SVD (SINGULAR VALUE DECOMPOSITION)

- One of the most important techniques for MIMO processing.
- WARNING! NOT TO CONFUSE WITH EIGEN VALUE DECOMPOSITION. (EVD)

  Eigen value Decomposition exists only for SQUARE matrix, where as SVD exists for ANY matrix.

- Given a matrix H with $r \geq t$,

  SVD is defined as

$$H = \underbrace{U}_{r \times r \text{ Square Matrix}} \ \underbrace{\Sigma}_{r \times t \text{ Tall Matrix}} \ \underbrace{V^H}_{t \times t \text{ Square Matrix}}$$

- Properties
  1. The matrices U and V are UNITARY matrices.

$$U^H U = U U^H = I$$

$$V^H V = V V^H = I.$$

  2. The matrix Σ has the following structure.

$$\Sigma = \left.\begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & \sigma_t \\ 0 & & & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix}\right\} r \times t \qquad \begin{matrix} \} \ t \times t \\ \\ \} \ (r-t) \times t \end{matrix}$$

The diagonal values are SINGULAR (Real & non Negative) $(\sigma_i \geq 0)$, which are arranged in DECREASING order.

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_t \geq 0.$$

No. of $\underline{\text{non-zero singular values}}$ σ_i equals the Rank of H.

$\underline{\underline{\text{Example}}}$ σ_1 σ_2 σ_3 σ_4

$3.8 \geq 1.6 \geq 0.35 \geq 0$

$\Rightarrow \boxed{\text{RANK} = 3}$

③ The columns of U are $\underbrace{\text{ORTHO}}$ $\underbrace{\text{NORMAL}}$.

$\bar{u}_i^H \bar{u}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$ $\|\bar{u}_i\|^2 = 1$

$\Rightarrow \|\bar{u}_i\| = 1$

$U = [\bar{u}_1\ \bar{u}_2 \ldots \bar{u}_r]$

$U \rightarrow$ LEFT SINGULAR Matrix

$\bar{u}_i \rightarrow$ LEFT SINGULAR Vectors

$\bar{u}_1 \rightarrow$ Dominant Left Singular Vector

(Each vector is $r \times 1$).

④ The columns of V are also $\underbrace{\text{ORTHO}}$ $\underbrace{\text{NORMAL}}$

$\bar{v}_i^H \bar{v}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$ $\|\bar{v}_i\|^2 = 1$

$\Rightarrow \|\bar{v}_i\| = 1$

$V = [\bar{v}_1, \bar{v}_2 \ldots \bar{v}_r]$

$V \rightarrow$ RIGHT SINGULAR Matrix

$\bar{v}_i \rightarrow$ RIGHT SINGULAR Vectors

$\bar{v}_1 \rightarrow$ Dominant Right Singular Vector

(Each Vector is $t \times 1$)

⑤ Relation to Eigen Value Decomposition (EVD).

$$H^H H = V \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & \sigma_t^2 \end{bmatrix}_{t \times t} V^H$$

- $\sigma_1^2, \sigma_2^2, \ldots \sigma_t^2$ are 't' Eigen Values of $H^H H$
- $\underbrace{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_t}$ are the Eigen vectors of $H^H H$

← RIGHT SINGULAR VECTORS

$$HH^H = U \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots & & \vdots \\ 0 & 0 & & \sigma_t^2 & 0 & 0 & & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & 0 & 0 & 0 & & 0 \end{bmatrix}_{r \times r} U^H \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} t \\ \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} (r-t) \end{array} \Bigg\} r$$

- $\sigma_1^2, \sigma_2^2, \ldots, \sigma_t^2$ are 't' non-zero Eigen values of HH^H. And rest $(r-t)$ eigen values of HH^H are zero!
- $\underbrace{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_r}$ are the Eigen vectors of HH^H.

← LEFT SINGULAR VECTORS

SVD EXAMPLES

Consider the Matrix $H = \begin{bmatrix} 1 & 2 \\ 1 & -2 \\ 1 & -2 \\ 1 & 2 \end{bmatrix}$.

The SVD is given as

$$\underbrace{\begin{bmatrix} 1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix}_{4\times4}}_{U} \underbrace{\begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{4\times2}}_{\Sigma} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{2\times2}}_{V^H}$$

Let us explore each of the above.

(i) $\Sigma = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{4\times2}$

→ $r \times t$ Tall Matrix

→ $\sigma_1 = 4 \geq 0$

$\sigma_2 = 2 \geq 0$

$\sigma_1 \geq \sigma_2 \geq 0$

(ie) σ_i are NON NEGATIVE, REAL and in DECREASING ORDER

→ Two non-zero singular values.

$\boxed{\text{RANK} = 2}$

(ii) $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

→ $t \times t$ Square Matrix

→ V is a UNITARY Matrix

(i) $VV^H = V^H V = 0$

$\Rightarrow VV^H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

(ii) $V^H V = I$.

→ Columns of V are ORTHONORMAL

$\Rightarrow \bar{v}_i^H \bar{v}_j = 0 \, , \, i \neq j$

(ie) $\bar{v}_1^H \bar{v}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$

$\Rightarrow \bar{v}_i^H \bar{v}_j = 1 \, , \, i = j$

(ie) $\bar{v}_1^H \bar{v}_1 = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1$

$\bar{v}_2^H \bar{v}_2 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1$

$\Rightarrow \| \bar{v}_i \|^2 = 1$

(ie) $\| \bar{v}_1 \|^2 = 0 + 1 = 1$

$\| \bar{v}_2 \|^2 = 1 + 0 = 1$

→ $\bar{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is the Dominant Right Singular vector.

(iii) $U = \begin{bmatrix} 1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix}$

→ $r \times r$ Square Matrix

→ U is a UNITARY Matrix.

(ie) $UU^H = U^H U = 0.$

$$U U^H = \begin{bmatrix} 1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

|||ly $U^H U = I.$

$\rightarrow$ Columns of U are ORTHONORMAL

$\Rightarrow \bar{u}_i^H \bar{u}_j = 0$, if $i \neq j$

(ie) $\bar{u}_1^H \bar{u}_2 = \begin{bmatrix} 1/2 & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = 0$

$\text{|||}^{ly}\ \bar{u}_2^H \bar{u}_3 = \bar{u}_3^H \bar{u}_4 = \bar{u}_4^H \bar{u}_1 = 0$

$\Rightarrow \bar{u}_i^H \bar{u}_j = 1$, if $i = j$

(ie) $\bar{u}_1^H \bar{u}_1 = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} = 1$

$\text{|||}^{ly}\ \bar{u}_2^H \bar{u}_2 = \bar{u}_3^H \bar{u}_3 = \bar{u}_4^H \bar{u}_4 = 1$

$\Rightarrow \|\bar{u}_i\|^2 = 1$

(ie) $\|\bar{u}_1\|^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{-1}{2}\right)^2 + \left(\frac{-1}{2}\right)^2 + \left(\frac{1}{2}\right)^2$

$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$

$\text{|||}^{ly}\ \|\bar{u}_2\|^2 = \|\bar{u}_3\|^2 = \|\bar{u}_4\|^2 = 1$

$\rightarrow \bar{u}_1 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$ is the Dominant

Left singular vector.

SVD MIMO Processing

- SVD can be used for MIMO processing as follows.

MIMO channel model,

$$\bar{y} = H\bar{x} + \bar{n}$$

$\bar{n} \rightarrow r\times 1$ Noise Vector

$\bar{x} \rightarrow t\times 1$ Transmit vector

$H \rightarrow r\times t$ MIMO channel Matrix

$\bar{y} \rightarrow r\times 1$ Output vector

Substitute the SVD of $H = U\Sigma V^H$

$$\Rightarrow \bar{y} = (U\Sigma V^H)\bar{x} + \bar{n}$$

At the receiver, process using U^H.

(ie) $\tilde{y} = U^H \bar{y}$

$$= U^H(U\Sigma V^H\bar{x} + \bar{n})$$

$$= \underbrace{U^H U}\Sigma V^H\bar{x} + \underbrace{U^H\bar{n}}$$

$$= I\,\Sigma V^H x + \tilde{n}$$

$$\boxed{\tilde{y} = \Sigma V^H\bar{x} + \tilde{n}}$$

At the receiver, we multiply by U^H.
U^H is called as a COMBINER (or)
RECEIVE BEAMFORMER.

At the transmitter, pre-process using V
(ie) $\bar{x} = V\tilde{x}$.

$$\tilde{y} = \Sigma V^H \bar{x} + \tilde{n}$$
$$= \Sigma V^H V \tilde{x} + \tilde{n}$$
$$= \Sigma I \tilde{x} + \tilde{n}$$
$$= \Sigma \tilde{x} + \tilde{n}$$

At the transmitter, we preprocess using the matrix V. V is called as PRECODER (OR) TRANSMIT BEAMFORMING MATRIX.

As a result, we now have the model

$$\tilde{y} = \Sigma \tilde{x} + \tilde{n}$$

This can be explicitly written as

$$\tilde{y} = \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \vdots \\ \tilde{y}_r \end{bmatrix}_{r\times 1} = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & & \\ & & \ddots & \sigma_t \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{r\times t} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_t \end{bmatrix}_{t\times 1} + \begin{bmatrix} \tilde{n}_1 \\ \tilde{n}_2 \\ \vdots \\ \tilde{n}_t \\ \vdots \\ \tilde{n}_r \end{bmatrix}_{r\times 1}$$

$$\Rightarrow \left.\begin{aligned} \tilde{y}_1 &= \sigma_1 \tilde{x}_1 + \tilde{n}_1 \\ \tilde{y}_2 &= \sigma_2 \tilde{x}_2 + \tilde{n}_2 \\ &\vdots \\ \tilde{y}_t &= \sigma_t \tilde{x}_t + \tilde{n}_t \end{aligned}\right\}$$

→ t parallel streams
→ t DECOUPLED CHANNELS
→ very high data rate
→ SPATIAL MULTIPLEXING

$$\left.\begin{aligned} \tilde{y}_{t+1} &= 0 + \tilde{n}_{t+1} \\ \tilde{y}_{t+2} &= 0 + \tilde{n}_{t+2} \\ &\vdots \\ \tilde{y}_r &= 0 + \tilde{n}_r \end{aligned}\right\}$$

ONLY NOISE...
THESE CAN BE IGNORED.

Consider the i^{th} channel,

$$\tilde{y}_i = \sigma_i \tilde{x}_i + \tilde{n}_i$$

The output SNR of the i^{th} channel is given as

$$SNR_o^i = \frac{\sigma_i^2 \cdot P_i}{N_0}$$

where,

$P_i = E\{|\tilde{x}_i|^2\} \rightarrow$ Transmit Power

$N_0 \rightarrow$ output Power

$\sigma_i^2 \rightarrow$ Gain for the i^{th} channel

The Shannon Capacity (Maximum rate for Error free Transmission over a communication channel for given SNR) is given as

$$\log_2(1 + SNR_o^i) = \log_2\left(1 + \sigma_i^2 \cdot \frac{P_i}{N_0}\right)$$

Therefore, the Sum Capacity (Sum of the maximum rates of all the i channels) of the MIMO channel is

$$\sum_{i=1}^{t} R_i = \sum_{i=1}^{t} \log_2(1 + SNR_o^i)$$

$$= \sum_{i=1}^{t} \log_2\left(1 + \sigma_i^2 \frac{P_i}{N_0}\right)$$

There is a maximum transmit Power (P_0) for every transmitter. This is the total permissible transmit power.

$$\underbrace{P_1 + P_2 + \dots + P_t}_{} = \sum_{i=1}^{t} P_i = P_0$$

Sum of all Powers of MIMO Channels.

Maximum possible rate at which we can transmit over the MIMO channel for a Unit Bandwidth (bits/sec/Hz) is given by the Optimization problem given below.

$$\boxed{\text{Max} \sum_{i=1}^{t} \log_2\left(1+\sigma_i^2 \frac{P_i}{N_0}\right), \text{ subject to the constraint } \sum_{i=1}^{t} P_i = P_0}$$

This is the constrained optimization problem to achieve maximum sum-rate of the MIMO system in (bits/sec/unit Bandwidth) i.e (bits/sec/Hz). This is the Capacity of the MIMO System.

The constrained optimization problem can be handled using Lagrange Multiplier Technique.

Solving this optimization problem gives the following expression for the optimal power.

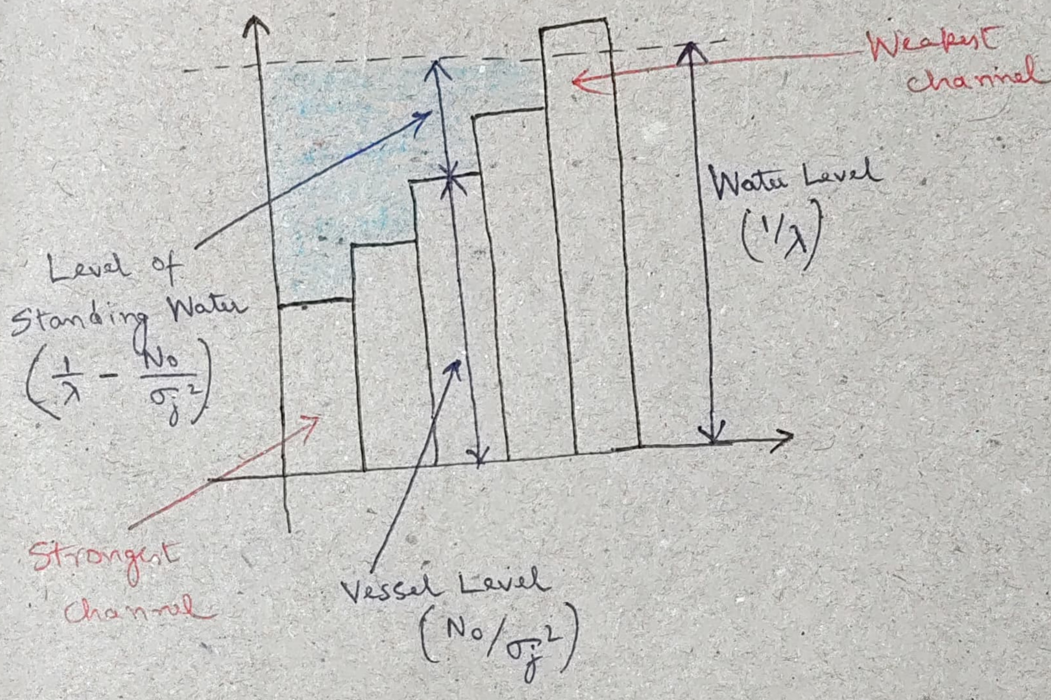
$$P_j = \left(\frac{1}{\lambda} - \frac{N_0}{\sigma_j^2}\right)^+$$

where, $\lambda \rightarrow$ Lagrange Multiplier

$$x^+ = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

$$\Rightarrow P_j = \begin{cases} \dfrac{1}{\lambda} - \dfrac{N_0}{\sigma_j^2}, & \text{if } \dfrac{1}{\lambda} - \dfrac{N_0}{\sigma_j^2} \geq 0 \\ & \Rightarrow \dfrac{1}{\lambda} \geq \dfrac{N_0}{\sigma_j^2} \\ 0, & \text{if } \dfrac{1}{\lambda} - \dfrac{N_0}{\sigma_j^2} < 0 \\ & \Rightarrow \dfrac{1}{\lambda} < \dfrac{N_0}{\sigma_j^2} \end{cases}$$

This is also termed as Water-Filling / water-pouring Power allocation.



(i) σ_i = Small

$\Rightarrow \dfrac{1}{\lambda} - \dfrac{N_0}{\sigma_j^2}$ = small

$\Rightarrow$ Weaker channels are allocated lower power

(ii) σ_i = large

$\Rightarrow \dfrac{1}{\lambda} - \dfrac{N_0}{\sigma_j^2}$ = large

$\Rightarrow$ Stronger channels are allocated more power.

MIMO Capacity Example

- Channel Matrix, $H = \begin{bmatrix} 1 & 2 \\ 1 & -2 \\ 1 & -2 \\ 1 & 2 \end{bmatrix}_{4\times 2}$ ← $r \times t$

- $N_0 = 12\,dB = 16$
- Total transmit power, $P_0 = 3\,dB = 2$
- SVD is given as $H = U \Sigma V^H$

$$\Rightarrow H = \underbrace{\begin{bmatrix} 1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix}_{4\times 4}}_{U} \underbrace{\begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{4\times 2}}_{\Sigma} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{2\times 2}}_{V^H}$$

where, $\sigma_1 = 4$, $\sigma_2 = 2$ are the Singular values

The Powers are given as

$$P_1 = \left(\frac{1}{\lambda} - \frac{N_0}{\sigma_1^2}\right)^+ = \left(\frac{1}{\lambda} - \frac{16}{4^2}\right)^+ = \left(\frac{1}{\lambda} - 1\right)^+$$

$$P_2 = \left(\frac{1}{\lambda} - \frac{N_0}{\sigma_2^2}\right)^+ = \left(\frac{1}{\lambda} - \frac{16}{2^2}\right) = \left(\frac{1}{\lambda} - 4\right)^+$$

Now, use the power constraint,

$$P_1 + P_2 = P_0 = 2$$

$$\Rightarrow \left(\frac{1}{\lambda} - 1\right)^+ + \left(\frac{1}{\lambda} - 4\right)^+ = 2$$

Negative (or) Positive ?

Assume $\left(\frac{1}{\lambda} - 4\right) \geq 0$, which also implies $\left(\frac{1}{\lambda} - 1\right) \geq 0$.

$$\Rightarrow \frac{1}{\lambda} - 1 + \frac{1}{\lambda} - 4 = 2$$

$$\Rightarrow \frac{2}{\lambda} = 7$$

$$\Rightarrow \frac{1}{\lambda} = 3.5$$

$$\therefore P_2 = \frac{1}{\lambda} - 4 = 3.5 - 4 = -0.5 < 0$$

(Negative Power value is NOT VALID)

Therefore, Set $\boxed{P_2 = 0}$

Now,

$$P_1 + P_2 = P_1 + 0 = 2$$

$$\boxed{\begin{aligned} P_1 &= 2 = 3\text{ dB} \\ P_2 &= 0 \end{aligned}}$$

This is the OPTIMAL POWER ALLOCATION.