ch- ordinary diffrential eqⁿ

~~Matrix~~

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→ ~~Defination of Matrix~~

→ what is differentiable eq^n :-

→ The eq which contain depended & independent or differentiation of depended variable it's called.

① $\frac{dy}{dx} + 2y = x$

② $\frac{d^2y}{dx^2} + \sin x \, y = \frac{dy}{dx}$

③ $\frac{d^3y}{dx^3} + 2\left(\frac{d^2y}{dx^2}\right)^2 = 5$

④ $\frac{dy}{dx} + \frac{1}{dy/dx} = x$

⑤ $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

degree = max power of higest derivative

order = max derivative

* Linear diffrential eq of the second order:-

→ consider

$$y'' + p(x)y' + q(x)y = r(x)$$

where p(x) and q(x) are coefficient of the eq^n

→ Also if $r(x) = 0$ then eq^n is called homogeneous

→ $r(x) \neq 0$ then eq^n is called non homogeneous

→ if p(x) & q(x) are constant then it is called eq^n with constant coefficient

↳ otherwise variable coefficient

Q-1 $(1-x^2)y'' - 2xy' + 6y = 0$ ⇒ linear & homogeneous → variable coefficient

② $y'' + 4y' + 3y = 0$ ⇒ linear & homo... → constant

③ $y''y + y' = 0$ ⇒ non linear & homo... → variable

④ $y'' + \sin x \, y = 0$ ⇒ linear & homo... → variable

* General form:-

→ A linear diff eq^n of n^{th} order is of the form

$$\boxed{\frac{d^n y}{dx^n} + P_1(x)\frac{d^{n-1}y}{dx^{n-1}} + P_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \dots + P_{n-1}(x)\frac{dy}{dx} + P_n(x)y = X}$$

$P_i(x)$ are fnx of x

$$\frac{d^n y}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \dots + a_{n-1}\frac{dy}{dx} + a_n y = X$$

* Diffrential operator D:-

diffine as $D = \frac{d}{dx}$

i.e $\frac{dy}{dx} = Dy$, $\frac{d^2y}{dx^2} = D^2y$

→ in notation form:-

$\therefore D^n y + a_1 D^{n-1}y + a_2 D^{n-2}y + \dots + a_{n-1}Dy + a_n y = X$

$\therefore (D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_{n-1}D + a_n)y = X$

$\therefore f(D)y = X$

↳ Find the sol^n

(C.F.) complimentary fxn

(P.I.) Particular ~~fxn~~ Integral

* Initial value problem

$y'' + P(x)y' + q(x)y = 0$

$y(x_0) = k_0$, $y'(x_0) = k_1$

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* **Boundary value Problems:**

$y'' + P(x)y' + q(x)y = 0$
$y(x_0) = k_0 \; , \; y(x_1) = k_1$

* Method to find C.F. of $f(D)y = X$

step ① Write the eq^n in notation form i.e. $f(D)\,y = X$
② Form the (auxiliary eq^n) → A.E. $f(D) = 0$
③ Find the roots of the A.E.
say the roots are $m_1, m_2, m_3, \ldots m_n$ for the A.E.

Before we go to finalstep letus call how we finding roots of $ax^2 + bx + c = 0$ for what, we were using Δ method to determine nature of the roots
If $\Delta > 0$ the roots are real & equal
$\Delta = 0$ " " real & coincident.
$\Delta < 0$ " " imaginary & pairs in

→ letus consider a first order linear ordinary eq^n with constant coefficient of the form $\frac{dy}{dx} - ay = X$

It's solution is $y(I.F.) = \int X(I.F.)\,dx + c$
for complimentary fxn $X = 0$
$\therefore \frac{dy}{dx} - ay = 0$ which is linear eq^n &
$I.F = e^{-\int a\,dx} = e^{-ax}$
It's sol^n is $ye^{-ax} = c \Rightarrow y = ce^{ax}$

Three cases
④ All the roots $m_1, m_2, \ldots, m_n$ are real & ~~distinct~~ different,
Case-1 then,

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$\therefore (D^2 - 5D + 6)y = 0$

$\therefore$ A.E. is $D^2 - 5D + 6 = 0$

$D = 2, 3$

$\boxed{C.F. = c_1 e^{2x} + c_2 e^{3x}}$

$\therefore P.I. = 0$

$\therefore$ soln is,

$y = C.F + P.I$

$y = c_1 e^{2x} + c_2 e^{3x}$

Q. solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$

→ in notation form,

$(D^2 + 2D + 2)y = 0$

$\therefore$ A.E. is $D^2 + 2D + 2 = 0$

$D = \frac{-2 \pm \sqrt{4-8}}{2}$ $\quad D = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= -1 \pm i$

$\therefore \alpha = -1, \beta = 1$

→ $C.F. = e^{-x}(c_1 \cos x + c_2 \sin x)$

$\therefore P.I = 0$

soln is

$y = C.F + PI$

$\boxed{y = e^{-x}(c_1 \cos x + c_2 \sin x)}$

Q. solve $\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = 0$

→ in notation form

$\therefore (D^4 + 2D^2 + 1)y = 0$

A.E. is $D^4 + 2D^2 + 1 = 0$

$\therefore (D^2+1)^2 = 0$ $\quad \alpha = 0, \beta = 1$

β positive levano

$\therefore D = i, i, -i, -i$

$C.F. = e^{0x}((c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x)$

$P.I = 0$

soln is $y = C.F. + P.I.$

$y = e^{0x}((c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x)$

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Q. solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$

→ in notation form,

$(D^2 + D - 2)y = 0$

$\therefore$ A.E. $= D^2 + D - 2 = 0$

$D = -2, 1$

$\therefore$ C.F. $= c_1 e^{-2x} + c_2 e^{x}$

$\therefore$ P.I $= 0$

$\therefore$ soln is $y = C.F + P.I$

$$\boxed{y = c_1 e^{-2x} + c_2 e^{x}}$$

Q. solve $\frac{d^4y}{dt^4} + \frac{4d^2y}{dt^2} = 0$

→ In notation form,

$(D^4 + 4D^2) = 0$

$\therefore$ A.E. form is $= D^4 + 4D^2 = 0$

$D^2(D^2 + 4) = 0$

$D = 0, 0, 2i, -2i$

$\alpha = 0, \beta = 2$

$\therefore$ C.F. $= c_1 \cos 2t + c_2 \sin 2t + (c_3 + c_4 t)e^{0 \times t}$

Q. solve $\frac{d^2x}{dt^2} + 3a\frac{dx}{dt} - 4a^2x = 0$

→ in notation form,

$\therefore (D^2 + 3aD - 4a^2)x = 0$

$\therefore$ A.E. eqn is $D^2 + 3aD - 4a^2 = 0$

$\therefore D = a, -4a$

$\therefore$ C.F $= c_1 e^{at} + c_2 e^{-4at}$

$\therefore$ also P.I $= 0$

$\therefore$ soln is,

$x = CF + PI$

$$\boxed{x = c_1 e^{at} + c_2 e^{-4at}}$$

$$C.F. = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$$

case-2 If all roots are real & diffrent except two roots are equal (i.e $m_1 = m_2$), then

$$C.F. = (c_1 x + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

If $m_1 = m_2 = m_3$ then

$$C.F. = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + \dots + c_n e^{m_n x}$$

case-3 If one pair of roots is imaginary (i.e. $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$), and the rest of the roots are real and diffrent, then

$$C.F. = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

If ($m_1 = m_2 = \alpha + i\beta$, $m_3 = m_4 = \alpha - i\beta$) then

$$C.F. = e^{\alpha x}[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x] + c_5 e^{m_5 x} + \dots + c_n e^{m_n x}$$

ex $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$ solve

→ In notation form,

$(D^2 + 6D + 9)y = 0$

A.E. is $D^2 + 6D + 9 = 0$

$D = -3, -3$

$C.F. = (c_1 + c_2 x) e^{-3x}$

P.I. = 0

∴ sol^n is

$y = C.F. + P.I.$

$$\boxed{y = (c_1 + c_2 x) e^{-3x}}$$

ex solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

→ In notation form

Q. $3\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 2y = 0$

→ In notation form,

$\therefore (3D^2 - 9D + 2)y = 0$

$\therefore$ A.E eq^n is $3D^2 - 9D + 2 = 0$

$\therefore D = \frac{9 \pm \sqrt{81 - 24}}{2(3)}$

$= \frac{9 \pm \sqrt{57}}{6}$ } real roots છે case-1 લેવો

→ $C.F = c_1 e^{\left(\frac{9+\sqrt{57}}{6}\right)x} + c_2 e^{\left(\frac{9-\sqrt{57}}{6}\right)x}$

$P.I = 0$

$\therefore$ sol^n is $y = CF + PI$

$\therefore y = c_1 e \ldots$

Q. solve the initial value problem $y'' - 9y = 0$ given that $y(0) = 2,\ y''(0) = -1$

→ In notation form,

$\therefore (D^2 - 9)y = 0$

$\therefore$ A.E. eq^n is $D^2 - 9 = 0$

$D = 3, -3$

$\therefore CF = c_1 e^{3x} + c_2 e^{-3x}$

$\therefore PI = 0$

→ sol^n is $y = CF + PI$

$\therefore y = c_1 e^{3x} + c_2 e^{-3x}$ $\therefore y' = 3c_1 e^{3x} - 3c_2 e^{-3x}$

$\therefore y(0) = 2,\quad y'(0) = -1$

$\therefore 2 = c_1 + c_2$ $6 = 3c_1 + 3c_2$

$\therefore -1 = 3c_1 - 3c_2$

$\therefore$ ① + ②

$6c_1 = 5$

$\therefore \boxed{c_1 = 5/6}$ $\boxed{c_2 = 7/6}$ → sol^n is, $\boxed{y = \frac{5}{6}e^{3x} + \frac{7}{6}e^{-3x}}$

viva આર્બિટરી કોન્સ્ટન્ટ ની value condition $y(0)=2$ પ્રમાણે બદલાઈ શકે

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Q. solve $y'' + 8y' + 25y = 0$ given that $y(0) = 2$, $y(\pi/6) = 5$

Hint: roots img होंगे।

→ In notation form,

$(D^2 + 8D + 25)y = 0$

$\therefore$ A.E. eqⁿ is $D^2 + 8D + 25 = 0$

$$D = \frac{-8 \pm \sqrt{64 - 100}}{2}$$

$$= \frac{-8 \pm \sqrt{-36}}{2}$$

$$= \frac{-8 \pm 6i}{2}$$

$$= -4 \pm 3i$$

$$\alpha = -4, \beta = 3$$

→ C.F. = ~~c_1~~ $e^{-4x} [\cancel{(c_1 + c_2x)} \cos 3x + c(c_3 + c_4x) \sin 3x]$

C.F. = $e^{-4x} [c_1 \cos 3x + c_2 \sin 3x]$

$\therefore$ P.I. = 0

soln is C.F + P.I

$y = e^{-4x} [c_1 \cos 3x + c_2 \sin 3x]$

$\therefore y(0) = 2$, $y(\pi/6) = 5$

$\therefore 2 = e^{(0)} [c_1 + 0]$

$\boxed{2 = c_1}$

$\therefore 5 = e^{-4\pi/6} [0 + c_2(1)]$

$\therefore \boxed{c_2 = 5e^{4\pi/6} = 5e^{2\pi/3}}$

→ $\boxed{y = e^{-4x} [2 \cos 3x + 5e^{2\pi/3} \sin 3x]}$

Q. solve $\frac{d^3y}{dx^3} + y = 0$

→ In notation form

$(D^3 + 1)y = 0$

$a^3+b^3=(a+b)(a^2-ab+b^2)$
$a^3-b^3=(a-b)(a^2$

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$\therefore$ A.E eq^n is $D^3+1=0$

$\therefore$ ~~$D^3=-1$~~ $(D+1)(D^2-D+1)=0$?

$\therefore$ ~~$D=$~~ $\hookrightarrow D=-1$ $\quad D=\frac{1\pm\sqrt{1-4}}{2}$

$=\frac{1\pm\sqrt{3}i}{2}$

$\alpha=\frac{1}{2},\ \beta=\sqrt{3}/2$

$\therefore$ C.F $= c_1e^{-x}+e^{3/2}\left(c_2\cos\frac{\sqrt{3}}{2}x+c_3\sin\frac{\sqrt{3}}{2}x\right)$

$\rightarrow$ $\downarrow$
$y=\ldots$

Q. $\frac{d^3y}{dx^3}+3\frac{d^2y}{dx^2}+3\frac{dy}{dx}+y=0$

$\rightarrow$ In notation form

$\therefore (D^3+3D^2+3D+1)y=0$

$\therefore$ A.E. eq^n is $D^3+3D^2+3D+1=0$

$\therefore (D+1)^3=0$ ચાંને round મેવા.
Noted

$\therefore D=-1,-1,-1$

$\therefore$ C.F $=(c_1+c_2x+c_3x^2)e^{-x}$

$\boxed{y=\ldots}$

Q. $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+10y=0$, $y(0)=1$, $y'(0)=1$

$\therefore$ In notation form,

$(D^2-2D+10)y=0$

$\therefore$ A.E. eq^n is $D^2-2D+10=0$

$\therefore D=1\pm3i$

$\alpha=1,\ \beta=3$

$\therefore$ C.F $=e^x(c_1\cos3x+c_2\sin3x)$

$y=e^x(c_1\cos3x+c_2\sin3x)$

$y'=e^x(-c_1\sin3x\cdot3+c_2\cos3x\cdot3)+e^x(c_1\cos3x+c_2\sin3x)$

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$\therefore y(0)=1$, $y'(0)=1$

$\therefore \boxed{c_1=1}$ $1=c_1+3c_2$

$\boxed{c_2=0}$

$\rightarrow y=e^{x}(1\cos 3x+0)$

$\boxed{y=e^{x}\cos 3x}$

Q. $\frac{d^3y}{dx^2}-\frac{3d^2y}{dx^2}+\frac{3dy}{dx}-y=0$

↓H.W
દાખલો

* Method to find P.I. of $f(D)y=X$:

→ Method of variation parameters

$$\frac{d^2y}{dx^2}+\frac{Pdy}{dx}+qy=X$$

when P, q, X are fx^n of x. then

$\boxed{P.I=u_1y_1+u_2y_2}$

→ where y_1, y_2 are the sol^n of $\frac{d^2y}{dx^2}+\frac{Pdy}{dx}+qy=0$

$\therefore u_1=-\int\frac{y_2X}{\omega}dx$, $u_2=\int\frac{y_1X}{\omega}dx$

$\therefore$ where $\omega=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is called the Wronskian of y_1, y_2

Q solve $\frac{d^2y}{dx^2}+4y=\sec 2x$.

આ શર્ત લખવાની.

→ In notation form,

$\therefore (D^2+4)y=\sec 2x$

$\therefore$ A.E. is $D^2+4=0$

$D=2i, -2i$ $\alpha=0, \beta=2$

$\therefore C.F=c_1\cos 2x+c_2\sin 2x$

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$\therefore y_1 = \cos 2x \quad , \quad y_2 = \sin 2x$

$\therefore W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -\sin 2x \cdot 2 & \cos 2x \cdot 2 \end{vmatrix}$

$= 2\cos^2 2x + 2\sin^2 2x$

$\boxed{W = 2}$

$\therefore u_1 = -\int \frac{y_2 X}{W} dx$

$= -\int \frac{\sin 2x \cdot \sec 2x}{2} dx \rightarrow -\frac{1}{2}\int \frac{\sin 2x}{\cos 2x} dx$

$= -\frac{1}{2}\int \tan x \, dx \propto \underline{\log}$ $\quad = +\frac{1}{4} \ln \cos 2x$

$\boxed{= \int \frac{f'}{f} dx = \ln f}$

$\therefore u_2 = \int \frac{y_1 X}{W} dx$

$= \int \frac{\cos 2x \cdot \sec 2x}{2} dx$

$u_2 = x/2$

$\rightarrow \therefore$ Therefore $P.I. = u_1 y_1 + u_2 y_2$

$= +\frac{1}{4} \ln \cos 2x \cdot \cos 2x + \frac{x}{2} \cdot \sin 2x$

$\therefore$ solⁿ is

$\rightarrow y = C.F. + P.I.$

$$\boxed{y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4} \ln \cos 2x (\cos 2x) + \frac{x}{2} \sin 2x}$$

common ex

$\frac{d^2y}{dx^2} + a^2 y = \sec ay$ or $\operatorname{cosec} ay$

Q. solve $\frac{d^2y}{dx^2} + \frac{3dy}{dx} - 10y = e^{2x}$

$\rightarrow$ In notation form,

$\therefore (D^2 + 3D - 10)y = e^{2x}$

$\therefore$ A.E is $D^2 + 3D - 10 = 0$

$D = -5, 2 \quad 2, -5$

$\therefore C.F = c_1 e^{2x} + c_2 e^{-5x}$

$\therefore y_1 = e^{2x}, \quad y_2 = e^{-5x}$

$\therefore \omega = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{-5x} \\ 2e^{2x} & -5e^{-5x} \end{vmatrix}$

$= -5e^{-3x} - 2e^{-3x}$

$= -7e^{-3x}$

$\therefore u_1 = -\int \frac{y_2 x}{\omega} dx$

$= -\int \frac{e^{-5x} e^{2x}}{-7e^{-3x}} dx$

$u_1 = x/7$

$\therefore u_2 = \int \frac{y_1 x}{\omega} dx$

$= \int \frac{e^{2x} \cdot e^{2x}}{-7e^{-3x}} dx$

$= \frac{-1}{7} \int e^{7x} dx$

$\therefore u_2 = \frac{-1}{49} e^{7x}$

$\rightarrow$ P.I $= x/7 \, e^{2x} + \frac{-1}{49} e^{7x} e^{-5x}$

$= \frac{x}{7} e^{2x} - \frac{1}{49} e^{2x}$

$\rightarrow$ soln is $y = C.F + P.I.$

$$\boxed{y = c_1 e^{2x} + c_2 e^{-5x} + \frac{x}{7} e^{2x} - \frac{1}{49} e^{2x}}$$

Q solve $\frac{d^2y}{dx^2} + 2y = e^x$.

$\rightarrow$ In notation form,

$\therefore (D^2 + 2)y = e^x$

$\therefore$ A.E eq is $D^2 + 2 = 0$

$\therefore D^2 = -2$

$D = \sqrt{2}i, -\sqrt{2}i \qquad \alpha = 0, \beta = \sqrt{2}$

$\therefore$ C.F $= c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x$

$\therefore y_1 = \cos\sqrt{2}x, \quad y_2 = \sin\sqrt{2}x$

Q. solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 3x^{3/2} e^x$

$\therefore$ In notation form

$\therefore (D^2 - 2D + 1) y = 3x^{3/2} e^x$

$\therefore$ A.E eqn is $D^2 - 2D + 1 = 0$

$D = 1, 1$

$\therefore$ C.F $= (c_1 + c_2 x) e^x$

$\therefore y_1 = e^x \quad y_2 = xe^x$

$\therefore \omega = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix}$

$= xe^{2x} + e^{2x} - xe^{2x}$

$\omega = e^{2x}$

$\therefore u_1 = -\int \frac{y_2 X}{\omega} dx \qquad \therefore u_2 = \int \frac{y_1 X}{\omega} dx$

$= -\int \frac{x\cancel{e^x} \cdot 3x^{3/2} \cancel{e^x}}{\cancel{e^{2x}}} dx \qquad = \int \frac{\cancel{e^x} \cdot 3x^{3/2} \cancel{e^x}}{\cancel{e^{2x}}}$

$= -3 \int x^{5/2} dx \qquad = 3 \int x^{3/2}$

$= \frac{-3 x^{7/2}}{7/2} \qquad = \frac{3 x^{5/2}}{5/2}$

$\rightarrow$ P.I $= u_1 y_1 + u_2 y_2$

$= \frac{-6}{7} x^{7/2} \cdot e^x + \frac{6}{5} x^{5/2} x e^x$

$= -6/7 \, x^{7/2} e^x + \frac{6}{5} x^{7/2} e^x \qquad \boxed{= \frac{12}{35} e^x x^{7/2}}$

$\rightarrow$ soln is $y = $ C.F + P.I

$= c_1 e^x + c_2 x e^x + \frac{-6}{7} x^{7/2} e^x + \frac{6}{5} x^{7/2} e^x$

$= (c_1 + c_2 x) e^x + \frac{12}{35} e^x x^{7/2}$

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Q-1 solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{-x}\ln x$

Q-2 solve $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$

Q-3 solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$

Q-4 $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \frac{1}{1+e^x}$

Ans-1 In notation form,

$(D^2 + 2D + 1)y = e^{-x}\ln x$

$\therefore$ A.E. eqn is $D^2 + 2D + 1 = 0$

$D = -1, -1$

$\therefore$ C.F. $= (c_1 + c_2x)e^{-x}$

$y_1 = e^{-x}, \quad y_2 = xe^{-x}$

$\therefore \omega = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & -xe^{-x} + e^{-x} \end{vmatrix} = e^{-2x}$

$\therefore u_1 = -\int \frac{y_2 X}{\omega}\, dx$

$= -\int \frac{x e^{-x} e^{-x} \ln x}{e^{-2x}}$

$= -\int x\ln x\, dx$

$= -\left[\ln x \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2}\, dx\right]$

$= -\left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]$

$\therefore u_2 = \int \frac{e^{-x} e^{-x} \ln x}{e^{-2x}}\, dx$

$\therefore u_2 = x\ln x - x$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right)$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\cos bx + b\sin bx\right)$$

$$\therefore\ \omega = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos\sqrt{2}x & \sin\sqrt{2}x \\ -\sin\sqrt{2}x\sqrt{2} & \cos\sqrt{2}x\sqrt{2} \end{vmatrix}$$

$$= \cos^2\sqrt{2}x + \sin^2\sqrt{2}x(\sqrt{2})$$

$$\omega = \sqrt{2}$$

$$\therefore\ u_1 = -\int \frac{y_2 X}{\omega}\,dx \qquad u_2 = +\int \frac{y_1 X}{\omega}\,dx$$

$$= -\int \frac{\sin\sqrt{2}x \cdot e^x}{\sqrt{2}}\,dx \qquad = \int \frac{\cos\sqrt{2}x\, e^x}{\sqrt{2}}\,dx$$

$$= \frac{-1}{\sqrt{2}}\int$$

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$\therefore (D^2-D)y + 4Dy + 2y = e^{3z}$

$\therefore (D^2+3D+2)y = e^{3z}$

$\rightarrow$ A.E. eq is $D^2+3D+2=0$

$D=-2,-1$

$\therefore$ C.F. $= c_1e^{-2z} + c_2e^{-z}$

$\therefore y_1 = e^{-2z}$, $y_2 = e^{-z}$

$\therefore \omega = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2z} & e^{-z} \\ -2e^{-2z} & -e^{-z} \end{vmatrix}$

$= -e^{-3z} + 2e^{-3z}$

$= e^{-3z}$

$\therefore u_1 = -\int \frac{y_2 x}{\omega} dz$

$= -\int \frac{e^{-z}\,\cancel{x^3}\,e^{3z}}{e^{-3z}} dz$ → ધ્યાન રાખવું.

$u_1 = \frac{-e^{5z}}{5}$

$u_2 = \int \frac{y_1 x}{\omega} dz$

$= \int \frac{e^{-2z}\cdot e^{3z}}{e^{-3z}} dz$

$= \int e^{4z} dz$

$u_2 = \frac{e^{4z}}{4}$

$\therefore$ P.I $= u_1y_1 + u_2y_2$

$= \frac{-e^{5z}}{5} \times e^{-2z} + \frac{e^{5z}}{5} e^{-z}$

$= \frac{-e^{3z}}{5} + \frac{e^{3z}}{4} = \frac{1}{20} e^{3z}$

$\therefore$ sol^n is,

$\therefore y =$ C.F + P.I.

$\therefore y = c_1e^{-2z} + c_2e^{-z} + \frac{e^{3z}}{20}$

$\therefore \boxed{y = \frac{c_1}{x^2} + \frac{c_2}{x} + \frac{x^3}{20}}$

Q solve $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 2\cos(\log x)$

→ given eq^n is cauchy's homogeneous eq^n

Let $x = e^z \Rightarrow z = \log x$

also $x\frac{dy}{dx} = Dy$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$\therefore (D^2 - D)y - 2Dy + 2y = 2\cos z$

$\therefore (D^2 - 3D + 2)y = 2\cos z$

$\therefore$ A.E. eq^n is $D^2 - 3D + 2 = 0$

$$D = 2, 1$$

$\therefore$ C.F $= c_1e^{2z} + c_2e^z$

$\therefore y_1 = e^{2z} \quad y_2 = e^z$

$\therefore \omega = \begin{vmatrix} e^{2z} & e^z \\ 2e^{2z} & e^z \end{vmatrix} = e^{3z} - 2e^{3z}$

$\boxed{\omega = -e^{3z}}$

$\therefore u_1 = -\int \frac{y_2 X}{\omega}\, dz$

$$= -\int \frac{e^z\, 2\cos z}{-e^{3z}}\, dz$$

$$= 2\int e^{-2z}\cos z\, dz$$

$$= 2\left[\frac{e^{-2z}}{5}(-2\cos z + \sin z)\right]$$

$\therefore u_2 = \int \frac{y_1 X}{\omega}\, dz$

$$= \int \frac{e^{2z}\, 2\cos z}{-e^{3z}}\, dz$$

$$= -2\int e^{-z}\cos z\, dz$$

$$= -2\left[\frac{e^{-z}}{2}(-\cos \cancel{bx}\, z + \sin \cancel{bx}\, z)\right]$$

Cauchy's homogeneous linear equations :

→ A linear differential eq^n with variable coefficient is of the form,

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1} x \frac{dy}{dx} + a_n y = \dots$$

where $a_1, a_2, \dots$ are constant and x is the fun of x, is called ....

→ This eq^n reduces to a linear differential eq^n with constant coefficient by taking $x = e^z$, i.e. $z = \log x$

using the notation $D = \frac{d}{dz}$ and chain rule, we have

$$x\frac{dy}{dx} = Dy$$

$$x^2 \frac{d^2 y}{dx^2} = D(D-1)y$$

$$x^3 \frac{d^3 y}{dx^3} = D(D-1)(D-2)y$$

Q Solve $x^2 \frac{d^2 y}{dx^2} + 4x\frac{dy}{dx} + 2y = x^3$.

→ Given eq^n is cauchy's homogeneous eq^n,

Let $x = e^z \Rightarrow z = \log x$

also, $x\frac{dy}{dx} = Dy$

$$x^2 \frac{d^2 y}{dx^2} = D(D-1)y$$

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$\therefore P.I = u_1 y_1 + u_2 y_2$

$= \frac{2}{5}\left[e^{-2z}(-2\cos z + \sin z)\right] \cancel{\frac{2}{5}} e^{2z} - \frac{2}{2}\left[\bar{e}^{z}(\cos z + \sin z)\right] e^{z}$

$= \left(-\frac{4}{5}\cos z + \frac{2}{5}\sin z\right) + \cos z - \sin z$

$= \frac{1}{5}\cos z - \frac{3}{5}\sin z$

$\therefore$ soln is $y = C.F + P.I.$

$= c_1 e^{2z} + c_2 e^{z} + \frac{1}{5}\cos z - \frac{3}{5}\sin z$

$\therefore y = c_1 x^2 + c_2 x + \frac{1}{5}\cos(\log x) - \frac{3}{5}\sin(\log x)$

Q. solve $x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = (1+x)^2$

$\rightarrow$ given eqn is cauchy's homogeneous eqn

also $x = e^z \Rightarrow z = \log x$

also $x\frac{dy}{dx} = Dy$

$x^2\frac{d^2y}{dx^2} = D(D-1)y$

$\therefore (D^2 - D)y - 3Dy + 4y = (1+e^z)^2$

$\therefore (D^2 - 4D + 4)y = (1+e^z)^2$

$\therefore$ A.E. eqn is $D^2 - 4D + 4 = 0$

$\cancel{D = 4 \pm \sqrt{16}}$

$D = 2, 2$

$\therefore C.F = (c_1 + c_2 z)e^{2z}$

$\therefore y_1 = e^{2z}, \; y_2 = ze^{2z}$

$$w = \begin{vmatrix} e^{2z} & ze^{2z} \\ 2e^{2z} & 2ze^{2z} + e^{2z} \end{vmatrix}$$

$= 2ze^{4z} + e^{4z} - 2ze^{4z}$

$\boxed{w = e^{4z}}$

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$\therefore u_1 = -\int \frac{y_2 x}{\omega}\, dz$

$= -\int \frac{z e^{2z}}{e^{4z}} (1+e^{z})^2$

$= -\int z e^{-2z} (1 + 2e^{z} + e^{2z})$

$= -\int z e^{-2z} + 2z e^{-z} + z \, dz$

$\Rightarrow$

$u_2 = \int \frac{y_1 x}{\omega}\, dz$

$= \int \frac{e^{2z} \cdot (1+e^{z})^2}{e^{4z}}\, dz$

$= \int e^{-2z} (1+e^{z})^2\, dz$

Q. H.W.

Solve $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + y = x\sin(\log x)$

→

* **Power series solⁿ :-**

→ $a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n + \dots$

Analytic function

A function $f(x)$ is said tobe analytic at $x = x_0$ if its tailor's series

$$\boxed{f(x_0) + (x-x_0)f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \dots + \frac{(x-x_0)^n}{n!}f^n(x_0) + \dots}$$

exists & it convergent to $f(x)$

Note :- analytic છે કે નહિ તે ચેક કરવા માટે પોઈન્ટ આગળ (ex = $x=2$) માટે taylor's series ($x-2$ ની) શોધવાની પછી તે convergent છે કે નહિ તે check કરવાનું.

* some analytic fxn or series

eg. ① every polynomial is analytic

② $\frac{p(x)}{q(x)}$ is analytic except $q(x) = 0$.

* **ordinary & singular point :-**

consider the eqⁿ

$$y'' + p(x)y' + Q(x)y = 0$$

The point $x = x_0$ is said be an ordinary point if $p(x)$ and $Q(x)$ are analytic at $x=x_0$ otherwise it is called singular point.

→ singular point are of two types :
① regular singular ② irregular singular

→ A singular point $x=x_0$ is said to be regular singular if
$(x-x_0)p(x)$ and $(x-x_0)^2 Q(x)$ are analytic at $x=x_0$

otherwise it is called irregular singular point.

Q. $2x^2y'' + 7x(x+1)y' - 3y = 0$ check $x=0$

→ divide by $2x^2$

$\therefore\ y'' + \frac{7}{2}\frac{(x+1)}{x}y' - \frac{3y}{2x^2} = 0$

$\therefore\ p(x) = \frac{7}{2}\frac{(x+1)}{x}$ $\qquad Q(x) = \frac{-\cancel{y}\,3}{2x^2}$

at $x=0$ p and Q are not analytic.

$\therefore\ x=0$ is a singular point

$\therefore\ (x-0)p(x) = \frac{7}{2}(x+1)$ $\qquad (x-0)^2 = \frac{-3}{2}$

$\therefore\ xp(x)$, $x^2Q(x)$ are analytic at $x=0$

$\therefore\ x=0$ is a regular singular point.

Q. $(x^2-1)y'' + xy' - y = 0$ check $x=0$, $x=1$

→ divide by x^2-1

$\therefore\ y'' + \frac{x}{x^2-1}y' - \frac{y}{(x^2-1)} = 0$

$\therefore\ p(x) = \frac{x}{x^2-1}$ $\quad Q(x) = \frac{-1}{x^2-1}$

→ ∴ at $x=0$ P and Q are analytic
$x=0$ is an ordinary point

at $x=1$
$p(x)$ and $Q(x)$ are not analytic
$x=1$ is a singular point

multiply

$\therefore (x-1)\,p(x) = \frac{x}{x+1}$

$\therefore (x-1)^2 Q(x) = \frac{-(x-1)}{x+1}$

→ $x=1$ $(x-1)p(x)$, $(x-1)^2 Q(x)$ are Analytic

Q. $x^2(x+1)^2 y'' + (x^2-1)y' + 2y = 0$, check $x = 0, 1, -1$

→ ∴ $y'' + \frac{(x^2-1)}{x^2(x+1)^2} y' + \frac{2y}{x^2(x+1)^2}$

$p(x) = \frac{(x^2-1)}{x^2(x+1)}$ $\quad Q(x) = \frac{2}{x^2(x+1)^2}$

at $x=0$ $p(x)$ and $Q(x)$ are not analytic
so it is singular point

multiply

$\therefore\ x\,p(x) = \frac{x-1}{x(x+1)}$ $\quad x^2 Q(x) = \frac{2}{(x^2+1)^2}$

→ at $x=0$ $x\,p(x)$ & $x^2 Q(x)$ are not analytic, પાછો ૦ આવે Irregular
so at $x=0$ is inregular singular point

→ at $x=1$ $x\,p(x)$ and $Q(x)$ are analytic ↳ ordinary
$x=1$ is ordinary point.

at $x=-1$ $p(x)$ and $Q(x)$ are not analytic
so at $x=-1$ is singular point

$\therefore (x+1)\,p(x) = \frac{x-1}{x^2}$ $\quad (x+1)\,Q(x) = \frac{2}{x^2}$

→ so, that $(x+1)p(x)$ and $(x+1)Q(x)$ are analytic
sothat $x=-1$ is regular singular point

Complex Integration

1. path of Integration :-
2. continous arc :-
   let $x = \phi_1(t)$, $y = \phi_2(t)$
   → It's called continous arc if fxn $\phi_1(t)$ or $\phi_2(t)$ are continous functions.

3. smooth arc :-
   → If $\phi_1(t)$ and $\phi_2(t)$ are diffrentiable then it's called smoot arc ex sin, cos, perabola

4. close curve :-

5. simple curve :- If it does not intersect itself it's call..

6. contour :- (simply connected domain)
   → If in a Region R of a complax plane every close curve contains point of that region only that's called.
   → otherwise it is said tobe multipely connect.. domain.

→ $\int_c f(z)\,dz$

$\int_c (u+iv)(dx+idy)$

$\int_c (u\,dx - v\,dy) + i(u\,dy - v\,dx)$

Q. $\int_0^{1+i} (x^2 - iy)\,dz$

→ ① along $y = x$ ② along $y = x^2$

$\therefore I = \int_0^{1+i} (x^2 - iy)(dx + i\,dy)$

$= \int_0^{1+i} (x^2 dx + y\,dy) + i(x^2 dy - y\,dx)$

①

$y = x$

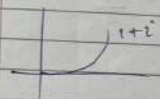
$dy = dx$

also limits are $x=0$ to $x=1$

$\therefore I = \int_0^1 (x^2 dx + x dx) + i(x^2 dx - x dx)$

$= \left[\frac{x^3}{3} + \frac{x^2}{2} + i\left(\frac{x^3}{3} - \frac{x^2}{2}\right)\right]_0^1 = \frac{5}{6} - \frac{i}{6}$

②



$y = x^2$

$dy = 2x\,dx$

also limits are $x=0$ to $x=1$

$\therefore I = \int_0^1 (x^2 dx + 2x^2 dx) + i(2x^3 dx - x^2 dx)$

$= \left[\frac{x^3}{3} + \frac{\cancel{2}x^3}{\cancel{3}} \right]_0^1 + i\left[\frac{x^4}{2} - \frac{x^3}{3} \right]_0^1$

$= \frac{5}{6} + \frac{i}{6}$

Q. $\int (z^2)\,dz$ from 0 to $2+i$

① along the line $2y = x$ ② along real axis from 0 to 2 and then 2 to $2+i$

→ $I = \int z^2 dz$

$= \int (x^2 - y^2 + 2ixy)(dx + i\,dy)$

$= \int_0^{2+i} (x^2 - y^2)\,dx - 2xy\,dy) + i[(x^2 - y^2)\,dy + 2xy\,dx]$

$= \frac{2}{3} + i\frac{11}{3}$

on c_1 $\therefore y=0$ $\therefore dy=0$

also $x=0$ to $x=2$

$$\int_{c_1} = \int_0^2 x^2 dx = \left(\frac{x^3}{3}\right)_0^2 = \frac{8}{3}$$

on c_2 $\therefore x=2$ $\therefore dx=0$

$\rightarrow$ $y=0$ to $y=1$

$$\therefore \int_{c_2} = \int_0^1 -4y\,dy + i(4-y^2)\,dy = \left[-2y^2 + i\left(4y - \frac{y^3}{3}\right)\right]_0^1$$

$$= -2 + \frac{11i}{3}$$

$$\therefore \int_c = \int_{c_1} + \int_{c_2}$$

$$= \frac{8}{3} - 2 + \frac{i11}{3}$$

Remark:- Integral is Independent of path frm $\int_c f(z)dz$ $f(z)$ is analytic.

Q Evaluate $\oint_c (z-a)^n dz$

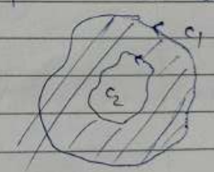
* <u>cauchy - Integral theorm</u> :-

→ If $f(z)$ is analytic & $f'(z)$ is continous everywhere with in and on a simple closed curve c.
then $\oint_c f(z)\,dz = 0$

cor 1 → $\int_p^{a} f(z)\,dz$ is independent of path joinaing p and a if $f(z)$ is analytic in a region & conditioning the path.

cor 2 If $f(z)$ is analytic in the region bounded by two close curve c_1 and c_2

$\int_{c_1} f(z)\,dz = \int_{c_2} f(z)\,dz$



0! = 1

Case 3: between the curve $c, c_1, c_2, \ldots, c_n$ when $c_1, c_2, \ldots, c_n$ lies inside c,

$$\therefore \int_c f(z)\,dz = \int_{c_1} + \int_{c_2} + \cdots \int_{c_n}$$

* Cachy - Integral Formula:

→ If $f(z)$ is analytic within and on a simple closed curve c, and a is any point inside c

$$\int_c \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

$$\int_c \frac{f(z)}{(z-a)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(a)$$

Q. $\int_c (4z^2 - 5z)\,dz$: c is circle in complex plane

→ Here $4z^2 - 5z$ is analytic everywhere