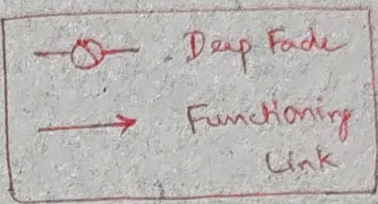
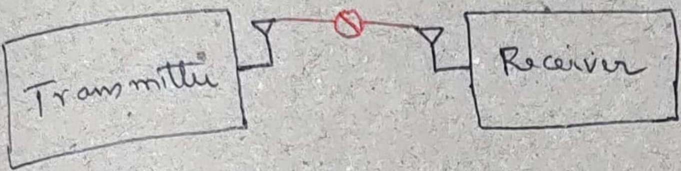
MULTIPLE ANTENNAS & DIVERSITY

- Problem with wireless system : BER is very high!!
- This arises because of DEEP FADE.
- In a deep fade, there is a sharp drop in the output signal power, because of which the reliability of communication is affected.

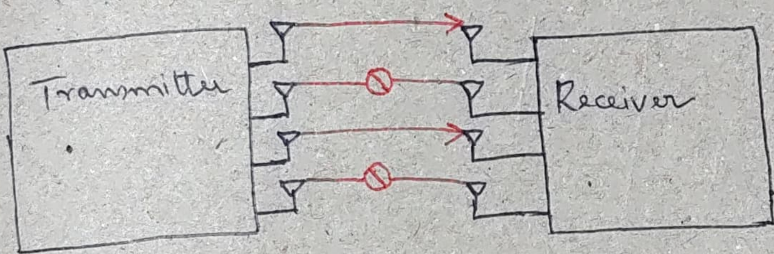


SINGLE LINK



- In typical wireless systems, there is a Single link
- When this link is in a deep fade, Communication is disrupted.

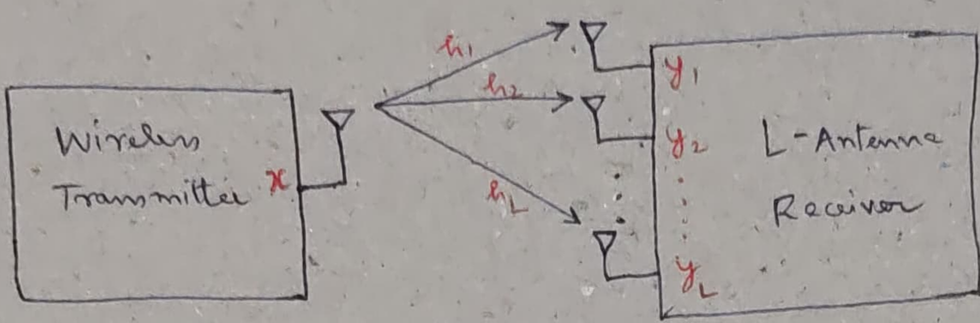
MULTIPLE LINKS



- In order to prevent Deep fade, one should have Multiple functioning Links.
- Even if one or few links are in deep fade, the rest can be used for signal communication.

- Essentially we need diverse links in a System. This principle is termed as DIVERSITY.
- One simple technique for Diversity is to use multiple antennas.

SIMO System



- Single Transmit Antenna, but multiple receive antennas
- There is one link b/w the Transmit antenna and each receive antenna.
- So, for L receive antennas, there are L links.
- This is known as a SINGLE INPUT MULTIPLE OUTPUT (SIMO) System.
- The SIMO system model is given by

$$\underbrace{y_i}_{\text{Received Symbol at Rx antenna } i} = \underbrace{h_i}_{\text{Channel coefficient b/w the Tx antenna and Rx antenna } i} \underbrace{x}_{\text{Transmitted Symbol}} + \underbrace{n_i}_{\text{Noise at antenna } i}$$

- For L antennas, the model is

$$y_1 = h_1 x + n_1$$

$$y_2 = h_2 x + n_2$$

$$\vdots$$

$$y_2 = h_L x + n_L$$

- This can be written in vector form as

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_L \end{bmatrix}_{L\times 1} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_L \end{bmatrix}_{L\times 1} x + \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_L \end{bmatrix}_{L\times 1}$$

- In compact representation,

$$\bar{y} = \bar{h}\, x + \bar{n}$$

$\bar{n}$ → Noise vector.

$\bar{h}$ → Channel vector

$\bar{y}$ → Output vector

- How to process these samples ?

The most popular and efficient technique to process the samples is via Linear Combining.

The output symbols are

$$\underbrace{y_1, y_2, \ldots, y_L}$$

$$\Downarrow$$

$$\underbrace{w_1^* y_1 + w_2^* y_2 + \ldots, w_L^* y_L}_{\text{Weighted Linear Combination}}$$

↖ FILTERING OPERATION.

- This can be represented in compact fashion in vector form as

$$w_1^* y_1 + w_2^* y_2 + \dots + w_L^* y_L$$

$$= \begin{bmatrix} w_1^* & w_2^* & \dots & w_L^* \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_L \end{bmatrix}$$

$$= \bar{w}^H \cdot \bar{y}$$

- $\bar{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix}$ where, $\bar{w} \rightarrow$ Beamforming vector (or) Beamformer

- How to choose the Beamformer $\bar{w}$?

An efficient way to choose beamformer is to maximize the SNR of the output.

(a) CHOOSE BEAMFORMER TO MAXIMIZE SNR.

$$\underbrace{\bar{w}^H \bar{y}}_{\text{BEAMFORMER OUTPUT}} = \bar{w}^H \cdot (\bar{h} x + \bar{n})$$

$$= \underbrace{\bar{w}^H \bar{h} x}_{\text{SIGNAL PART}} + \underbrace{\bar{w}^H \bar{n}}_{\text{NOISE PART}}$$

$$\boxed{\begin{aligned} \text{Signal Power} &= |\bar{w}^H \bar{h}|^2 \cdot E\{|x|^2\} \\ &= |\bar{w}^H \bar{h}|^2 \cdot P \\ \text{Noise Power} &= E\{|\bar{w}^H \cdot \bar{n}|^2\} \end{aligned}}$$

Evaluating the Output Noise Power

$$E\{|\bar{w}^H \bar{n}|^2\} = E\left\{ \left| [w_1^* \cdots w_L^*] \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_L \end{bmatrix} \right|^2 \right\}$$

$$= E\{|w_1^* n_1 + \cdots + w_L^* n_L|^2\}$$

Let the noise samples $(n_1, n_2, \ldots, n_L)$ be INDEPENDENT IDENTICALLY DISTRIBUTED (iid) across the different antennas, then

$$\boxed{E\{n_i n_j^*\} = \begin{cases} 0, & \text{if } i \neq j \\ N_0, & \text{if } i = j \end{cases}}$$

→ $i \neq j$

$$E\{n_i n_j^*\} = E\{n_i\} \cdot E\{n_j^*\}$$

(Since they are independent)

$$= 0 \times 0 \quad \left(\begin{array}{c} \text{Zero Mean Noise} \\ E\{n_i\} = E\{n_j^*\} = 0 \end{array} \right)$$

$$= 0$$

→ $i = j$

$$E\{n_i n_j^*\} = E\{|n_i|^2\}$$

$$= N_0.$$

- Using this property, Output noise power can be derived as

$$E\{|\bar{w}^H . \bar{n}|^2\} = N_0 \|\bar{w}\|^2$$

where, $\|\bar{w}\|^2 = |w_1|^2 + |w_2|^2 + \cdots + |w_L|^2$

- And, the Output SNR is given as

$$SNR_o = \frac{|\bar{w}^H \bar{h}|^2 \cdot P}{N_0 \cdot \|\bar{w}\|^2}$$

- Using Cauchy Schwarz inequality,

$$|\bar{w}^H \bar{h}|^2 \le \|\bar{w}\|^2 \cdot \|\bar{h}\|^2$$

- The output SNR can be simplified as

$$SNR_o = \frac{|\bar{w}^H \bar{h}|^2 \cdot P}{N_o \|\bar{w}\|^2} \le \frac{\cancel{\|\bar{w}\|^2} \cdot \|\bar{h}\|^2 \cdot P}{N_o \cdot \cancel{\|\bar{w}\|^2}}$$

$$\boxed{\begin{aligned} SNR_o &= \|\bar{h}\|^2 \cdot \frac{P}{N_o} \quad (\text{For QPSK}) \\ &\|\bar{h}\|^2 \cdot \frac{2P}{N_o} \quad (\text{For BPSK}) \end{aligned}}$$

- The Maximum Output SNR occurs when

$$\boxed{\bar{w} \propto \bar{h}}$$

$$\bar{w} = \frac{\bar{h}}{\|\bar{h}\|} \longrightarrow \text{Unit - Norm Beamformer}$$

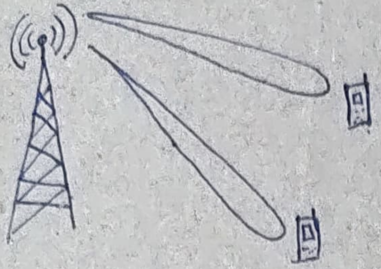
(i.e) $\|\bar{w}\| = 1$.

- Since, $\frac{\bar{h}}{\|\bar{h}\|}$ is the Combiner / Beamformer, that maximizes the SNR_o, this is known as the MAXIMAL RATIO COMBINER (MRC)

- The output SNR of maximal ratio combiner is

$$\boxed{SNR_o = \frac{P}{N_o} \|\bar{h}\|^2}$$

BEAMFORMER



- Beamformer forms a narrow beam to a particular user
- Therefore, it maximizes the SNR
- And also cut-off the unwanted users / Malicious users.

EXAMPLES

① Consider the channel vector $\bar{h} = \begin{bmatrix} \sqrt{2} - \sqrt{2}j \\ \sqrt{2} + \sqrt{2}j \end{bmatrix}$.

What is the MRC beamformer $\bar{w}$?

What is the Processing at the receiver ?

If SNR = 3dB, what is the Output SNR ?

Solution

$$\bar{h} = \begin{bmatrix} \sqrt{2} - \sqrt{2}j \\ \sqrt{2} + \sqrt{2}j \end{bmatrix}$$

- $\bar{h} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$, where $h_1 = \sqrt{2} - \sqrt{2}j$
  $h_2 = \sqrt{2} + \sqrt{2}j$

$$\|\bar{h}\| = \sqrt{|h_1|^2 + |h_2|^2}$$
$$= \sqrt{(2+2) + (2+2)}$$
$$= \sqrt{8} = 2\sqrt{2}.$$

- MRC beamformer, $\bar{w} = \frac{\bar{h}}{\|\bar{h}\|} = \frac{1}{2\sqrt{2}} \bar{h}$

$$\Rightarrow \bar{w} = \frac{1}{2\sqrt{2}} \begin{bmatrix} \sqrt{2} - \sqrt{2}j \\ \sqrt{2} + \sqrt{2}j \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{2} - \frac{1}{2}j \\ \frac{1}{2} + \frac{1}{2}j \end{bmatrix}$$

⊙ Since, $\overline{w}$ is Unit-Norm Beamformer,

$$\|\overline{w}\| = 1.$$

$$\Rightarrow \|\overline{w}\| = \sqrt{|w_1|^2 + |w_2|^2}$$

$$= \sqrt{\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{4}+\frac{1}{4}\right)}$$

$$= \sqrt{1}$$

$$= 1.$$

we have,

$$\overline{w} = \begin{bmatrix} 1/2 - 1/2 j \\ 1/2 + 1/2 j \end{bmatrix}$$

$\overline{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$, where

$$w_1 = \frac{1}{2} - \frac{1}{2} j$$

$$w_2 = \frac{1}{2} + \frac{1}{2} j$$

⊙ The processing at the receiver,

$$\overline{w}^H \cdot \overline{y} = \begin{bmatrix} w_1^* & w_2^* \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} + \frac{1}{2} j & \frac{1}{2} - \frac{1}{2} j \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \left(\frac{1}{2} + \frac{1}{2} j\right) y_1 + \left(\frac{1}{2} - \frac{1}{2} j\right) y_2$$

⊙ SNR = 3 dB ≈ 2.

The output SNR of MRC is given by

$$SNR_o = \frac{P}{N_o} \cdot \|\overline{h}\|^2$$

$$= SNR \cdot \|\overline{h}\|^2$$

$$= 2 \times (2\sqrt{2})^2$$

$$= 16$$

$$\approx 12 \text{ dB}.$$

BER of Multiple Antenna System
(For BPSK)

BER of multiple antenna system is given as

$$\boxed{{}^{2L-1}C_{L-1} \times \frac{1}{2^L} \times \frac{1}{SNR^L}}$$

$$\boxed{{}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}}$$

BER Table

# Antennas	BER Formula	BER @ 20dB	BER @ 60 dB
① L = 1	$\frac{1}{2} \times \frac{1}{SNR}$	5×10^{-3}	5×10^{-7}
② L = 2	$\frac{3}{4} \times \frac{1}{SNR^2}$	7.5×10^{-5}	7.5×10^{-13}
③ L = 3	$\frac{5}{4} \times \frac{1}{SNR^3}$	1.25×10^{-6}	1.25×10^{-18}

Inference: As # Antennas is increasing, the BER is decreasing at a faster rate.

For L antennas, BER $\propto \frac{1}{SNR^L}$

DIVERSITY ORDER

If the BER decreases as $\frac{1}{SNR^L}$, then the diversity order is L.

Example. L = 2 Antennas,

$$BER \propto \frac{1}{SNR^2}$$

$\Rightarrow$ Diversity order = 2.

Deep Fade analysis

- Deep Fade occurs when Signal is buried in Noise.

(ie) Output Signal Power $\leq$ Output Noise Power

$$\Rightarrow SNR_0 \leq 1$$

$$\Rightarrow \frac{P}{N_0} \|\bar{h}\|^2 \leq 1$$

CHI-SQUARED RANDOM VARIABLE

$$\Rightarrow \underbrace{\|\bar{h}\|^2 = g}_{} \leq \frac{1}{SNR}$$

- Probability of deep fade, P_{DF} is given as

$$P_{DF} = P_r\left(g \leq \frac{1}{SNR}\right)$$

- The Probability Density Function (PDF) of 'g' is

$$f_G(g) = \frac{g^{L-1} \cdot e^{-g}}{(L-1)!}$$

- Probability of deep fade,

$$P_{DF} = \int_0^{1/SNR} f_G(g)\, dg$$

$$= \int_0^{1/SNR} \frac{g^{L-1} \cdot e^{-g}}{(L-1)!} \cdot dg$$

$$= \int_0^{1/SNR} \frac{g^{L-1}}{(L-1)!}\, dg$$

At high SNR, $\frac{1}{SNR} \rightarrow 0$

$\Rightarrow e^{-g} \approx 1$

$$= \frac{g^L}{L!}\Bigg|_0^{1/SNR}$$

$$= \frac{1}{L!} \times \frac{1}{SNR^L}$$

- **Inference**:
  Therefore, once again, not surprisingly, we observe

$$BER \propto P_{DF} \propto \frac{1}{SNR^L}$$

- **Justification**
  - → Consider the multiple antenna system (SIMO)
  - → E_i → Event that link 'i' is in deep fade
  - → Probability that a single link is in deep fade

$$P_r(E_i) \propto \frac{1}{SNR}$$

  - → If the multiple antenna system has to be in deep fade, then all the links are in deep fade.

(ie) $E_1 \cap E_2 \cap E_3 \ldots \cap E_L$

$$P_{DF} = P_r(E_1 \cap E_2 \cap \ldots \cap E_L)$$

$$= P_r(E_1) \times P_r(E_2) \times \ldots \times P_r(E_L)$$

(Assuming Links are Independently fading)

$$P_{DF} \propto \frac{1}{SNR} \times \frac{1}{SNR} \times \ldots \times \frac{1}{SNR}$$

$$\Rightarrow P_{DF} \propto \frac{1}{SNR^L}$$

  - → If $E_1, E_2, \ldots, E_L$ are independent, then the different links are independently fading. For this, the antenna spacing has to be large.

Good rule of thumb is $> \frac{\lambda}{2}$.

ANTENNA SPACING

λ = Wavelength of Signal

$$\lambda = \frac{c}{f_c}$$

where, $c \rightarrow$ Velocity of EM wave (or) Light

$c = 3 \times 10^8$ m/s

$f_c \rightarrow$ Carrier frequency.

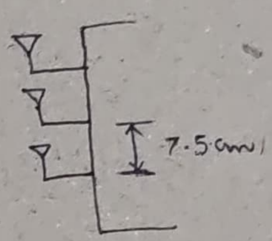
$$\boxed{\text{Antenna Spacing} \geq \frac{\lambda}{2}}$$

Example 1

At $f_c = 2$ GHz, Calculate the minimum Antenna Spacing.

$$\lambda = \frac{c}{f_c} = \frac{3 \times 10^8}{2 \times 10^9}$$

$$\frac{\lambda}{2} = \frac{1}{2} \times \frac{3 \times 10^8}{2 \times 10^9} = 7.5 \text{ cm.}$$

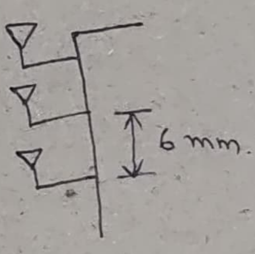


Example 2

At $f_c = 25$ GHz, Calculate the minimum Antenna spacing.

$$\lambda = \frac{c}{f_c} = \frac{3 \times 10^8}{25 \times 10^9}$$

$$\frac{\lambda}{2} = \frac{1}{2} \times \frac{3 \times 10^8}{25 \times 10^9} = 6 \text{ mm}$$



Note:

As f_c increases, Antenna Spacing decreases!

(i.e) More Antennas can be embedded on device of same size.