2. Linear Algebra for ML

Vectors

Consider 2 n-dimensional vectors

$$\bar{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad \bar{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

The corresponding 2 n-dimensional <u>row vectors</u> are given by <u>transpose</u>

$$\bar{u}^T = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}$$

$$\bar{v}^T = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$$

Inner product

The inner product is defined as

$$\langle \bar{u}, \bar{v} \rangle = \bar{u}^T \bar{v}$$

$$= \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$= \sum_{i=1}^{n} u_i v_i$$

Norm of a vector

Norm of a vector, denoted by $\|\bar{u}\|$, is defined as

$$\|\bar{u}\|^2 = \langle \bar{u}, \bar{u} \rangle = \bar{u}^T u = u_1^2 + u_2^2 + \dots + u_n^2$$

$$\|u\| = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$$

Matrices

A matrix is a $m \times n$ array of scalars.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Identity Matrix

I is the Identity matrix.

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Matrix inverse

- Inverse is defined only for square ($n \times n$) matrices.
- If inverse of matrix A exists, it is denoted by A^{-1}, and satisfies the property

$$A^{-1}A = AA^{-1} = I$$

- If inverse exists, A is termed INVERTIBLE (or) NON-SINGULAR.

  If inverse does not exist, A is termed SINGULAR.

- Let $|A|$ denote the determinant of A.

  Inverse of A exists if and only if

$$|A| = \det(A) \neq 0$$

Linear system of equations

- consider the system of linear equations

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

- This can be expressed using matrices and vectors as

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\bar{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}_{\bar{b}}$$

- If A is invertible, this can be solved as

$$A\bar{x} = \bar{b}$$
$$\Rightarrow \bar{x} = A^{-1}\bar{b}$$

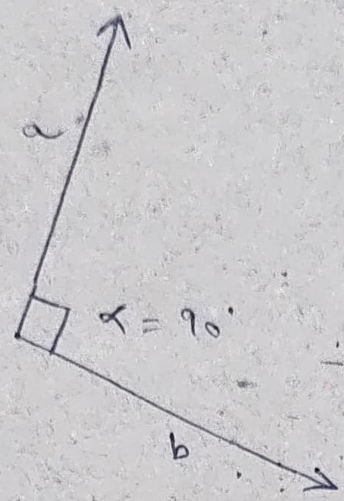
Inner product (for complex vectors)

The inner product is defined as

$$\langle \bar{u}, \bar{v} \rangle = \bar{u}^H \bar{v}$$
$$= \begin{bmatrix} u_1^* & u_2^* & \dots & u_n^* \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$
$$= \sum_{i=1}^{n} u_i^* v_i$$

Perpendicular vectors

orthogonal vectors.



orthogonality

Two vectors are orthogonal, if their inner product is zero.

$$\langle \bar{u}, \bar{v} \rangle = 0$$

$$\Rightarrow \bar{u}^T \bar{v} = 0 \quad (\text{Real vectors})$$

$$\bar{u}^H \bar{v} = 0 \quad (\text{complex vectors})$$

CS inequality

$$\boxed{\begin{array}{c}\text{Cauchy - Schwarz (CS) inequality} \\ |\bar{u}^T \bar{v}|^2 \leq \|\bar{u}\|^2 \, \|\bar{v}\|^2\end{array}}$$

$$|\bar{u}.\bar{v}|^2 = \|\bar{u}\|^2 \, \|\bar{v}\|^2 \cos^2\theta$$

$$\leq \|\bar{u}\|^2 \, \|\bar{v}\|^2$$

Problem 1 :

Consider $\bar{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $\bar{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. $\bar{u}^T \bar{v} = ?$

$$\bar{u}^T \bar{v} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= (1)(1) + (2)(1) + (3)(1) + (4)(1)$$

$$= 10$$

Problem 2 :

consider $\bar{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$. $\|\bar{u}\|^2 = ?$

$\|\bar{u}\| = ?$

$$\|\bar{u}\|^2 = (1)^2 + (2)^2 + (3)^2 + (4)^2$$

$$= 1 + 4 + 9 + 16$$

$$= 30$$

$$\|\bar{u}\| = \sqrt{30}.$$

Problem 3 :

consider $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. $\det(A) = ?$

$A^{-1} = ?$

Check $AA^{-1} = I$.

$$\det(A) = 4 - 6 = -2 \neq 0.$$

$$A^{-1} = \frac{1}{\det(A)} \times \operatorname{Adj}(A) = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Problem 4 :

consider $2x + 3y = 1$ $\quad$ $A = ?$ $\quad \bar{b} = ?$ $\quad A^{-1} = ?$

$4x - y = 9$. $\quad \bar{x} = ?$

$$\underbrace{\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\bar{x}} = \underbrace{\begin{bmatrix} 1 \\ 9 \end{bmatrix}}_{\bar{b}}$$

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

$$= \frac{1}{-14} \begin{bmatrix} -1 & -3 \\ -4 & 2 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 1 & 3 \\ 4 & -2 \end{bmatrix}$$

$$\bar{x} = A^{-1} \bar{b} = \frac{1}{14} \begin{bmatrix} 1 & 3 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 9 \end{bmatrix}$$

$$= \frac{1}{14} \begin{bmatrix} 28 \\ -14 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Problem 5 :

Which is the vector orthogonal to $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

To find a vector orthogonal to $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, we need a vector $\begin{bmatrix} a \\ b \end{bmatrix}$ such that $\begin{bmatrix} -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = 0$

$$\Rightarrow -2a + 1b = 0$$

$$\Rightarrow b = 2a$$

Therefore, any vector of the form $\begin{bmatrix} a \\ 2a \end{bmatrix}$ is orthogonal to $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Let $a = 1$, then Orthogonal vector $= \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Check : $\begin{bmatrix} -2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = (-2)(1) + (1)(2) = 0 \Rightarrow$ Orthogonal.

We can also scale it. Eg. $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$, $\begin{bmatrix} -1 \\ -2 \end{bmatrix}$, etc., are all orthogonal to $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Problem 6 :

- Consider $\bar{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $\bar{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. $|\bar{u}^T\bar{v}|^2 = ?$

  $\|\bar{u}\|^2 \|\bar{v}\|^2 = ?$

(i) Compute $|\bar{u}^T\bar{v}|^2$

The dot product, $\bar{u}^T \cdot \bar{v} = (1)(1) + (2)(1) + (3)(1) + (4)(1)$

$= 1 + 2 + 3 + 4 = 10$

Then, $|\bar{u}^T\bar{v}|^2 = |10|^2 = 100$

(ii) Compute $\|\bar{u}\|^2 \cdot \|\bar{v}\|^2$

$\|\bar{u}\|^2 = 1^2 + 2^2 + 3^2 + 4^2 = 1 + 4 + 9 + 16 = 30$

$\|\bar{v}\|^2 = 1^2 + 1^2 + 1^2 + 1^2 = 4$

$\|\bar{u}\|^2 \cdot \|\bar{v}\|^2 = (30) \cdot (4) = 120$

Final answer :

$|\bar{u}^T\bar{v}|^2 = 100$

$\|\bar{u}\|^2 \cdot \|\bar{v}\|^2 = 120$

This shows :-

$$|\bar{u}^T\bar{v}|^2 \leq \|\bar{u}\|^2 \cdot \|\bar{v}\|^2$$

which is Cauchy - Schwarz inequality.