7. Linear Discriminant Analysis.

Discriminant functions.

Consider a classifier built using functions

$$g_i(\bar{x}), \quad i = 1, 2, \ldots, L$$

such that the input vector $\bar{x}$ is assigned to class l if

$$g_l(\bar{x}) = \max_{1 \leq i \leq L} g_i(\bar{x})$$

These $g_i(\bar{x})$ are termed discriminant functions.

Gaussian density.

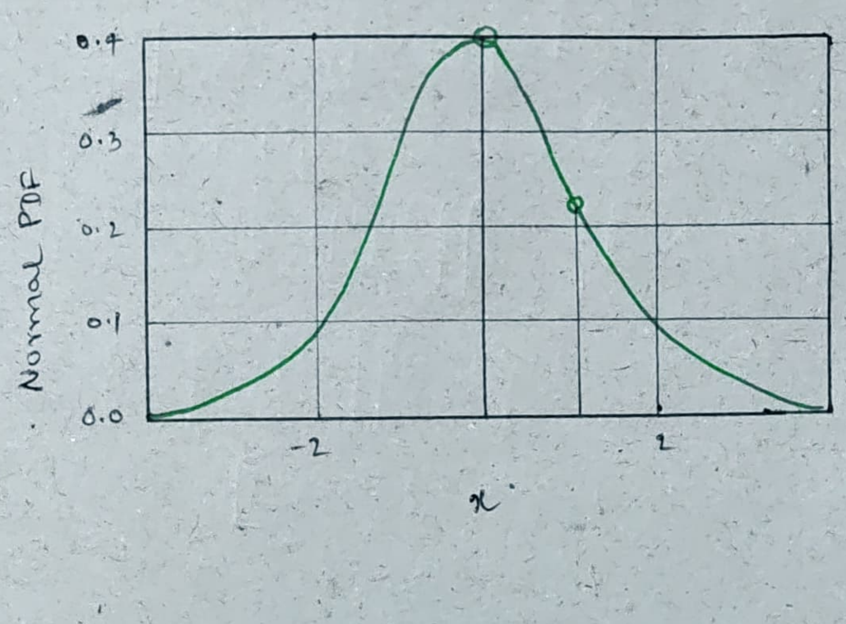
Recall, the expression for the Gaussian PDF is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

The Mean and Variance of the Gaussian RV are

$$E\{X\} = \mu$$

$$E\{(X-\mu)^2\} = \sigma^2$$



<u>Multivariate Gaussian</u>

The PDF of a Gaussian random vector is given as

$$f_{\bar{x}}(\bar{x}) = \frac{1}{\sqrt{(2\pi)^n |R|}} e^{-\frac{1}{2}(\bar{x}-\bar{\mu})^T R^{-1} (\bar{x}-\bar{\mu})}$$

The Mean and Covariance Matrix are defined as

$$E\{\bar{x}\} = \bar{\mu}$$

$$E\{\underbrace{(\bar{x}-\bar{\mu})(\bar{x}-\bar{\mu})^T}_{\text{outer product}}\} = R$$

<u>Problem 1</u>

Given R below, find $|R|$ and R^{-1}.

$$R = \begin{bmatrix} 7 & 2 \\ 2 & 1 \end{bmatrix}$$

Covariance Matrix, $R = \begin{bmatrix} 7 & 2 \\ 2 & 1 \end{bmatrix}$

$$|R| = (7)(1) - (2)(2) = 3$$

$$R^{-1} = \frac{1}{|R|} \operatorname{adj}(R) = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ -2 & 7 \end{bmatrix}$$

<u>Problem 2</u>

Find $(\bar{x}-\bar{\mu})^T R^{-1} (\bar{x}-\bar{\mu})$, given $\bar{\mu} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

The quantity in exponent can be simplified as

$$\frac{1}{3}\begin{bmatrix} x_1-1 & x_2-2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} x_1-1 \\ x_2-2 \end{bmatrix}$$

$$= \frac{1}{3}\left[(x_1-1)^2 + 7(x_2-2)^2 - 2(x_1-1)(x_2-2) - 2(x_1-1)(x_2-2)\right]$$

$$= \frac{1}{3}\left[x_1^2 + 7x_2^2 + 6x_1 - 24x_2 - 4x_1x_2 + 21\right]$$

Problem 3

Find the multivariate Gaussian PDF. $(n=2)$

$$f_{\bar{X}}(\bar{x}) = \frac{1}{\sqrt{(2\pi)^n |R|}} e^{-\frac{1}{2}(\bar{x}-\bar{\mu})^T R^{-1} (\bar{x}-\bar{\mu})}$$

$$= \frac{1}{\sqrt{12\pi^2}} e^{-\frac{1}{6}(x_1^2 + 7x_2^2 + 6x_1 - 24x_2 - 4x_1x_2 + 21)}$$

Gaussian discriminant

* Consider the input vectors $\bar{x}$ drawn from two Gaussian classes

C_0 : Mean $\bar{\mu}_0$ and Covariance R

C_1 : Mean $\bar{\mu}_1$ and Covariance R

* Also termed Gaussian Discriminant Analysis.
* The Likelihoods of the two classes are

$$p(\bar{x}; C_0) = \frac{1}{\sqrt{(2\pi)^n |R|}} e^{-\frac{1}{2}(\bar{x}-\bar{\mu}_0)^T R^{-1} (\bar{x}-\bar{\mu}_0)}$$

$$p(\bar{x}; C_1) = \frac{1}{\sqrt{(2\pi)^n |R|}} e^{-\frac{1}{2}(\bar{x}-\bar{\mu}_1)^T R^{-1} (\bar{x}-\bar{\mu}_1)}$$

* Choose the class that maximizes the likelihood.

Choose C_0 if

$$p(\bar{x}; C_0) \geq p(\bar{x}; C_1)$$

$$\Rightarrow \cancel{\frac{1}{\sqrt{(2\pi)^n |R|}}} e^{-\frac{1}{2}(\bar{x}-\bar{\mu}_0)^T R^{-1} (\bar{x}-\bar{\mu}_0)} \geq \cancel{\frac{1}{\sqrt{(2\pi)^n |R|}}} e^{-\frac{1}{2}(\bar{x}-\bar{\mu}_1)^T R^{-1} (\bar{x}-\bar{\mu}_1)}$$

$$\Rightarrow (\bar{x}-\bar{\mu}_0)^T R^{-1} (\bar{x}-\bar{\mu}_0) \leq (\bar{x}-\bar{\mu}_1)^T R^{-1} (\bar{x}-\bar{\mu}_1)$$

This discriminant function can be simplified as

$$\text{Choose } C_0 : \bar{h}^T(\bar{x}-\bar{\mu}) \geq 0$$

$$\text{Choose } C_1 : \bar{h}^T(\bar{x}-\bar{\mu}) < 0$$

where,

$$\bar{\mu} = \frac{1}{2}(\bar{\mu}_0 + \bar{\mu}_1)$$

$$\bar{h} = R^{-1}(\bar{\mu}_0 - \bar{\mu}_1)$$

Linear classifier

Thus, the classifier is linear, and is characterized by the hyperplane.

$$\bar{h}^T(\bar{x}-\bar{\mu}) = 0.$$

Problem 4:

Given the two classes C_0, C_1 are distributed as

$$C_0 \sim \mathcal{N}\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1/2 & 0 \\ 0 & 1/4 \end{bmatrix}\right)$$

$$C_1 \sim \mathcal{N}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/2 & 0 \\ 0 & 1/4 \end{bmatrix}\right)$$

Calculate $\bar{h}, \bar{\mu}$.

$$\bar{\mu} = \frac{1}{2}(\bar{\mu}_0 + \bar{\mu}_1) = \frac{1}{2}\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\bar{h} = R^{-1}(\bar{\mu}_0 - \bar{\mu}_1) = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 4 \end{bmatrix}$$

Problem 5 :

For the same classes C_0 and C_1, determine the classifier.

$$\bar{h}^T(\bar{x}-\bar{\mu}) = \begin{bmatrix} -6 & 4 \end{bmatrix}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} -6 & 4 \end{bmatrix}\left(\bar{x} - \frac{1}{2}\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) \geq 0$$

$$\boxed{\begin{array}{ll} \text{Choose } C_0 \text{ if} & -6x_1 + 4x_2 - 3 \geq 0 \\ \text{Choose } C_1 \text{ if} & -6x_1 + 4x_2 - 3 < 0. \end{array}}$$

$$\max\left[(-6x_1 + 4x_2 - 3),\ (6x_1 - 4x_2 + 3)\right]$$