Week 4 : Session 1.

5. MIMO Channel Estimation

In the previous chapter, we looked at Multiple Antenna Channel estimation where there are multiple Tx. antennas and a single Rx. Antenna, which we called as MISO System/channel. Let us now look at what happens when we have MIMO channel.

MIMO denotes Multiple Input Multiple Output, which means there exist multiple Tx. Antennas and multiple Rx. Antennas.

MIMO enables Spatial Multiplexing, which is nothing but Parallel Transmission of multiple information streams over the same Time and frequency resources, which leads to significantly very high data rates !! This is why, MIMO is a key technology in 4G / 5G.

MIMO technology is used in several systems include

- LTE (4G)
- NR (5G)
- 802.11 n/ac/ax (wifi)

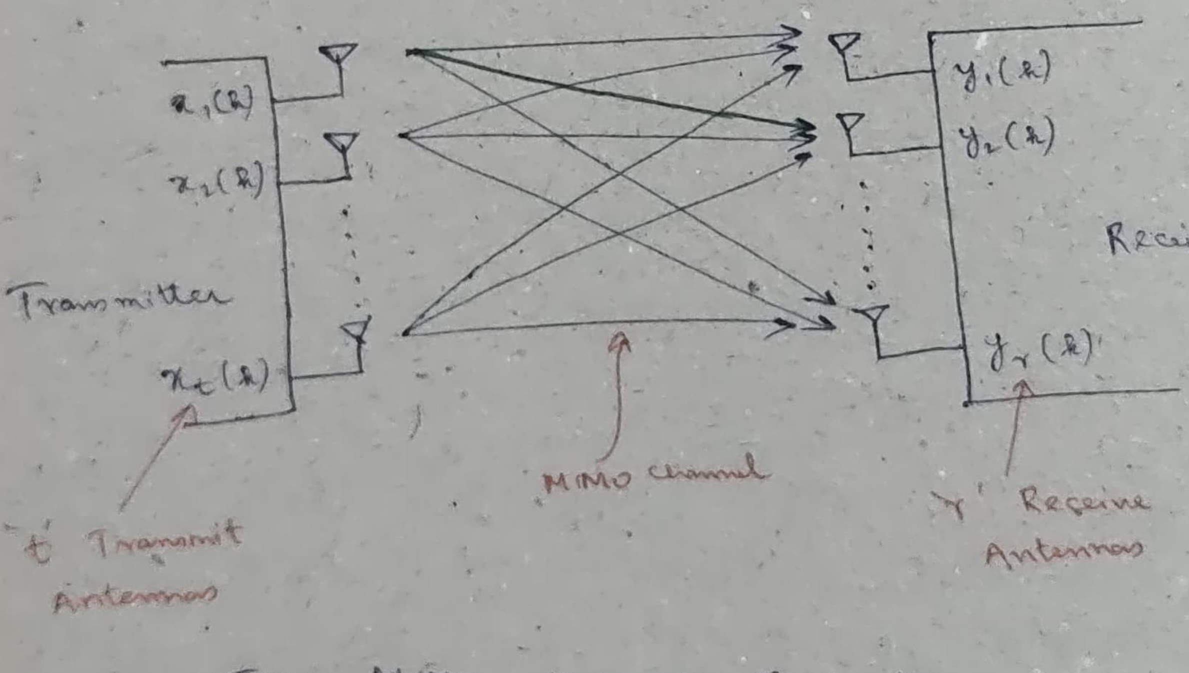

Fig. MIMO Channel Schematic

- $\underbrace{x_1(k), x_2(k), \ldots, x_t(k)}$ Transmit Symbols over the 't' Transmit Antennas at time 'k'

$k \rightarrow$ Time instant

- $\underbrace{y_1(k), y_2(k), \ldots, y_r(k)}$ Output Symbols over the 'r' receive antennas at time 'k'

- $r \rightarrow$ No. of Receive antennas
- $t \rightarrow$ No. of Transmit antennas

MIMO System model is given as

$$\underbrace{\begin{bmatrix} y_1(k) \\ y_2(k) \\ \vdots \\ y_r(k) \end{bmatrix}}_{\substack{r \times 1 \\ \text{output} \\ \text{vector}}} = \underbrace{\begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1t} \\ h_{21} & h_{22} & \cdots & h_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ h_{r1} & h_{r2} & \cdots & h_{rt} \end{bmatrix}}_{\substack{r \times t \\ \text{MIMO channel} \\ \text{Matrix}}} \underbrace{\begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_t(k) \end{bmatrix}}_{\substack{t \times 1 \\ \text{Pilot} \\ \text{vector}}} + \underbrace{\begin{bmatrix} v_1(k) \\ v_2(k) \\ \vdots \\ v_r(k) \end{bmatrix}}_{\substack{r \times 1 \\ \text{Noise} \\ \text{vector}}}$$

$$\Rightarrow \boxed{\bar{y}(k) = H\bar{x}(k) + \bar{v}(k)}$$ in compact fashion.

MIMO Channel Matrix $(r \times t)$

'h_{ij}' in the i^{th} row and j^{th} column denotes the channel between i^{th} Rx. Antenna and j^{th} Tx. Antenna.

Example

$h_{21} \rightarrow$ Channel between 2^{nd} Rx. Antenna and 1^{st} Tx. Antenna.

Now, for MIMO Channel Estimation, we transmit the PILOTS. Since we've multiple Tx. Antennas, naturally we'll transmit Pilot vectors. And correspondingly, we'll have the output vectors.

Consider the transmission of <u>N pilot vectors</u>,

N output vectors:

$$
\begin{aligned}
\bar{y}(1) &= H\,\bar{x}(1) + \bar{v}(1)\\
\bar{y}(2) &= H\,\bar{x}(2) + \bar{v}(2)\\
&\;\;\vdots\\
\bar{y}(N) &= H\,\bar{x}(N) + \bar{v}(N)
\end{aligned}
$$

N pilot vectors

$\bar{y}(N) \rightarrow$ output vector at time instant N

$\bar{x}(N) \rightarrow$ Pilot vector at time instant N

We can concatenate (Placing side-by-side) as

$$
\underbrace{\begin{bmatrix} \vdots & \vdots & & \vdots \\ \bar{y}(1) & \bar{y}(2) & \cdots & \bar{y}(N) \\ \vdots & \vdots & & \vdots \end{bmatrix}}_{\substack{\text{'N' output vectors at}\\ \text{each of the 'r'}\\ \text{Rx. Antenna}\\ (r \times N)}}
= H \underbrace{\begin{bmatrix} \vdots & \vdots & & \vdots \\ \bar{x}(1) & \bar{x}(2) & \cdots & \bar{x}(N) \\ \vdots & \vdots & & \vdots \end{bmatrix}}_{\substack{\text{'N' Pilot vectors at}\\ \text{each of the 't'}\\ \text{Tx. Antenna}\\ (t \times N)}}
+ \underbrace{\begin{bmatrix} \vdots & \vdots & & \vdots \\ \bar{v}(1) & \bar{v}(2) & \cdots & \bar{v}(N) \\ \vdots & \vdots & & \vdots \end{bmatrix}}_{\substack{\text{'N' Noise vectors at}\\ \text{each of the 'r'}\\ \text{Rx. Antenna}\\ (H \times N)}}
$$

$$\Rightarrow \boxed{Y = HX + V}$$ in compact fashion

Y → Output Matrix

H → MIMO Channel Matrix

X → Pilot Matrix

V → Noise Matrix

Note that, X is of size $t \times N$, $N \geq t$.

This is known as <u>Wide Matrix</u>.

$t \rightarrow$ # Rows $\rightarrow$ # Tx. Antennas

$N \rightarrow$ # columns $\rightarrow$ # Pilot vectors.

columns $\geq$ # Rows.

$\Rightarrow$ WIDE Matrix.

For the wide Matrix, the pseudo-inverse of X is given as $X^T(XX^T)^{-1}$. (ie) $X * X^T(XX^T)^{-1} = I$

Here, X is Wide Matrix.

$\Rightarrow$ X is NOT invertible unless # Rows $\neq$ # Columns.

Thus $X^T(XX^T)^{-1}$ acts as a Right inverse of X.

Hence the name "Pseudo-inverse of X".

The MIMO channel estimate is

$$\boxed{\hat{H} = Y X^T (X X^T)^{-1}}$$

Week 4 : Session 2.

Example :

(i) Consider the MIMO channel estimation problem with pilot vectors $\bar{x}(1) = [3 \ -2]^T$, $\bar{x}(2) = [-2 \ 3]^T$, $\bar{x}(3) = [4 \ 2]^T$, $\bar{x}(4) = [2 \ 2]^T$. What is the Pilot Matrix ?

$$\bar{x}(1) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \quad \bar{x}(2) = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$\bar{x}(3) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \quad \bar{x}(4) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

N = 4 pilot vectors

The Pilot Matrix is

$$X = [\bar{x}(1) \quad \bar{x}(2) \quad \bar{x}(3) \quad \bar{x}(4)]$$

$$\boxed{X = \begin{bmatrix} 3 & -2 & 4 & 2 \\ -2 & 3 & 2 & 2 \end{bmatrix}_{t \times N}}$$

where,

$t \rightarrow$ No. of Transmit antennas = 2

$N \rightarrow$ No. of Pilot vectors = 4.

(ii) The output vectors are

$$\bar{y}(1) = [-2 \quad 1 \quad -3]^T$$
$$\bar{y}(2) = [-1 \quad 3 \quad 3]^T$$
$$\bar{y}(3) = [-1 \quad -2 \quad 2]^T$$
$$\bar{y}(4) = [-3 \quad -1 \quad 1]^T$$

What is the Output matrix ?

The <u>Output Matrix</u> is

$$Y = [\bar{y}(1) \quad \bar{y}(2) \quad \bar{y}(3) \quad \bar{y}(4)]$$

$$\boxed{Y = \begin{bmatrix} -2 & -1 & -1 & -3 \\ 1 & 3 & -2 & -1 \\ -3 & 3 & 2 & 1 \end{bmatrix}_{r \times N}}$$

where,

$r \rightarrow$ No. of Receive Antennas $= 3$

We got $r = 3$, $t = 2$, which implies the given System is 3×2 MIMO System.

(iii) The channel Estimate is given as follows.

$$\hat{H} = Y X^T (X X^T)^{-1}.$$

$$Y X^T = \begin{bmatrix} -2 & -1 & -1 & -3 \\ 1 & 3 & -2 & -1 \\ -3 & 3 & 2 & 1 \end{bmatrix}_{3\times4} \begin{bmatrix} 3 & -2 \\ -2 & 3 \\ 4 & 2 \\ 2 & 2 \end{bmatrix}_{4\times2} = \begin{bmatrix} -14 & -7 \\ -13 & 1 \\ -5 & 21 \end{bmatrix}_{3\times2}$$

$$X X^T = \begin{bmatrix} 3 & -2 & 4 & 2 \\ -2 & 3 & 2 & 2 \end{bmatrix}_{2\times4} \begin{bmatrix} 3 & -2 \\ -2 & 3 \\ 4 & 2 \\ 2 & 2 \end{bmatrix}_{4\times2} = \underbrace{\begin{bmatrix} 33 & 0 \\ 0 & 21 \end{bmatrix}_{2\times2}}$$

Inner Product of Rows of XX^T is Zero.

$\Rightarrow$ Rows are othogonal

$\Rightarrow$ orthogonal PILOT Matrix

Orthogonal Pilot Matrix is preferred typically. This is because, the accuracy of the channel Estimation obtained is Higher.

Since, XX^T is a diagonal matrix, inversion is easy!

$$XX^T = \begin{bmatrix} 33 & 0 \\ 0 & 21 \end{bmatrix}, \quad (XX^T)^{-1} = \begin{bmatrix} 1/33 & 0 \\ 0 & 1/21 \end{bmatrix}$$

Therefore, the MIMO channel estimate is

$$\hat{H} = YX^T (XX^T)^{-1}$$

$$= \begin{bmatrix} -14 & -7 \\ -13 & 1 \\ -5 & 21 \end{bmatrix}_{3\times 2} \begin{bmatrix} 1/33 & 0 \\ 0 & 1/21 \end{bmatrix}_{2\times 2}$$

$$= \begin{bmatrix} -\frac{14}{33} & -\frac{7}{21} \\ -\frac{13}{33} & \frac{1}{21} \\ -\frac{5}{33} & \frac{21}{21} \end{bmatrix}_{3\times 2}$$