Week 7 : Session 5

10. Online / Sequential Estimation

Let us now look at another estimation paradigm, which is Online / Sequential Estimation. This is one of the most practically applicable estimation paradigm.

Consider the SISO channel estimation problem

$$y = hx + v$$

The corresponding model is

$$y(1) = h\,x(1) + v(1)$$

$$y(2) = h\,x(2) + v(2)$$

$$\vdots$$

$$y(N) = h\,x(N) + v(N)$$

The channel estimate is

$$\hat{h} = \frac{\sum_{k=1}^{N} x(k)\,y(k)}{\sum_{k=1}^{N} x^2(k)} = \frac{\bar{x}^T \bar{y}}{\bar{x}^T \bar{x}}$$

ML Estimate / LS Estimate

where, $\bar{x} = \begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(N) \end{bmatrix}$, $\bar{y} = \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}$

This can also be represented as Estimate at time N.

$$\hat{h}(N) = \frac{\sum_{k=1}^{N} x(k)\,y(k)}{\sum_{k=1}^{N} x^2(k)} = \frac{\bar{x}^T \bar{y}}{\bar{x}^T \bar{x}} = \frac{\bar{x}^T \bar{y}}{\|\bar{x}\|^2}$$

We are making the estimate as a time varying quantity, as more and more observations keep coming, we need to keep updating the estimate. We can't wait for all the observations to come coz we've to start the decoding and so on. So, we want to keep refining the estimate as the observations keep coming in, so that the estimate becomes more and more accurate. This is why the estimate is updated with time. And hence the name ONLINE...

Estimate is evolving with time (ie) Estimate is carried out with sequential fashion $(\hat{h}(1), \hat{h}(2), \ldots, \hat{h}(N))$ one after the other, so we'll have the sequence of estimates.

Consider now the next output at time $N+1$.

$$y(N+1) = h\, x(N+1) + v(N+1).$$

Instead of repeating the entire estimation process, we can simply <u>update</u> the previous estimate.

(ie) $\hat{h}(N) \xrightarrow{\text{update}} \hat{h}(N+1)$.

This update process is termed as <u>Sequential Estimation</u>. The estimation is carried out sequentially $(\hat{h}(N), \hat{h}(N+1), \ldots)$ as the outputs $(y(N+1), y(N+2), \ldots)$ arrive.

This is also termed as <u>Online estimation</u>, as the estimation is being carried out continuously and never stops. $\Rightarrow$ Estimator is ONLINE!

This can be achieved as follows.

The estimate at time $N+1$ is

$$\hat{h}(N+1) = \frac{\sum_{k=1}^{N+1} x(k)\, y(k)}{\sum_{k=1}^{N+1} x^2(k)}$$

The MSE at time N is

$$p(N) = \frac{\sigma^2}{\|\bar{x}\|^2}$$

$$\Rightarrow \|\bar{x}\|^2 = \frac{\sigma^2}{p(N)}$$

Therefore, the estimate at time N is

$$\hat{h}(N) = \frac{\bar{x}^T \bar{y}}{\bar{x}^T \bar{x}} = \frac{\bar{x}^T \bar{y}}{\|\bar{x}\|^2}$$

$$\Rightarrow \bar{x}^T \bar{y} = \hat{h}(N) \cdot \|\bar{x}\|^2$$

$$= \hat{h}(N) \cdot \frac{\sigma^2}{p(N)}.$$

And, the estimate at time N+1 is

$$\hat{h}(N+1) = \frac{\sum_{k=1}^{N} x(k)\,y(k) + x(N+1)\,y(N+1)}{\sum_{k=1}^{N} x^2(k) + x^2(N+1)}$$

$$= \frac{\bar{x}^T \bar{y} + x(N+1)\,y(N+1)}{\|\bar{x}\|^2 + x^2(N+1)}$$

$$= \frac{\hat{h}(N)\,\frac{\sigma^2}{p(N)} + x(N+1)\,y(N+1)}{\frac{\sigma^2}{p(N)} + x^2(N+1)}$$

Add & Subtract $x^2(N+1)$

$$= \frac{\hat{h}(N)\left\{\frac{\sigma^2}{p(N)} + x^2(N+1) - x^2(N+1)\right\} + x(N+1)\,y(N+1)}{\frac{\sigma^2}{p(N)} + x^2(N+1)}$$

$$= \frac{\hat{h}(N)\left\{\frac{\sigma^2}{p(N)} + x^2(N+1)\right\} - \hat{h}(N)\,x^2(N+1) + x(N+1)\,y(N+1)}{\frac{\sigma^2}{p(N)} + x^2(N+1)}$$

update to be added to $\hat{h}(N)$

$$= \hat{h}(N) \boxed{- \frac{\hat{h}(N)\,x^2(N+1)\,p(N)}{\sigma^2 + p(N)\,x^2(N+1)} + \frac{x(N+1)\,y(N+1)\,p(N)}{\sigma^2 + p(N)\,x^2(N+1)}}$$

$$= \hat{h}(N) + \underbrace{\frac{p(N)\,x(N+1)}{\sigma^2 + p(N)\,x^2(N+1)}}_{\text{Gain } k(N+1)} \underbrace{\left(y(N+1) - \hat{h}(N)\,x(N+1)\right)}_{\text{Prediction error } e(N+1)}$$

$$\boxed{\hat{h}(N+1) = \hat{h}(N) + k(N+1)\,e(N+1)}$$

This is the Update Rule!

We dont need to recalculate $\hat{h}(N+1)$ again. Instead update $\hat{h}(N)$ to obtain $\hat{h}(N+1)$.

The Prediction error $e(N+1)$ tells, how well we are able to predict $y(N+1)$ using the estimate at time N. (ie) $\hat{h}(N)$

Summarizing

The Estimate at time N+1 is

$$\hat{h}(N+1) = \hat{h}(N) + k(N+1)\, e(N+1)$$

where,

$$k(N+1) = \frac{p(N)\, x(N+1)}{\sigma^2 + p(N)\, x^2(N+1)} \Bigg\} \text{ Gain}$$

$$e(N+1) = y(N+1) - \underbrace{\hat{h}(N)}\, x(N+1) \Big\} \text{ Prediction Error / Innovation}$$

If the estimate at time N (i.e.) $\hat{h}(N)$ is good, then the error $e(N+1)$ will be very small.

If the estimate at time N (i.e.) $\hat{h}(N)$ is poor, then the error $e(N+1)$ will be High.

As the time keeps increasing, $\hat{h}(N)$ becomes more and more closer to h (i.e.) Error keeps progressively decreasing and Gain also keeps decreasing because of the variance σ^2.

Thus, we have derived the Online estimator

$$\hat{h}(N+1) = \hat{h}(N) + k(N+1)\, e(N+1)$$

The MSE can be updated as follows.

we have,

$$p(N) = \frac{\sigma^2}{\|\bar{x}\|^2} = \frac{\sigma^2}{\sum_{k=1}^{N} x^2(k)}$$

$$\Rightarrow p(N+1) = \frac{\sigma^2}{\sum_{k=1}^{N+1} x^2(k)}$$

$$= \frac{\sigma^2}{\sum_{k=1}^{N} x^2(k) + x^2(N+1)}$$

$$= \frac{\sigma^2}{\|\bar{x}\|^2 + x^2(N+1)}$$

$$= \frac{\sigma^2}{\frac{\sigma^2}{p(N)} + x^2(N+1)}$$

$$= \frac{\sigma^2 p(N)}{\sigma^2 + p(N) x^2(N+1)}$$

$$= \left(\frac{\sigma^2}{\sigma^2 + p(N) x^2(N+1)} \right) p(N)$$

$$= \left(1 - \frac{p(N) x^2(N+1)}{\sigma^2 + p(N) x^2(N+1)} \right) p(N)$$

$$= \left(1 - \underbrace{\frac{p(N) x(N+1)}{\sigma^2 + p(N) x^2(N+1)}}_{\text{Gain } k(N+1)} x(N+1) \right) p(N)$$

$$\boxed{p(N+1) = \left(1 - k(N+1)\, x(N+1) \right) p(N)}$$

This is the MSE update Rule!

MSE at time N+1

MSE at time N

Week 8 : Session 1

Online Estimation - Vector Parameter

Consider now the MISO channel estimation problem.

$$y(1) = \bar{x}^T(1)\,\bar{h} + v(1)$$
$$y(2) = \bar{x}^T(2)\,\bar{h} + v(2)$$
$$\vdots$$
$$y(N) = \bar{x}^T(N)\,\bar{h} + v(N)$$

$$\Rightarrow \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} = \begin{bmatrix} \bar{x}^T(1) \\ \bar{x}^T(2) \\ \vdots \\ \bar{x}^T(N) \end{bmatrix} \bar{h} + \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}$$

$N \times M$ Pilot matrix.

$$\Rightarrow \bar{y} = X\,\bar{h} + \bar{v}$$

The ML Estimate / LS Estimate at time N is

$$\hat{h}(N) = (X^T X)^{-1} X^T \bar{y}$$

Consider now a new output at time N+1.

$$y(N+1) = \bar{x}^T(N+1)\,\bar{h} + v(N+1)$$

How to update $\hat{h}(N)$?

(ii) How to obtain $\hat{h}(N+1)$?

We try to deduce from the Scalar model.

The Scalar model can be modified as follows.

$$(i)\ \underbrace{\hat{h}(N+1) = \hat{h}(N) + k(N+1)\,e(N+1)}_{\text{Scalar Parameter}}$$

$$\Downarrow$$

Gain becomes Vector of Size $M \times 1$

$$\underbrace{\hat{h}(N+1) = \hat{h}(N) + \boxed{\bar{k}(N+1)}\,e(N+1)}_{\text{Vector Parameter}}$$

(ii) $\underbrace{k(N+1) = \dfrac{p(N)\, x(N+1)}{\sigma^2 + p(N)\, x^2(N+1)}}_{\text{Scalar Parameter}}$

$$\Downarrow$$

Matrix

$$\underbrace{\bar{k}(N+1) = \frac{P(N)\, \bar{x}(N+1)}{\sigma^2 + \bar{x}^T(N+1)\, P(N)\, \bar{x}(N+1)}}_{\text{Vector Parameter}}$$

(iii) $\underbrace{e(N+1) = y(N+1) - x(N+1)\, \hat{h}(N)}_{\text{Scalar Parameter}}$

$$\Downarrow$$

$$\underbrace{e(N+1) = y(N+1) - \bar{x}^T(N+1)\, \hat{h}(N)}_{\text{Vector Parameter}}$$

Therefore, the net model is,

$$\hat{h}(N+1) = \hat{h}(N) + \bar{k}(N+1)\, e(N+1)$$

$$\bar{k}(N+1) = \frac{P(N)\, \bar{x}(N+1)}{\sigma^2 + \bar{x}^T(N+1)\, P(N)\, \bar{x}(N+1)}$$

$$e(N+1) = y(N+1) - \bar{x}^T(N+1)\, \hat{h}(N)$$

The Error covariance at time N is

$$P(N) = \sigma^2 (X^T X)^{-1}$$

Inspired by the scalar case, this can be updated for N+1 as

$$\underbrace{p(N+1) = \left(1 - k(N+1)\, x(N+1)\right) p(N)}_{\text{Scalar Parameter}}$$

$$\Downarrow$$

$$\underbrace{P(N+1) = \left(I - \bar{k}(N+1)\, \bar{x}^T(N+1)\right) P(N)}_{\text{Vector Parameter}}$$

Week 8 : Session 2

Online Estimation - Example

Consider $\bar{y} = \begin{bmatrix} -2 \\ -3 \\ 1 \\ -2 \end{bmatrix}$, $X = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \end{bmatrix}$. $N = 4$

- The ML estimate is

$$\hat{h}(N) = (X^T X)^{-1} X^T \bar{y}$$

$$X^T X = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \end{bmatrix} = 4I$$

$$\Rightarrow \hat{h}(N) = \frac{1}{4} I \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1/2 \end{bmatrix}$$

Therefore, Estimate at time $N = 4$

(ie) $\boxed{\hat{h}(4) = \begin{bmatrix} -1 \\ -1/2 \end{bmatrix}}$.

- Let $\sigma^2 = 4$. The Error Covariance is

$$P(N) = \sigma^2 (X^T X)^{-1} = 4 \times \frac{1}{4} I = I.$$

Therefore, Error Covariance Matrix at time $N = 4$

(ie) $\boxed{P(4) = I}$.

- Consider now a new input - output

$$y(N+1) = -2 \quad , \quad \bar{x}(N+1) = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

(ie) $y(5) = -2$, $\bar{x}(5) = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

The estimate $\hat{h}(N+1)$ can be evaluated as follows.

$$- \bar{k}(N+1) = \frac{P(N)\,\bar{x}(N+1)}{\sigma^2 + \bar{x}^T(N+1)\,P(N)\,\bar{x}(N+1)}$$

$$\text{(i.e.) } k(5) = \frac{P(4)\,\bar{x}(5)}{\sigma^2 + \bar{x}^T(5)\,P(4)\,\bar{x}(5)}$$

$$= \frac{I \times \begin{bmatrix} -2 \\ 2 \end{bmatrix}}{4 + \begin{bmatrix} -2 & 2 \end{bmatrix} I \times \begin{bmatrix} -2 \\ 2 \end{bmatrix}}$$

$$\boxed{k(5) = \frac{1}{12}\begin{bmatrix} -2 \\ 2 \end{bmatrix}}$$

$$- e(N+1) = y(N+1) - \bar{x}^T(N+1)\,\hat{h}(N)$$

$$\text{(i.e.) } e(5) = y(5) - \bar{x}^T(5)\,\hat{h}(4)$$

$$= -2 - \begin{bmatrix} -2 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ -1/2 \end{bmatrix}$$

$$= -2 - 1$$

$$\boxed{e(5) = -3}$$

Now, the estimate $\hat{h}(N+1) = \hat{h}(N) + \bar{k}(N+1)\,e(N+1)$

$$\text{(i.e.) } \hat{h}(5) = \hat{h}(4) + \bar{k}(5)\,e(5)$$

$$= \begin{bmatrix} -1 \\ -1/2 \end{bmatrix} + \frac{1}{12}\begin{bmatrix} -2 \\ 2 \end{bmatrix}(-3)$$

$$\boxed{\hat{h}(5) = \begin{bmatrix} -1/2 \\ -1 \end{bmatrix}}$$

The update procedure has very low complexity as there is no Matrix inversion. Hence it is very well suitable for practical implementation.

Week 8 ; Session 3.

The error covariance update for time N+1 is

$$P(N+1) = \left(I - \bar{k}(N+1)\,\bar{x}^T(N+1)\right) P(N)$$

2x2

(ii) $$P(5) = \left(I - \bar{k}(5)\,\bar{x}^T(5)\right) P(4)$$

$$= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{12}\begin{bmatrix} -2 \\ 2 \end{bmatrix}\begin{bmatrix} -2 & 2 \end{bmatrix}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1/3 & -1/3 \\ -1/3 & 1/3 \end{bmatrix}\right)$$

$$\boxed{P(5) = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}}$$