Classical Logic in a Judgemental Natural Deduction Style

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1 Propositional Logic Proof Theory

We begin with the syntax of sentential aka propositional formulae and proof contexts. The set $\mathcal{A}$ of propositional symbols is countably infinite.

$$
\begin{array}{llll}
P \in \mathcal{A} & & & \textit{propositional symbols} \\
\phi & ::= & P \mid \neg\phi \mid \phi \vee \phi \mid \phi \wedge \phi \mid \phi \Rightarrow \phi & \textit{well formed formulae (wffs)} \\
\Delta & ::= & \varnothing \mid \Delta, \phi & \textit{contexts}
\end{array}
$$

Binding strengths of operators: $\neg$ binds stronger than $\Rightarrow$ binds stronger than $\wedge$ binds stronger than $\vee$. Each of $\neg, \vee, \Rightarrow$ are right-associative. We define the relation $\phi \in \Delta$ inductively as follows, allowing lookup of a formula in a context.

$$
\phi \in \Delta, \phi \qquad\qquad \frac{\phi \in \Delta}{\phi \in \Delta, \phi'}
$$

Judgements are written $\Delta \vdash \phi$. Provability of a judgement is obtained by its finite derivation, given the following set of *natural deduction* inference rules. Note that each connective has an introduction (I) and elimination (E) form (except negation, which we'll get to in a second). Intuitively, introduction forms show how to build formulae, and elimination forms show how to use them.

$$
\begin{array}{cccc}
\textsc{Context} & \wedge\text{-I} & \wedge\text{-E}_1 & \vee\text{-E}_2 \\[2pt]
\dfrac{\phi \in \Delta}{\Delta \vdash \phi} & \dfrac{\Delta \vdash \phi_1 \quad \Delta \vdash \phi_2}{\Delta \vdash \phi_1 \wedge \phi_2} & \dfrac{\Delta \vdash \phi_1 \wedge \phi_2}{\Delta \vdash \phi_1} & \dfrac{\Delta \vdash \phi_1 \wedge \phi_2}{\Delta \vdash \phi_2}
\end{array}
$$

$$
\begin{array}{ccc}
\vee\text{-I}_1 & \vee\text{-I}_2 & \vee\text{-E} \\[2pt]
\dfrac{\Delta \vdash \phi_1}{\Delta \vdash \phi_1 \vee \phi_2} & \dfrac{\Delta \vdash \phi_2}{\Delta \vdash \phi_1 \vee \phi_2} & \dfrac{\Delta \vdash \phi_1 \vee \phi_2 \quad \Delta, \phi_1 \vdash \phi \quad \Delta, \phi_2 \vdash \phi}{\Delta \vdash \phi}
\end{array}
$$

$$
\begin{array}{ccc}
\Rightarrow\text{-I} & \Rightarrow\text{-E (aka Modus Ponens)} & \textsc{Lem} \\[2pt]
\dfrac{\Delta, \phi_1 \vdash \phi_2}{\Delta \vdash \phi_1 \Rightarrow \phi_2} & \dfrac{\Delta \vdash \phi_1 \Rightarrow \phi_2 \quad \Delta \vdash \phi_1}{\Delta \vdash \phi_2} & \dfrac{}{\Delta \vdash \phi \vee \neg\phi}
\end{array}
$$

The exclusion of the rule LEM, commonly called *law of the excluded middle*, obtains *intuitionistic* or *constructive* logic, versus *classical* logic. Formally, this is because it is

the only rule form that allows an existentially bound proposition to be asserted without a witness of the existential being constructable (this is the fundamental requirement of constructivism). Consequently, intuitionistic logic has a deep relation with computation (as evinced by e.g. the Curry Howard isomorphism). Introduction and elimination rules for negation can be formulated that are sound and complete for LEM, in order to obtain a consistent rule style for all connectives, but this doesn't avoid the fundamentally non-constructive nature of LEM.

Definition 1 *A context Δ is* inconsistent *iff $\Delta \vdash \phi \wedge \neg\phi$ is provable for some ϕ. A logic is inconsistent if $\varnothing$ is inconsistent.*

1.1 Derived Rules

Some familiar principles that are *derivable* from the above, i.e. that do not add any more expressive power, include *double negation elimination*, rejected by intuitionists and not derivable in intuitionistic logic, but inter-derivable with LEM:

$$\frac{\Delta \vdash \neg\neg\phi}{\Delta \vdash \phi}$$

and *reductio ad absurdum* or *proof by contradiction*, also rejected by intuitionists and not derivable in intuitionistic logic, but inter-derivable with LEM:

$$\frac{\Delta, \neg\phi \vdash \phi' \qquad \Delta, \neg\phi \vdash \neg\phi'}{\Delta \vdash \phi}$$

though interestingly enough, a negative form of reductio ad absurdum can be derived in intuitionistic logic:

$$\frac{\Delta, \phi \vdash \phi' \qquad \Delta, \phi \vdash \neg\phi'}{\Delta \vdash \neg\phi}$$

and *contraposition*:

$$\frac{\Delta \vdash \phi_1 \Rightarrow \phi_2}{\Delta \vdash \neg\phi_2 \Rightarrow \neg\phi_1}$$

and *syllogism*:

$$\frac{\Delta \vdash \phi_1 \Rightarrow \phi_2 \qquad \Delta \vdash \phi_2 \Rightarrow \phi_3}{\Delta \vdash \phi_1 \Rightarrow \phi_3}$$

and the *principle of explosion*:

$$\frac{\Delta \vdash \phi' \wedge \neg\phi'}{\Delta \vdash \phi}$$

2 Propositional Logic Model Theory

Model theory interprets validity of judgements. The basis of the model theory is an *interpretation*, which assigns truth values to propositions. This is a reflection of the

idea that in any given possible world, any fact may be either true or false. Given a particular interpretation, the standard meaning of the logical connectives induces truth valuation on arbitrary formulae, as follows.

Definition 2 Interpretations *ρ are total mappings from propositional symbols to truth values $\{T, F\}$. Interpretations are extended to formulae as follows:*

$$
\begin{aligned}
\rho(\neg\phi) &= T \text{ iff } \rho(\phi) = F \\
\rho(\phi_1 \wedge \phi_2) &= T \text{ iff } \rho(\phi_1) = T \text{ and } \rho(\phi)_2 = T \\
\rho(\phi_1 \vee \phi_2) &= T \text{ iff } \rho(\phi_1) = T \text{ or } \rho(\phi)_2 = T \\
\rho(\phi_1 \Rightarrow \phi_2) &= T \text{ iff } \rho(\phi_1) = F \text{ or both } \rho(\phi_1) = T \text{ and } \rho(\phi_2) = T
\end{aligned}
$$

Contexts allow us to restrict logical consideration to a subset of all possible worlds, as made clear in the definition of semantic entailment $\Delta \models \phi$, as follows. Note that inconsistent contexts vacuously entail any formulae.

Definition 3 *We write $\rho \models \Delta$ iff $\rho(\phi) = T$ for all $\phi \in \Delta$. We write $\Delta \models \phi$ iff $\rho \models \Delta$ implies $\rho(\phi) = T$ for every ρ.*

2.1 Properties

Soundness and *Completeness* relate the proof and model theories. Naturally, we want to show that anything that is provable is true, and anything that is true is provable. Soundness is easily proven by induction on proof derivations. A corollary of this result is consistency of propositional logic.

Lemma 1 (Soundness of Propositional Logic) *If $\Delta \vdash \phi$ then $\Delta \models \phi$.*

Corollary 1 *Propositional logic is consistent.*

Completeness is a lot harder to prove. Intuitively, one approach is to show how to construct a proof from the truth table of an arbitrary formula.

Lemma 2 (Completeness of Propositional Logic) *If $\Delta \models \phi$ then $\Delta \vdash \phi$.*

3 Proof Theory of First Order Logic

The syntactic extension to first order logic is obtained by parameterizing propositional symbols with *terms*, that may be *universally* or *existentially quantified*. For simplicity in this presentation, we omit functions from the term language, but they can easily be added to obtain the complete standard definition of terms. Note that first-orderly restrictions on quantification (only over terms, not formulae) is obtained by the grammatical definition of wffs. The sets $\mathcal{T}$ and $\mathcal{V}$ of term constants and variables are countably infinite.

$$
\begin{array}{lll}
t \in \mathcal{T} & & \textit{term constants} \\
x \in \mathcal{V} & & \textit{term variables} \\
k & ::= \quad t \mid x & \textit{terms} \\
\phi & ::= \quad P(k_1, \ldots, k_n) \mid \forall x.\phi \mid \exists x.\phi \mid \cdots & \textit{well formed formulae (wffs)}
\end{array}
$$

Instantiation of quantified formulae is obtained by an appropriate definition of *substitution* of term constants for variables.

Definition 4 *The symbols $\exists$ and $\forall$ are variable binders, and formulae are equated up to α-renaming of bound variables. We write $\phi[t/x]$ to denote the capture-avoiding substitution of t for free occurrences of x in ϕ.*

The proof system is extended with introduction and elimination forms for the quantifiers. The terms t that are "hidden" in the preconditions of the $\exists$-E and $\forall$-I rules are called *eigenvalues*. Note the the definition of these rules allow $\forall$- and $\exists$- bound formulae to behave like infinite conjuncts and disjuncts, respectively. The correspondance is especially clear in the $\exists$-E rule.

$$\forall\text{-I} \quad \frac{\Delta \vdash \phi[t/x] \qquad t \text{ does not occur in } \Delta, \phi}{\Delta \vdash \forall x.\phi} \qquad\qquad \forall\text{-E} \quad \frac{\Delta \vdash \forall x.\phi}{\Delta \vdash \phi[t/x]}$$

$$\exists\text{-I} \quad \frac{\Delta \vdash \phi[t/x]}{\Delta \vdash \exists x.\phi} \qquad\qquad \exists\text{-E} \quad \frac{\Delta \vdash \exists x.\phi' \qquad \Delta, \phi'[t/x] \vdash \phi \qquad t \text{ does not occur in } \Delta, \phi, \phi'}{\Delta \vdash \phi}$$