MIMO TECHNOLOGY

(MULTIPLE INPUT MULTIPLE OUTPUT)

- Multiple Input → Multiple Transmit Antennas
  → Multiple Input Symbols
  $x_1, x_2, \ldots$
- Multiple Output → Multiple Receive Antennas
  → Multiple Output Symbols
  $y_1, y_2, \ldots$

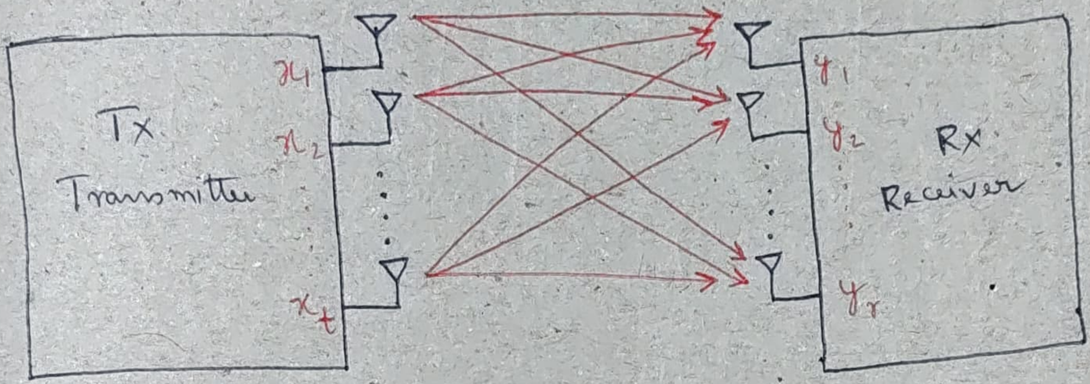

Fig: MIMO WIRELESS SYSTEM

- MIMO is a key technology in 4G LTE and 5G NR
- It is also extensively used in Wi-Fi
  – 802.11 n/ac/ax WLAN Standards.
- MIMO can lead to significant increase in data rates via parallel transmission of multiple streams. This is termed as SPATIAL MULTIPLEXING!

MULTIPLEX SEVERAL INFORMATION STREAMS IN SPATIAL DOMAIN IN SAME TIME AND SAME BANDWIDTH.

- Capacity increases manifold.

MIMO System model

$r \rightarrow$ No. of Receive antennas
$t \rightarrow$ No. of Transmit antennas $\Bigg\} \Rightarrow r \times t$ MIMO System

Example.

$\left.\begin{array}{l}\#\text{ Rx. antennas} = 4 \\ \#\text{ Tx. antennas} = 3\end{array}\right\} \Rightarrow 4 \times 3$ MIMO System

MIMO system model is mathematically described as follows.

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_r \end{bmatrix}_{r \times 1}}_{\substack{\text{Output Vector} \\ (\bar{y})}} = \underbrace{\begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1t} \\ h_{21} & h_{22} & \cdots & h_{2t} \\ \vdots & & & \\ h_{r1} & h_{r2} & \cdots & h_{rt} \end{bmatrix}_{r \times t}}_{\substack{\text{MIMO channel Matrix} \\ (H)}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_t \end{bmatrix}_{t \times 1}}_{\substack{\text{Transmit Vector} \\ (\bar{x})}} + \underbrace{\begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_r \end{bmatrix}}_{\substack{\text{Noise Vector} \\ (\bar{n})}}$$

$$\Rightarrow \bar{y} = H\bar{x} + \bar{n}$$

where, $h_{ij} \rightarrow$ Channel coefficient b/w i^{th} receive antenna and j^{th} transmit antenna.

Eg. $h_{32} \rightarrow$ channel coefficient b/w $\underline{3^{rd}}$ receive antenna and $\underline{2^{nd}}$ transmit antenna.

MIMO Example

Consider 3×2 MIMO system.

\# Rx. Antennas = 3

\# Tx. Antennas = 2

outputs : y_1, y_2, y_3

inputs : x_1, x_2

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \\ h_{31} & h_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

$$\bar{y} = H \bar{x} + \bar{n}$$

MIMO System of Equations.

$$\left.\begin{aligned} y_1 &= h_{11} x_1 + h_{12} x_2 + n_1 \\ y_2 &= h_{21} x_1 + h_{22} x_2 + n_2 \\ y_3 &= h_{31} x_1 + h_{32} x_2 + n_3 \end{aligned}\right\} \begin{array}{c} 3 \times 2 \\ \text{MIMO} \\ \text{system} \\ \text{model.} \end{array}$$

MIMO Receiver.

Design the MIMO receiver.

(i) Given $\bar{y}$, how to determine $\bar{x}$?

MIMO System of linear equations can be written as

$$\begin{array}{c} \text{'r'} \\ \text{Equations} \\ (y_1, y_2, \dots y_r) \end{array} \left\{\begin{aligned} y_1 &= h_{11} x_1 + h_{12} x_2 + \dots + h_{1t} x_t \\ y_2 &= h_{21} x_1 + h_{22} x_2 + \dots + h_{2t} x_t \\ &\vdots \\ y_r &= \underbrace{h_{r1} x_1 + h_{r2} x_2 + \dots + h_{rt} x_t}_{\substack{\text{'t' unknowns.} \\ (x_1, x_2, \dots, x_t)}} \end{aligned}\right.$$

Case (i) $r = t$. (ie) # Equations = # Unknowns

- In this case, H is a SQUARE matrix
- If H is non-singular / invertible / $\text{Det}(H) \neq 0$, H^{-1} exists.
- (ie) $\bar{y} = H\bar{x}$ has UNIQUE solution.
- The unique solution is given as

$$\boxed{\hat{x} = H^{-1}\bar{y}}$$

Estimated vector.

Case (ii) $r > t$. (ie) # Equations > # Unknowns

- In this case, H is NOT A SQUARE MATRIX. Rather, H is a TALL Matrix
- H is not invertible.
- Typically, in such case, NO solution. Hence we need to find an approximate solution!

$$\bar{y} = H\bar{x}$$

$$\Rightarrow \bar{y} - H\bar{x} = \bar{e} \longrightarrow \text{Error}.$$

Find $\hat{x}$ such that Error is minimum.

(ie) $\min \|\bar{e}\|^2 = \min \|\bar{y} - H\bar{x}\|^2$

This is termed as Least-Squares problem.

The solution to LS problem is given by

$$\boxed{\hat{x} = (H^H H)^{-1} H^H \bar{y}}$$

This is termed as Zero Forcing (ZF) receiver).

Note $\hat{x} = \underbrace{(H^H H)^{-1} H^H}\,\bar{y}$

Even though, H is a TALL matrix and is not invertible;

$(H^H H)^{-1} H^H$ is acting as an inverse of H.

(ie) $\underbrace{(H^H H)^{-1} H^H} \times H = I$.

Therefore, $(H^H H)^{-1} H^H$ is a Pseudo-inverse.

This is represented as H^{+} (H Dagger).

MIMO ZF RECEIVER EXAMPLE.

Consider $\bar{y} = \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix}$ and $H = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}_{4\times2}$

What is $\hat{x}$?

Since, H is not a Square Matrix (TALL MATRIX)

(ie) $r > t$, the Estimate of the input vector $\bar{x}$ can be done using LS solution (ZF receiver).

(ie) $\hat{x} = (H^H H)^{-1} H^H \bar{y}$

$\Rightarrow \boxed{\hat{x} = (H^T H)^{-1} H^T \bar{y}}$ (For Real Matrix, $H^H = H^T$)

$$H^T H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}_{2\times4} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}_{4\times2} = \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix}$$

$$(H^T H)^{-1} = \frac{1}{\det(H^T H)} (\text{Adjoint of } H^T H)$$

$\det(H^T H) = 4 \times 30 - 10 \times 10 = 20$

$$(H^T H)^{-1} = \frac{1}{20} \begin{bmatrix} 30 & -10 \\ -10 & 4 \end{bmatrix}.$$

$$(H^T H)^{-1} H^T = \frac{1}{20} \underbrace{\begin{bmatrix} 30 & -10 \\ -10 & 4 \end{bmatrix}}_{2\times 2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}_{2\times 4}$$

$$= \frac{1}{20} \begin{bmatrix} 20 & 10 & 0 & -10 \\ -6 & -2 & 2 & 6 \end{bmatrix}$$

Pseudo-Inverse (Quick check)

$$\underline{(H^T H)^{-1} H^T \times H} = I = \underline{H \times (H^T H)^{-1} H^T}$$

$$\frac{1}{20} \begin{bmatrix} 20 & 10 & 0 & -10 \\ -6 & -2 & 2 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\left.\begin{array}{l}\text{The estimate of the} \\ \text{input vector } \bar{x}\end{array}\right\} = \hat{x} = (H^T H)^{-1} H^T \bar{y}$$

$$= \frac{1}{20} \begin{bmatrix} 20 & 10 & 0 & -10 \\ -6 & -2 & 2 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix}$$

$$\boxed{\hat{x} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}}$$

LMMSE receiver (LINEAR MINIMUM MEAN SQUARE ERROR)

- Another popular MIMO receiver is the LMMSE receiver
- Linear Estimate : $\hat{x} = C^H \bar{y}$
- LMMSE explained :

$$\underset{\text{(2) Minimum}}{\min}\ \underset{\text{(3) Mean}}{E}\left\{\underbrace{\left\|\overbrace{C^H\bar{y}}^{\text{(1) Linear Transformation}}-\bar{x}\right\|^2}_{\text{(4) Square Error}}\right\}$$

- Note the following quantities.

  (1) COVARIANCE MATRIX of $\bar{x}$

  $\rightarrow R_{xx} = E\{\bar{x}\,\bar{x}^H\}$

  $\rightarrow$ $t \times t$ Matrix

  (2) COVARIANCE MATRIX of $\bar{y}$

  $\rightarrow R_{yy} = E\{\bar{y}\,\bar{y}^H\}$

  $\rightarrow$ $r \times r$ Matrix

  (3) CROSS COVARIANCE MATRIX

  $\rightarrow R_{xy} = E\{\bar{x}\,\bar{y}^H\}$

  $\rightarrow$ $t \times r$ Matrix

- LMMSE receiver is given as

$$\boxed{\hat{x} = R_{xy} R_{yy}^{-1} \bar{y}}$$

⊙ The Covariance Matrix of $\bar{x}$ can be derived as

$$R_{xx} = E\{\bar{x}\bar{x}^H\} = PI.$$

∵ the transmit symbols are iid with mean 0 and power P.

$$\text{iid} \quad E\{x_i x_j^*\} = 0 \quad , \text{if } i \neq j$$
$$\left.\begin{aligned} &= E\{|x_i|^2\} \\ &= P \end{aligned}\right\} \text{if } i = j$$

⊙ The Covariance Matrix of $\bar{y}$ can be derived as

$$R_{yy} = E\{\bar{y}\bar{y}^H\}$$
$$= E\{(H\bar{x}+\bar{n})(H\bar{x}+\bar{n})^H\}$$
$$= E\{(H\bar{x}+\bar{n})(\bar{x}^H H^H + \bar{n}^H)\}$$
$$= E\{H\bar{x}\bar{x}^H H^H + \bar{n}\bar{x}^H H^H + H\bar{x}\bar{n}^H + \bar{n}\bar{n}^H\}$$
$$= H\underbrace{E\{\bar{x}\bar{x}^H\}}_{PI}H^H + \underbrace{E\{\bar{n}\bar{x}^H\}}_{0}H^H + \underbrace{E\{\bar{x}\bar{n}^H\}}_{0}H + \underbrace{E\{\bar{n}\bar{n}^H\}}_{N_0 I}$$
$$= H \cdot PI \cdot H^H + N_0 I$$
$$= P \cdot HH^H + N_0 I$$

Note.

① $E\{\bar{n}\bar{x}^H\} = E\{\bar{x}\bar{n}^H\} = 0$

Noise and Transmit vector are uncorrelated. Hence, cross covariance b/w noise and Transmit vector is zero.

② Noise samples are iid across the antennas with Power N_0.

$$(a)\ E\{n_i n_j^*\} = 0 \quad , \text{if } i \neq j$$
$$\left.\begin{aligned} &= E\{|n_i|^2\} \\ &= N_0 \end{aligned}\right\} \text{if } i = j$$

⊙ The Cross covariance Matrix of $\bar{x}, \bar{y}$ can be derived as

$$R_{xy} = E\{\bar{x}\bar{y}^H\} = E\{\bar{x}(H\bar{x}+\bar{n})^H\}$$

$$= E\{\bar{x}(\bar{x}^H H^H + \bar{n}^H)\}$$

$$= \underbrace{E\{\bar{x}\bar{x}^H\}}_{PI} H^H + \underbrace{E\{\bar{x}\bar{n}^H\}}_{0}$$

$$= PI \cdot H^H + 0$$

$$= P \cdot H^H$$

Therefore, the LMMSE receiver

$$\hat{x} = R_{xy} R_{yy}^{-1} \bar{y}$$

$r \times r$ Matrix

$$= P H^H \cdot \overbrace{(PHH^H + N_0 I)}^{}{}^{-1} \cdot \bar{y}$$

$$= H^H \left(HH^H + \frac{N_0}{P} I\right)^{-1} \bar{y}$$

$$\boxed{\hat{x} = \underbrace{\left(HH^H + \frac{1}{SNR} I\right)^{-1}}_{} H^H \bar{y}}$$

MUCH LOWER COMPUTATIONAL COMPLEXITY.

$t \times t$ Matrix

Note: Inversion of $t \times t$ Matrix is much easier than inversion of $r \times r$ Matrix, since $r \geq t$ (typically)

At high SNR ($SNR \to \infty$), $\frac{1}{SNR} \to 0$.

$$\Rightarrow \hat{x} = (HH^H)^{-1} H^H \bar{y} \rightarrow \text{ZF receiver.}$$

∴ At very high SNR, LMMSE $\rightarrow$ ZF

MIMO LMMSE RECEIVER EXAMPLE

Consider $\bar{y} = \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix}$ and $H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$.

What is $\hat{x}$ when SNR $= -3\,\text{dB} = 0.5$.

The LMMSE Estimate is given by

$$\hat{x} = \left(H^H H + \frac{1}{SNR} I\right)^{-1} H^H \bar{y}$$

$$\Rightarrow \boxed{\hat{x} = \left(H^T H + \frac{1}{SNR} I\right)^{-1} H^T \bar{y}}$$ (For Real Matrix, $H^H = H^T$)

$$H^T H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}_{2\times4} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}_{4\times2} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = 4I$$

$$H^T H + \frac{1}{SNR} I = 4I + \frac{1}{0.5} I = 4I + 2I = 6I.$$

$$\left(H^T H + \frac{1}{SNR} I\right)^{-1} = (6I)^{-1} = \frac{1}{6} I$$

$$\left(H^T H + \frac{1}{SNR} I\right)^{-1} H^T \bar{y} = \frac{1}{6} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\boxed{\hat{x} = \frac{1}{6} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$