Week 4 ; Session 3

6. Channel Equalization

Typically in wireless channel, we have Intersymbol Interference (ISI). (ie) Past symbols interfere with the current symbols. ISI arises due to the Delay spread of the wireless channel. (ie) In Multipath propagation, there are different multipath components with different delays. The delayed signal components interfere with the current signal, which leads to ISI.

Equalization is used to remove the effect of Inter symbol Interference (ISI).

Consider the ISI channel model (Multi-Tap Wireless channel) as below.

$$y(k) = h(0)\,x(k) + h(1)\,x(k-1) + \dots + h(L-1)\,x(k-L+1) + v(k).$$

where,

$y(k) \rightarrow$ output symbol at time k

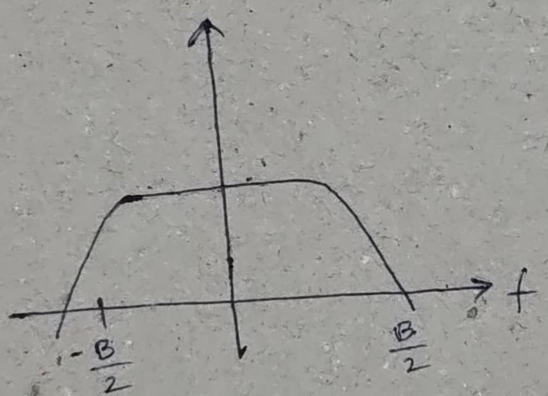
$x(k) \rightarrow$ Input symbol at time k

$x(k-1), x(k-2), \dots, x(k-L+1) \rightarrow$ Past / Delayed Symbols

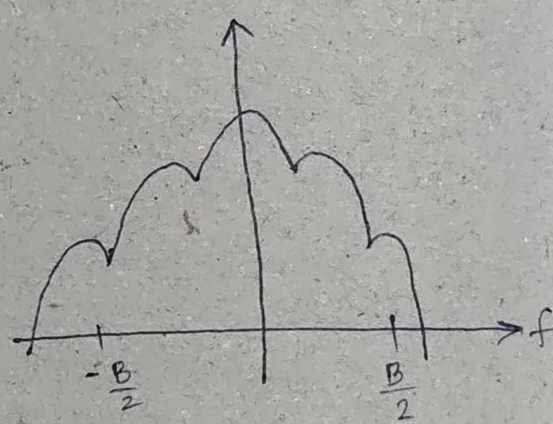
$h(0), h(1), \dots, h(L-1) \rightarrow$ Channel Taps.

The output symbol $y(k)$ depends not only on the Input symbol $x(k)$, but also on the Past / Delayed symbols, which causes interference with the current Input symbol.

Flat Fading channel / Frequency Selective channel



- Channel Response is Flat
- NO distortion of Signal spectrum
- NO ISI
- Flat Fading channel



- Channel Response is NOT Flat
- Signal spectrum gets distorted
- ISI occurs
- Frequency Selective channel

Gain of the channel depends on particular frequency bands, which implies Attenuation is Frequency selective.

When ISI occurs, we need to remove the same. (i.e.) we need to make this channel appear like a 'flat' fading channel. The Frequency Selective channel has non uniform gain over the frequency band, which we need to make it Uniform. (i.e.) we need to Equalize the gain of the Frequency Selective channel over the frequency band.

Once we have equalized the gain (i.e.) Once the gain becomes flat over the frequency band, that naturally removes the ISI. Therefore, this process is known as Channel Equalization.

Channel Taps / (a.k.a) FIR channel Filters

$$y(k) = h(0)\,x(k) + h(1)\,x(k-1) + \dots + h(L-1)\,x(k-L+1) + v(k)$$

$$\Rightarrow \boxed{y = \underbrace{h * x}_{\text{Linear convolution, leading to ISI, causing distortion.}} + v}$$

Convolutive channel

To understand Equalization, let us consider $L=2$ tap Channel.

$$\text{(i.e.)}\quad y(k) = h(0)\,x(k) + h(1)\,x(k-1) + v(k)$$

interference from 1 past Symbol

Now, how to eleminate / remove ISI due to $x(k-1)$ and extract $x(k)$ alone? This is known as the Equalization problem.

We perform Equalization using $y(k+1)$ and $y(k)$.

$$\begin{bmatrix} y(k+1) \\ y(k) \end{bmatrix} = \underbrace{\begin{matrix} h(0)\,x(k+1) + h(1)\,x(k) \\ h(0)\,x(k) + h(1)\,x(k-1) \end{matrix}}_{[\quad][\quad]} + \begin{bmatrix} v(k+1) \\ v(k) \end{bmatrix}$$

obtained by replacing 'k' by 'k+1'.

Representing ISI channels in vector form.

$$\underbrace{\begin{bmatrix} y(k+1) \\ y(k) \end{bmatrix}}_{2\times1} = \underbrace{\begin{bmatrix} h(0) & h(1) & 0 \\ 0 & h(0) & h(1) \end{bmatrix}}_{2\times3} \underbrace{\begin{bmatrix} x(k+1) \\ x(k) \\ x(k-1) \end{bmatrix}}_{3\times1} + \underbrace{\begin{bmatrix} v(k+1) \\ v(k) \end{bmatrix}}_{2\times1}$$

Channel Matrix / Filter Matrix
(Channel is acting as an FIR filter)

$$\Rightarrow \bar{y} = H\bar{x} + \bar{v} \quad \text{(compact vector form)}$$

Now, the Linear Algebra tool is going to simplify the very complex estimation process. In this case, "Channel Equalization".

Let the equalizer weights be c_0, c_1. Now, we linearly combine $y(k+1)$, $y(k)$ with the equalizer weights.

$$\Rightarrow c_0\, y(k+1) + c_1\, y(k)$$

$$\Rightarrow \begin{bmatrix} c_0 & c_1 \end{bmatrix} \begin{bmatrix} y(k+1) \\ y(k) \end{bmatrix}$$

$$\Rightarrow \bar{c}^T \bar{y}(k).$$

$$\Rightarrow \bar{c}^T \left(H\bar{x}(k) + \bar{v}(k)\right)$$

$$\Rightarrow \underbrace{\bar{c}^T H \bar{x}(k) + \bar{c}^T \bar{v}(k)}$$

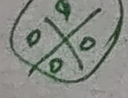
This is the Equalizer Output.

Now, we substitute the model.

$$\Rightarrow \bar{c}^T H \begin{bmatrix} x(k+1) \\ x(k) \\ x(k-1) \end{bmatrix} + \bar{c}^T \bar{v}(k)$$

$$\Rightarrow \underbrace{\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}} \begin{bmatrix} x(k+1) \\ x(k) \\ x(k-1) \end{bmatrix} + \underbrace{0}$$

Assuming ideal scenario with zero noise.



Assume, $\bar{c}^T H = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$, which suppresses $x(k+1)$ & $x(k-1)$ and recover $x(k)$ alone.

$$\Rightarrow x(k).$$

In order to recover $x(k)$, we choose

$$\bar{c}^T H = [0 \; 1 \; 0]$$

$$\Rightarrow H^T \bar{c} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Therefore, we've to design in such a way that $H^T\bar{c}$ approaches $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ vector as closely as possible. (ie) We've to minimize the error between the vector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $H^T\bar{c}$. So, we formulate the Least Squares (LS) problem and solve it.

$$\min \left\| \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - H^T \bar{c} \right\|^2$$

$H^T\bar{c}$ should approximate $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ vector as closely as possible, so that we'll be able to suppress $x(k+1)$ & $x(k-1)$ and recover $x(k)$.

And, the Least Squares (LS) Solution (ie) the Equalizer vector is given as

$$\bar{c} = \left((H^T)^T H^T \right)^{-1} (H^T)^T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \boxed{\bar{c} = (H H^T)^{-1} H \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$$

This is the Equalizer vector / Equalizer Filter.

We have derived the Equalizer vector $\bar{c}$ for a simple 2 Tap channel, and we can extend the same to any arbitrary number of Taps of the channel. So, this process illustrates the principle of how to design the Linear equalizer using the Least Squares (LS) principle.

For larger number of channel Taps, we have to construct the channel matrix. And once we got larger number of equalizer taps, we've to appropriately consider the $\bar{c}^T H$ (check which element to be 1), so that we can recover $x(k)$ by suppressing the interference from all the symbols. And formulate the LS problem and design the appropriate Equalizer vector.

Example : Week 5 : Session 2

Consider the ISI channel

$$y(k) = x(k) + \frac{1}{3} x(k-1) + v(k).$$

Design a 2 tap equalizer for this channel.

WKT : $y(k) = h(0)\, x(k) + h(1)\, x(k-1) + v(k).$

$\Rightarrow h(0) = 1 \quad , \quad h(1) = \frac{1}{3}$

Therefore, the channel Matrix, H is given by :

$$H = \begin{bmatrix} h(0) & h(1) & 0 \\ 0 & h(0) & h(1) \end{bmatrix} = \begin{bmatrix} 1 & 1/3 & 0 \\ 0 & 1 & 1/3 \end{bmatrix}$$

The Equalizer vector $\bar{c}$ is given by

$$\bar{c} = (HH^T)^{-1} H \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$HH^T = \begin{bmatrix} 1 & 1/3 & 0 \\ 0 & 1 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/3 & 1 \\ 0 & 1/3 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 10 & 3 \\ 3 & 10 \end{bmatrix}$$

$$(HH^T)^{-1} = \left(\frac{1}{9} \begin{bmatrix} 10 & 3 \\ 3 & 10 \end{bmatrix} \right)^{-1}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{9}{91} \begin{bmatrix} 10 & -3 \\ -3 & 10 \end{bmatrix}$$

$$\bar{c} = \frac{9}{91} \begin{bmatrix} 10 & -3 \\ -3 & 10 \end{bmatrix} \begin{bmatrix} 1 & 1/3 & 0 \\ 0 & 1 & 1/3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/91 \\ 81/91 \end{bmatrix}$$

$\Rightarrow c_0 = 3/91 \quad , \quad c_1 = 81/91$

Equalizer Output :

$$c_0\, y(k+1) + c_1\, y(k).$$

$$\Rightarrow \boxed{\frac{3}{91} y(k+1) + \frac{81}{91} y(k)}$$

Note :

* As the No. of Taps increases, the Performance increases

* But at the same time, the complexity also increases.