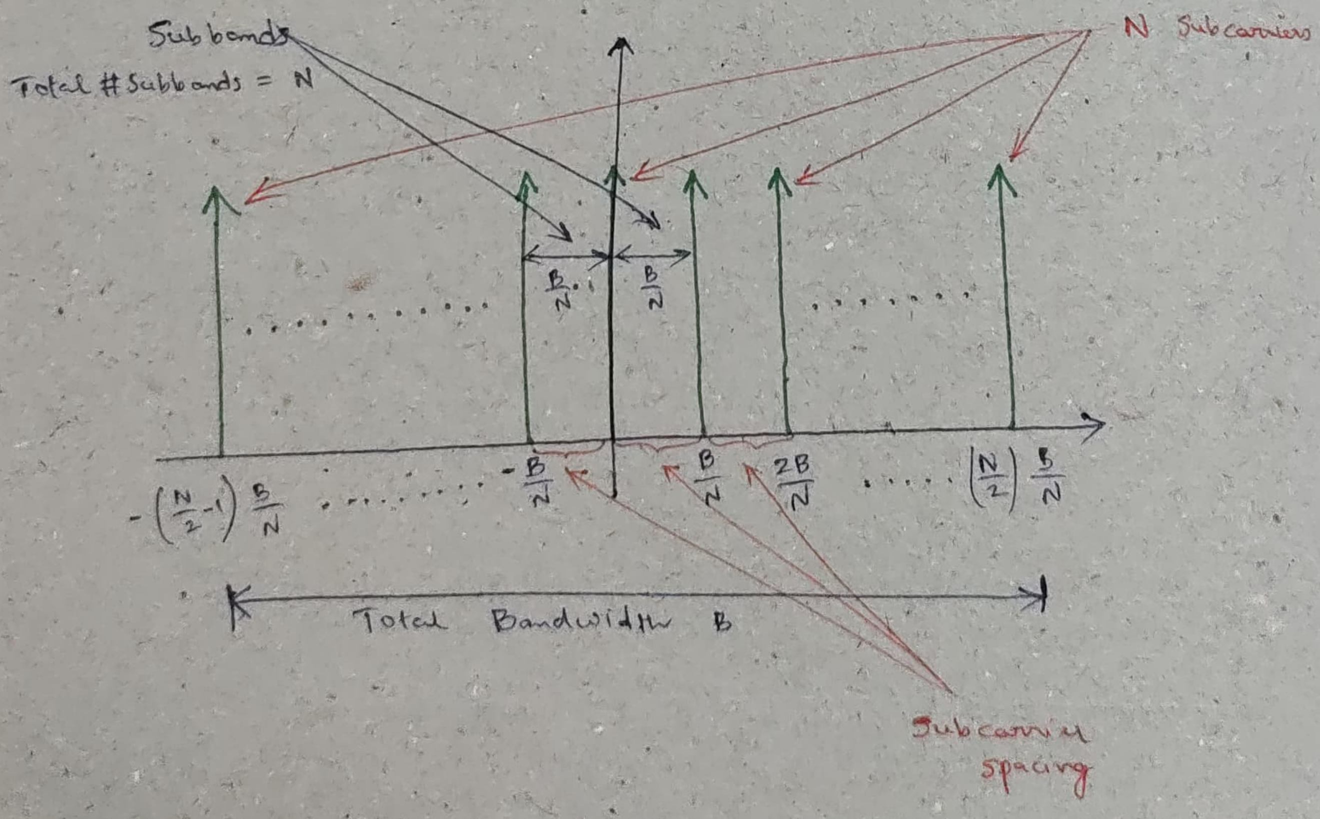
⊙ How to avoid ISI ?

⊙ One Solution is, Instead of using One carrier, we use N sub carriers.

(i) Dividing Total Bandwidth into N subbands.

And in each Subband, we're placing a Subcarrier.

This is known as Multi carrier System.



- As there exists multiple subcarriers, this is also known as Multi Carrier Modulation.

- Total Bandwidth is divided into N subbands.
  Bandwidth of each subband $= \frac{B}{N}$

- Subcarrier Spacing: The N subcarriers are placed with subcarrier spacing of $\frac{B}{N}$ with each other.

$$\frac{B}{N} = f_0$$

$\Rightarrow$ Subcarriers are placed at $\frac{B}{N}$ and multiples of $\frac{B}{N}$.

$$\ldots, -\frac{2B}{N}, -\frac{B}{N}, 0, \frac{B}{N}, \frac{2B}{N}, \frac{3B}{N}, \ldots$$

$$\ldots, -2f_0, -f_0, 0, f_0, 2f_0, 3f_0, \ldots$$

- We load the symbol X_k on the k^{th} subcarrier.

Modulating the symbol X_k on the k^{th} subcarrier $\Big\} \Rightarrow X_k e^{j2\pi k f_0 t}$

We take the SUM across all the subcarriers. The Transmit signal $x(t)$ is given as $\Bigg\}$ $x(t) = \frac{1}{N} \sum_k X_k e^{j2\pi k f_0 t}$

Normalization factor

- Constraint:

The Subcarriers are ORTHOGONAL.

(i.e.) $\underbrace{e^{j2\pi k f_0 t}}_{k^{th}\text{ subcarrier}}, \underbrace{e^{j2\pi l f_0 t}}_{l^{th}\text{ subcarrier}}$ are Orthogonal

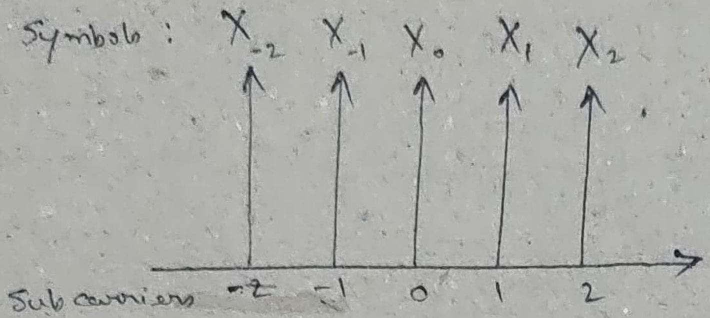
$$\Rightarrow f_0 \int_0^{1/f_0} e^{j2\pi k f_0 t} \cdot \left(e^{j2\pi l f_0 t}\right)^* dt$$ (INNER PRODUCT)

$$\Rightarrow f_0 \int_0^{1/f_0} e^{j2\pi (k-l) f_0 t} dt = \begin{cases} 0, & \text{if } k \neq l \\ 1, & \text{if } k = l. \end{cases}$$

The Subcarriers corresponding to the different Harmonics has to be orthogonal. Hence the name OFDM.

OFDM Explanation.

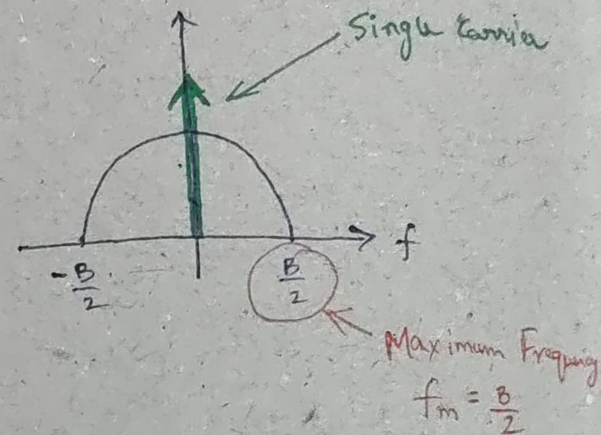
- ORTHOGONAL ⇒ Subcarriers are orthogonal
- FREQUENCY DIVISION ⇒ Dividing the frequency into N subbands.
- MULTIPLEXING ⇒ simultaneous transmission of N Symbols over N parallel sub carriers.



← Loading the Symbols over the Subcarriers.

OFDM Generation.

- Signal is Bandlimited to $\frac{B}{2}$.
- Minimum Sampling frequency (f_s)



$$f_s = 2 \times f_{max}$$

$$= 2 \times \frac{B}{2}$$

$$\boxed{f_s = B}$$ (According to Nyquist Criterion)

- Sampling duration, (T_s)

$$T_s = \frac{1}{f_s}$$

$$\boxed{T_s = \frac{1}{B}}$$

Nyquist Criterion

A Bandlimited Analog signal can be sampled at Twice the maximum frequency, without losing any information

Therefore, when we sample the OFDM signal / Multicarrier signal, the l^{th} sample will be at $t = l \cdot T_s = l \times \frac{1}{B}$.

$$\Rightarrow \boxed{t = \frac{l}{B}}$$

The l^{th} sample is given by

$$x(l) = \frac{1}{N} \sum_{k} X_k \, e^{j2\pi \cdot k \frac{B}{N} \cdot \frac{l}{B}}$$

$$= \underbrace{\frac{1}{N} \sum_{k} X_k \, e^{j2\pi \frac{k l}{N}}}_{\text{IDFT operation}}$$

So, the sampled OFDM signal can be generated via IDFT.

(i.e) the OFDM samples can be generated via IDFT.

The Advantage of IDFT is that, it can be implemented very efficiently using IFFT. To recover Symbols at the receiver, we use inverse of IFFT, which is FFT!!

In OFDM, we use $\begin{cases} \text{IFFT at the Transmitter} \\ \text{FFT at the Receiver} \end{cases}$

The Time Domain samples of the OFDM signal can be generated very efficiently using the IFFT algorithm!!

$$\boxed{\underbrace{X_0, X_1, \ldots, X_{N-1}}_{\text{Frequency Domain Symbols}}}$$

(Frequency Domain Symbols Loaded on Subcarriers!)

$$\Big\downarrow \text{IFFT}$$

$$\boxed{\underbrace{x(0), x(1), \ldots, x(N-1)}_{\text{Time Domain Samples}}}$$

(Time Domain Samples transmitted over channel)

Channel Taps / channel Filter

ISI channel Model

The channel is Frequency Selective / ISI channel. The ISI channel Model is given as

$$y(k) = h(0)\,x(k) + h(1)\,x(k-1) + \ldots + h(L-1)\,x(k-L+1) + v(k)$$

$$= \sum_{l=0}^{L-1} h(l)\,x(k-l) + v(k)$$

Frequency Selective ISI channel.

$$= \underbrace{h * x}\ + v(k)$$

The channel performs LINEAR CONVOLUTION between channel Taps and the transmitted TD samples of the OFDM symbols.

We have N Time Domain OFDM samples corresponding to the N subcarriers, we perform the N-point IFFT. (ie) The Original TD OFDM samples are generated by N-point IFFT by loading the corresponding N subcarriers.

Cyclic Prefix (CP)

Prior to transmission, we take $\tilde{L}$ samples from Tail of the Time Domain OFDM samples and Prefix them at Head. This is called as Cyclic Prefix (CP).

$$\underbrace{x(N-\tilde{L}), \ldots, x(N-2), x(N-1)}_{\text{Cyclic Prefix}},\ \underbrace{x(0), x(1), x(2), \ldots, x(N-1)}_{\text{Original Samples}}$$

Why do we do this? Why CP?

(*) The CP introduces Circular Symmetry. Because of that, after we remove the CP at the receiver, the Linear Convolution of the channel becomes Circular Convolution.

(ie) $y(l) = h \circledast x + v(l)$

Now, we take FFT at the Receiver.

$$\boxed{y(l) = h \circledast x + v(l)}$$

$$\Big\downarrow \text{FFT}$$

Symbol loaded on subcarrier k.

$$\boxed{Y_k = H_k \times X_k + V_k}$$

When we take FFT at the Receiver, the Circular convolution in Time Domain becomes the Product in Frequency domain.

Therefore, we have $Y_k = H_k X_k + V_k$.

$$\Rightarrow \quad Y_0 = H_0 \times X_0 + V_0$$

$$Y_1 = H_1 \times X_1 + V_1$$

$$\vdots$$

$$Y_{N-1} = H_{N-1} \times X_{N-1} + V_{N-1}$$

N Symbols loaded on Subcarriers.

Thus, OFDM converts a Big Wideband Frequency Selective channel into 'N' smaller parallel Narrowband Flat Fading channel.

It is evident that, in each subcarrier, there is no ISI.

(ie) Y_0 depends only on X_0, Y_1 depends only on X_1, and so on.

Week 5 : Session 4

CHANNEL ESTIMATION IN OFDM

To perform channel estimation in OFDM, we transmit PILOTS on each subcarrier. This can be modeled as

$$
\begin{aligned}
Y_k(1) &= H_k X_k(1) + V_k(1) \\
Y_k(2) &= H_k X_k(2) + V_k(2) \\
&\vdots \\
Y_k(N_p) &= H_k X_k(N_p) + V_k(N_p)
\end{aligned}
$$

Pilots (Pilot Symbols)

where, $N_p \rightarrow$ No. of Pilot symbols on each subcarrier.

(i.e.) we transmit 'N_p' pilots on the k^{th} subcarrier.

This can be written in vector form as

$$
\begin{bmatrix} Y_k(1) \\ Y_k(2) \\ \vdots \\ Y_k(N_p) \end{bmatrix} = H_k \underbrace{\begin{bmatrix} X_k(1) \\ X_k(2) \\ \vdots \\ X_k(N_p) \end{bmatrix}}_{\text{Pilot vector}} + \begin{bmatrix} V_k(1) \\ V_k(2) \\ \vdots \\ V_k(N_p) \end{bmatrix}
$$

$$\Rightarrow \bar{Y}_k = H_k \bar{X}_k + \bar{V}_k$$

The channel Estimate $(\hat{H}_k)$ for the k^{th} subcarrier is given by

$$\hat{H}_k = \frac{\sum_{i=1}^{N_p} X_k^*(i) \cdot Y_k(i)}{\sum_{i=1}^{N_p} X_k^*(i) X_k(i)}$$

$$= \frac{\sum_{i=1}^{N_p} X_k^*(i) \cdot Y_k(i)}{\sum_{i=1}^{N_p} |X_k(i)|^2}$$

$$\Rightarrow \boxed{\hat{H}_k = \left(\bar{X}_k^H \cdot \bar{X}_k\right)^{-1} \bar{X}_k^H \, \bar{Y}_k}$$

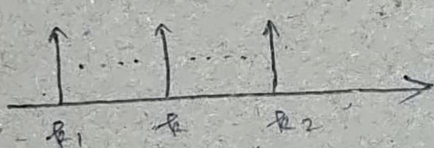
Interestingly, as the subcarriers are closely spaced, we don't need to transmit pilots on all the subcarriers of the OFDM system. So, we can estimate the rest of the coefficients of the subcarrier via Linear Interpolation.

We dont need to transmit the pilots on all the subcarriers, which is very inefficient. Coz the Pilots carry no information. So, we load pilots only on few evenly spaced subcarriers. For the rest of the subcarriers, the channel is estimated via Linear Interpolation.

$$(ii)\quad \hat{H}_k = \hat{H}_{k1} + \frac{(k - k1)}{(k2 - k1)} \left(\hat{H}_{k2} - \hat{H}_{k1}\right)$$

where, $k_1, k_2 \rightarrow$ Pilot subcarriers

$k \rightarrow$ Intermediate subcarriers



Example: Consider $\bar{X}_k = \begin{bmatrix} 1-j \\ 1+j \\ 1-j \\ 1+j \end{bmatrix}$, $\bar{Y}_k = \begin{bmatrix} -1 \\ -j \\ j \\ -1 \end{bmatrix}$.

$N_p = 4$ Pilot symbols.

The channel estimate $\hat{H}_k$ is given as

$$\hat{H}_k = \frac{\sum_{i=1}^{N_p} X_k^*(i)\, Y_k(i)}{\sum_{i=1}^{N_p} |X_k(i)|^2}$$

$$= \frac{(1+j)(-1) + (1-j)(-j) + (1+j)(j) + (1-j)(-1)}{2 + 2 + 2 + 2} = \frac{-1}{2}$$

$\boxed{\hat{H}_k = -1/2}$ $\leftarrow$ Channel Estimate for the subcarrier k.

We estimate the channels on few of the subcarriers, rest of the channel coefficients can be obtained via linear Interpolation

(or)

We can transmit the pilots on a few subcarriers in One OFDM symbol and then we can transmit the pilots on few other subcarriers in the next OFDM symbol.

So, the pattern chosen for the pilot subcarriers can keep changing through different OFDM symbols.