Astr 511: Galaxies as galaxies

University of Washington
Mario Jurić & Željko Ivezić

Dynamics I: Potentials and Orbits

Motivation: Understanding the Structure of Galaxies

Over the past few weeks we've discussed the phenomenology of galaxies: their shapes, sizes, luminosities, and luminosity profiles, their kinematics (velocities of their constituent stars and gas).

We've seen that there are *empirical relationships* between many of these properties (e.g., the Faber-Jackson relation and the fundamental plane, the Tully Fisher relation, $M - \sigma$ relation, etc.).

These connections are not surprising: many of them are a manifestation of the underlying *dynamics* of the system. In a series of lectures to follow we will look at using dynamics to understand the observational relationships, but **also tease out otherwise hidden implications** and insight on the internal structure of galaxies from the observational data.

Outline

1. **Introduction to potentials and orbits**
    - **Spherical potentials**
    - **Axial potentials**
    - **Stellar orbits**
    - **Epicycle approximation**

Reading:

- Binney & Merrifield: ch. 10
- Binney & Tremaine: chs. 2 & 3

Matter distribution, potentials, orbits ...

1. **Matter distribution gives rise to a potential**

2. **Potential gives rise to forces**

3. **Forces give rise to motions (orbits)**

4. **Motions change the distribution of matter; GOTO 1**

The above is just a restatement, in words, of the equations of motion governing matter in the universe. To fully understand galaxies, we need to selfconsistently solve these; in general, this is only possible by numerical integration.

Matter distribution, potentials, orbits ...

There's evidence that galaxies are approximately in *dynamic equilibrium*, i.e., they don't change in time (over $\sim Gyr$ timescales). Therefore:

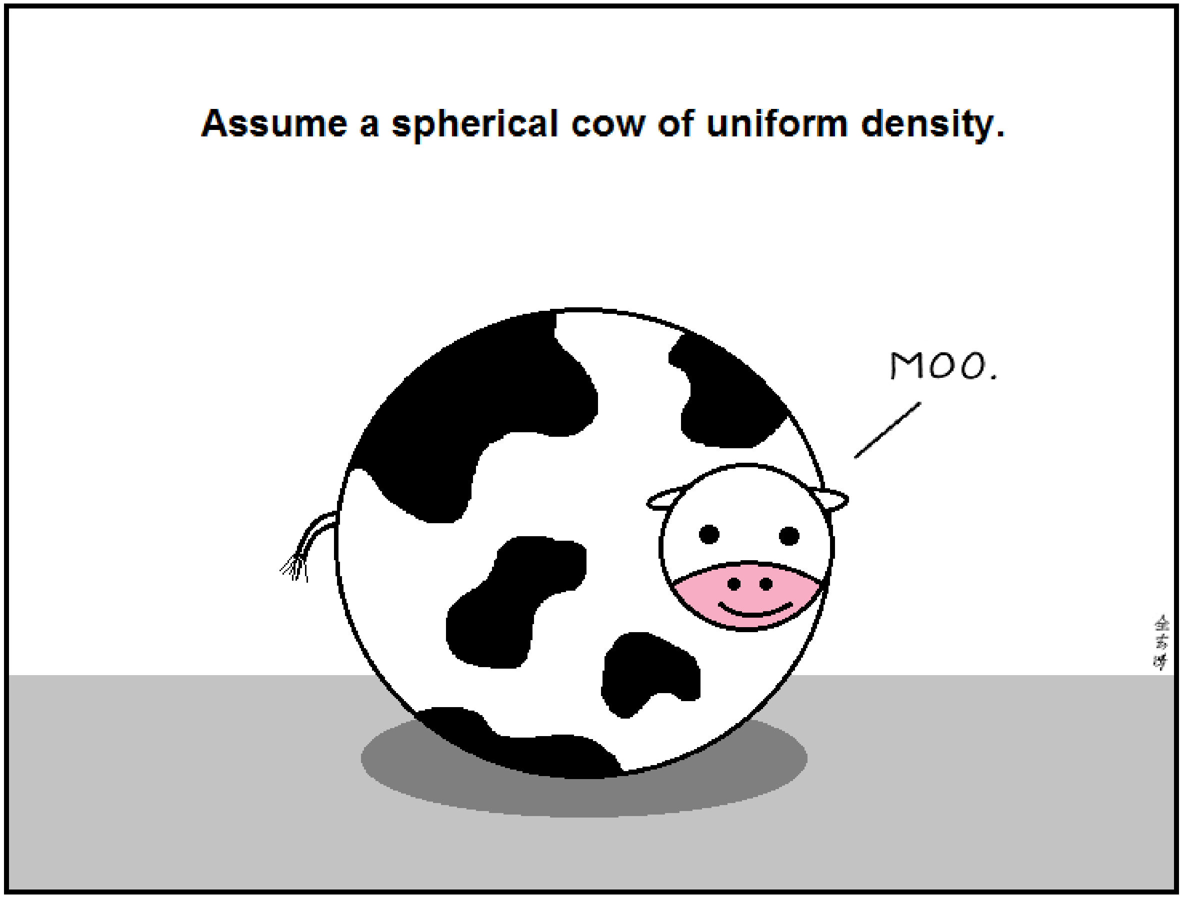
1. the equations of motion must have *self-consistent solutions* where the matter moves in just the right way to keep its distribution unchanged (and the potential) static, and

2. we can separately study possible potentials and the distributions that give rise to them, and the orbits that those potentials admit.

The former gives us an idea of the overall structure of the distribution of mass in galaxies. E.g., this is where one piece of evidence for dark matter comes from.

The latter allows us to examine and understand the allowed pathways for transport of matter through the galaxy, providing insight into the evolution of its content.

Spherical Systems



http://abstrusegoose.com/406

Spherical Systems – Important Quantities

The velocity of a test particle on a circular orbit is the **circular speed**, v_c. Setting the centripetal acceleration equal to the force we get

$$\frac{mv_c^2}{r} = |\mathbf{F}| = \frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{GM(r)m}{r^2} \quad \Rightarrow \quad v_c = \sqrt{\frac{GM(r)}{r}} \tag{1}$$

The circular speed is a measure of the mass interior to r, $M(r)$.

If v_c as a function of r is known, and we *assume that the potential is spherical*, we can compute the mass as a function of r (not the case for a non-spherical distribution.)

Spherical Systems – Important Quantities

Another important quantity is the **escape speed**, v_e, defined by

$$v_e(r) = \sqrt{2|\Phi(r)|}. \tag{2}$$

This definition comes from setting the kinetic energy of a star equal to the absolute value of its potential energy. That is, stars with positive total energy are not bound to the system. In order for a star to escape from from the gravitational field represented by Φ, it is necessary that its speed be greater than v_e. This can be used to get the local Φ of the galaxy.

Spherical Systems – Simple Examples

Let's look at something we're familiar with – a point mass:

$$\rho(r) = M\delta(r) \quad \Rightarrow \quad \Phi(r) = -\frac{GM}{r} \tag{3}$$

Plugging it into equations from previous slides, we find:

$$v_c(r) = \sqrt{\frac{GM}{r}} \quad ; \quad v_e(r) = \sqrt{\frac{2GM}{r}}. \tag{4}$$

Whenever the circular speed declines as $r^{-1/2}$, it is referred to as **Keplerian**. It usually implies that there is no significant mass at that radius.

Spherical Systems – Simple Examples

Homogeneous sphere:

$$M = \frac{4}{3}\pi r^3 \rho \quad ; \quad v_c = \sqrt{\frac{4\pi G\rho}{3}} r. \tag{5}$$

The equation of motion for a particle in such a body is

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{GM(r)}{r^2} = -\frac{4\pi G\rho}{3} r, \tag{6}$$

which describes a harmonic oscillator with period

$$T = \sqrt{\frac{3\pi}{G\rho}}. \tag{7}$$

Spherical Systems – Simple Examples

Independent of r, if a particle is started at r, it will reach the center in a time

$$t_{dyn} = \frac{T}{4} = \sqrt{\frac{3\pi}{16G\rho}}, \tag{8}$$

known as the **dynamical time**. Although this result is only true for a homogeneous sphere, it is a common practice to use this definition with any system of density ρ.

By integrating the density for a homogeneous sphere, we can get the potental:

$$\Phi = \begin{cases} -2\pi G\rho(a^2 - \frac{1}{3}r^2), & r < a \\ -\frac{4\pi G\rho a^3}{3r}, & r > a. \end{cases}$$

One would expect the center of a galaxy to have a potential of this type if there is no cusp in the central density (implying a linear rise in v_c).

Spherical Systems – Simple Examples

Isochrone potential:

$$\Phi(r) = -\frac{GM}{b + \sqrt{b^2 + r^2}}. \tag{9}$$

This potential has the nice property of going from a harmonic oscillator in the core to a Keplerian potential at large r, with the transition occurring at a scale b.

The circular speed is

$$v_c^2 = \frac{GMr^2}{(b+a)^2 a}, \tag{10}$$

where $a \equiv \sqrt{b^2 + r^2}$.

Spherical Systems – Simple Examples

Using Poisson's equation ($\nabla^2\Phi = 4\pi G\rho$), we can find the density:

$$\rho(r) = \frac{1}{4\pi G r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = M\left[\frac{3(b+a)a^2 - r^2(b+3a)}{4\pi(b+a)^3a^3}\right]. \quad (11)$$

So the central density is

$$\rho(0) = \frac{3M}{16\pi b^3}, \quad (12)$$

and the asymptotic density is

$$\rho(r) \approx \frac{bM}{2\pi r^4}. \quad (13)$$

Potential–Density Pairs

The isochrone potential and density shown in previous slide was an example of a **potential-density pair**. These are models admitting **equilibrium** solutions to Newtonian equations of motion that are similar to observed density and kinematic profiles.

Other frequently used models include:

Plummer's (1911) model: spherically symmetric, originally used to model globular clusters
Kuzmin's (1956) model: infinitely thin disk (aka *Toomre's model 1*)
Plummer–Kuzmin models: introduced by Miyamoto & Nagai (1975), smooth transition from Plummer's to Kuzmin's models
Logarithmic potentials: the circular speed is a constant at large radii (very frequently used because of this property).

The Milky Way Density and Potential

The most popular Milky Way *stellar density* models are double exponential disk (thin and thick in the Z direction, also exponential dependence in the R direction), with a power-law or logarithmic halo.

But when kinematic information is taken into account, we discover that these distributions would not be in dynamic equilibrium all by themselves. This leads us to postulate the existence of an additional, dark, matter component (more in an upcoming lecture).

Some recent good reviews:

Bahcall (1986, ARA&A 24, 577)

Gilmore, Wyse & Kuijken (1989, ARA&A 27, 555)

Majewski (1993, ARA&A 31, 575)

Freeman & Bland-Hawthorn (2002, ARA&A 40, 487)

Orbits in Static Spherical Potentials

The problem: given the initial conditions $\mathbf{x}(t_o)$ and $\dot{\mathbf{x}}(t_o)$, and the potential $\Phi(r)$, find $\mathbf{x}(t)$.

Orbits in spherical potentials are easy to consider and lead to some important concepts.

- Some general considerations
- Example 1: Spherical harmonic oscillator: $\Phi(r) = A + B\,r^2$
- Example 2: Point mass potential: $\Phi(r) = \frac{-GM}{r}$
- Example 3: Isochrone potential: $\Phi(r) = \frac{-GM}{b+\sqrt{b^2+r^2}}$

General considerations

The initial conditions are 6-dimensional and thus a general solution includes six orbital parameters. (aka *constants of motion*)

The equation of motion in a spherical potential is:

$$\ddot{\mathbf{r}} = F(r)\hat{\mathbf{e}}_r, \tag{14}$$

i.e. the force is always (unsurprisingly) radial.

Crossing through by $\mathbf{r}$, we show that the angular momentum vector, $\mathbf{L} \equiv \mathbf{r} \times \dot{\mathbf{r}}$ is conserved:

$$\frac{\mathrm{d}}{\mathrm{dt}}(\mathbf{r} \times \dot{\mathbf{r}}) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{dt}} \times \frac{\mathrm{d}\mathbf{r}}{\mathrm{dt}} + \mathbf{r} \times \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{dt}^2} = \mathbf{r} \times F(r)\hat{\mathbf{e}}_r = 0 \tag{15}$$

Therefore, the motion is constrained to the plane perpendicular to $\mathbf{L}$, and can be fully described in cylindrical coordinate system, r and ψ ($\mathbf{v} = \dot{r}\,\hat{\mathbf{e}}_r + r\dot{\psi}\,\hat{\mathbf{e}}_\psi$)

General considerations

The equations of motion in the plane are

$$\begin{aligned} \ddot{r} - r\dot{\psi}^2 &= F(r) \\ 2\dot{r}\dot{\psi} + r\ddot{\psi} &= 0. \end{aligned}$$

Integrating the second equation gives us $r^2\dot{\psi} = L =$ const. (note that this is the second Kepler's law!)

$\dot{\psi}$ can be eliminated using $\dot{\psi} = L/r^2$, leading to a one-dimensional equation of motion:

$$\ddot{r} - L^2/r^3 = F(r). \tag{16}$$

This equation motivates a definition of an effective potential

$$-\nabla\Phi_{\rm eff} \equiv F(r) + L^2/r^3, \tag{17}$$

and thus

$$\Phi_{\rm eff}(r) \equiv \Phi(r) + \frac{L^2}{2r^2}. \tag{18}$$

General considerations

The energy per unit mass is

$$E = \frac{1}{2}v^2 + \Phi(r) = \frac{1}{2}(\dot{r}^2 + r^2\dot{\psi}^2) + \Phi(r) = \frac{1}{2}\dot{r}^2 + \Phi_{\text{eff}}(r). \quad (19)$$

We can transform this expression into $\dot{r} = dr/dt = f(E, L, r)$, and integrate over r to find the period. For bound orbits r oscillates between an inner radius, or pericenter (r_{min}), and an outer radius, or apocenter (r_{max}). The **radial period** is

$$T_r = 2\int_{r_{\text{min}}}^{r_{\text{max}}} (\sqrt{2[E - \Phi_{\text{eff}}(r)]})^{-1}\, dr \quad (20)$$

The pericenter and apocenter are the solutions of $\Phi_{\text{eff}}(r) = E$.

General considerations

The **azimuthal period** is

$$T_\psi = \frac{2\pi}{\Delta\psi} T_r \tag{21}$$

where

$$\Delta\psi = 2L \int_{r_{\text{min}}}^{r_{\text{max}}} \left(r^2 \sqrt{2[E - \Phi_{\text{eff}}(r)]}\right)^{-1} dr \tag{22}$$

The orbit is closed only for $\Delta\psi = k(2\pi)$ – in general case, the orbit forms a rosette.

The orbital precession rate:

$$\Omega_p = \frac{\Delta\psi - 2\pi}{T_r} \tag{23}$$

General considerations

If we eliminate t rather than ψ, then we have an equation for the orbit's shape. In terms of the variable $u \equiv 1/r$

$$\frac{\mathrm{d}^2u}{\mathrm{d}\psi^2} + u = -\frac{F(u)}{L^2u^2} \quad \Rightarrow \quad \frac{\mathrm{d}^2u}{\mathrm{d}\psi^2} = \zeta(u). \tag{24}$$

This is a second order differential equation for $u(\psi)$, where $\zeta(u)$ and the initial conditions are presumably specified. It is not analytically soluble in the general case, but analytic solutions exist for a few commonly encountered potentials.

Let's now look at specific examples.

The harmonic potential

$$\Phi = \Phi_0 + \frac{1}{2}\Omega^2 r^2. \tag{25}$$

Generated by homogeneous density distribution.

The motion decouples in cartesian co-ordinates to $\ddot{x} = -\Omega^2 x$ and $\ddot{y} = -\Omega y$, and the solution is:

$$x = X\cos(\Omega t + \phi_x), \quad y = Y\sin(\Omega t + \phi_y), \tag{26}$$

where X, Y, ϕ_x and ϕ_y are arbitrary constants (determined from initial conditions).

This is the equation for an ellipse **centered** on the origin.

Orbits are closed since the periods for x and y oscillations are identical.

Point mass (Keplerian) potential

$$\Phi(r) = \frac{-GM}{r} \tag{27}$$

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\psi^2} + u = \frac{GM}{L^2} \quad \Rightarrow \quad u = \frac{GM}{L^2}[1 + e\cos(\psi - \psi_0)]. \tag{28}$$

This is the equation for an ellipse with one **focus at the origin** and eccentricity e (the first Kepler's law). The semi-major axis is $a = L^2/GM(1 - e^2)$.

The motion is periodic in ψ.This gives a closed orbit with

$$T_r = T_\psi = 2\pi\sqrt{\frac{a^3}{GM}} = 2\pi GM(2|E|)^{-3/2} \tag{29}$$

Note that $T^2 \propto a^3$ – the third Kepler's law!

Isochrone Potential

$$\Phi(r) = \frac{-GM}{b + \sqrt{b^2 + r^2}} \tag{30}$$

More extended than point mass, less extended than harmonic potential.

T_r same as for the Keplerian case ($T_r = 2\pi GM(2|E|)^{-3/2}$).

However,

$$\Delta\psi = \pi \left[1 + \frac{L}{\sqrt{L^2 + 4GMb}} \right] \tag{31}$$

i.e. $\pi < \Delta\psi < 2\pi$, and hence **the orbits are not closed!**

Axisymmetric Potentials

The problem: given the initial conditions $\mathbf{x}(t_o)$ and $\dot{\mathbf{x}}(t_o)$, and the potential $\Phi(R, z)$, find $\mathbf{x}(t)$.

A better description of real galaxies than spherical potentials, and the orbital structure is much more interesting.

- Poisson's equation for axisymmetric potentials, meridional plane

- Non-axisymmetric examples

- Epicycle approximation

Axisymmetric Potentials

The equations of motion in an axisymmetric potential (cylindrical coordinates) are

$$\ddot{R} = -\frac{\partial \Phi_{\mathrm{eff}}}{\partial R} \tag{32}$$

and

$$\ddot{z} = -\frac{\partial \Phi_{\mathrm{eff}}}{\partial z} \tag{33}$$

where

$$\Phi_{\mathrm{eff}} \equiv \Phi + \frac{L_z^2}{2R^2} \tag{34}$$

The total angular momentum is not conserved any more (we've broken the rotational symmetry!), but its $\hat{z}$ component is:

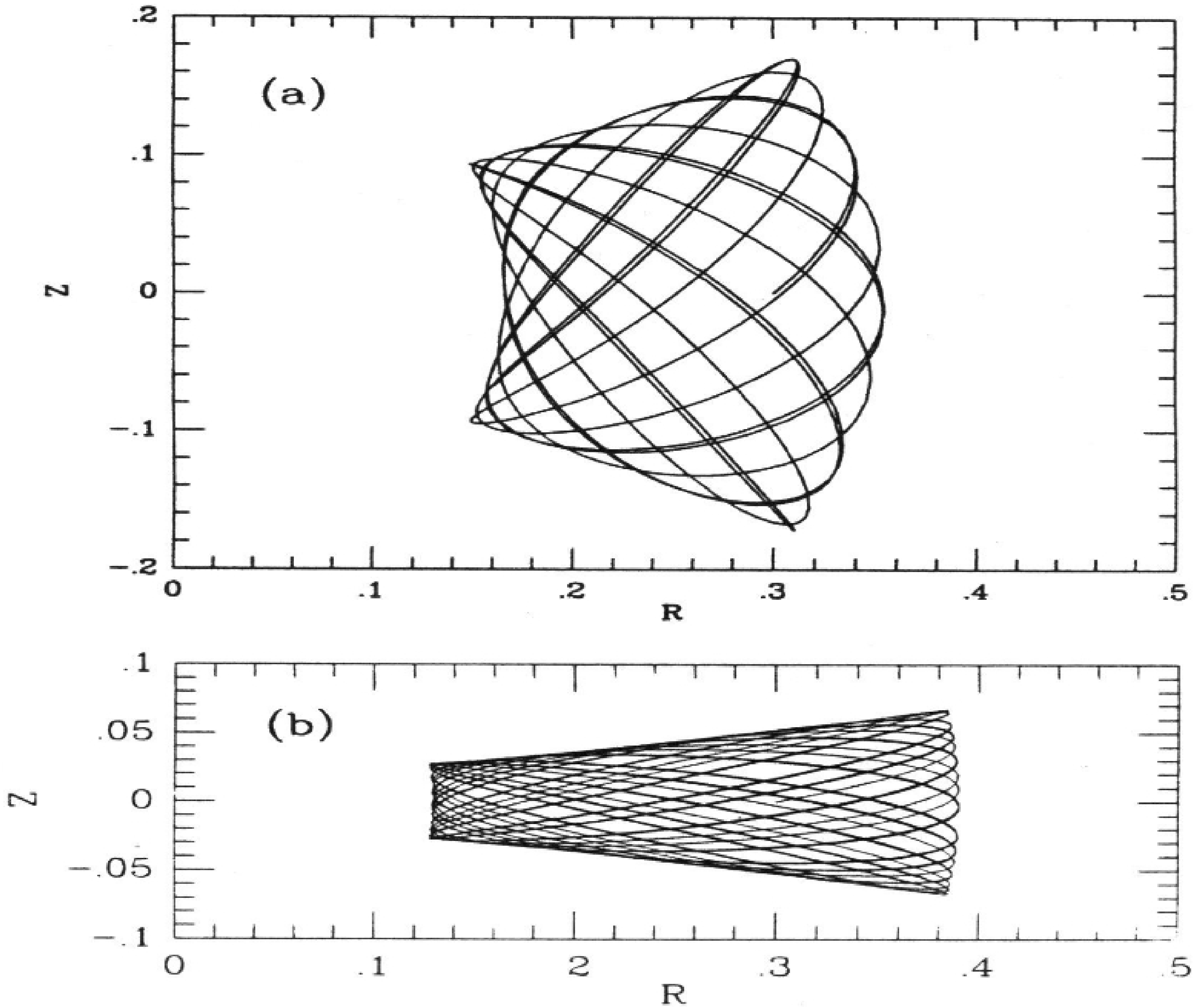
$$\frac{\mathrm{d}}{\mathrm{d}t}(R^2\dot{\phi}) = 0 \quad \Rightarrow \quad R^2\dot{\phi} = L_z. \tag{35}$$

Axisymmetric Potentials

Hence, if we solve the first two equations, the solution for ϕ can be obtained from the last equation as

$$\phi(t) = \phi(t_o) + \dot{\phi}(t_o)\, R^2(t_o)\, \int_{t_o}^{t} dt'/R^2(t') \tag{36}$$

The three-dimensional motion in the cylindrical (R, z, ϕ) space is reduced to a two-dimensional problem in **Cartesian** coordinates R and z. Therefore, it's sufficient to study the motion of the particle in the **meridional plane** defined by (R, z) coordinates.


(a)
.2
.1
0
−.1
−.2
Z
0
.1
.2
.3
.4
.5
R
(b)
.1
.05
0
−.05
−.1
Z
0
.1
2
.3
.4
.5
R

Axisymmetric Logarithmic Potential

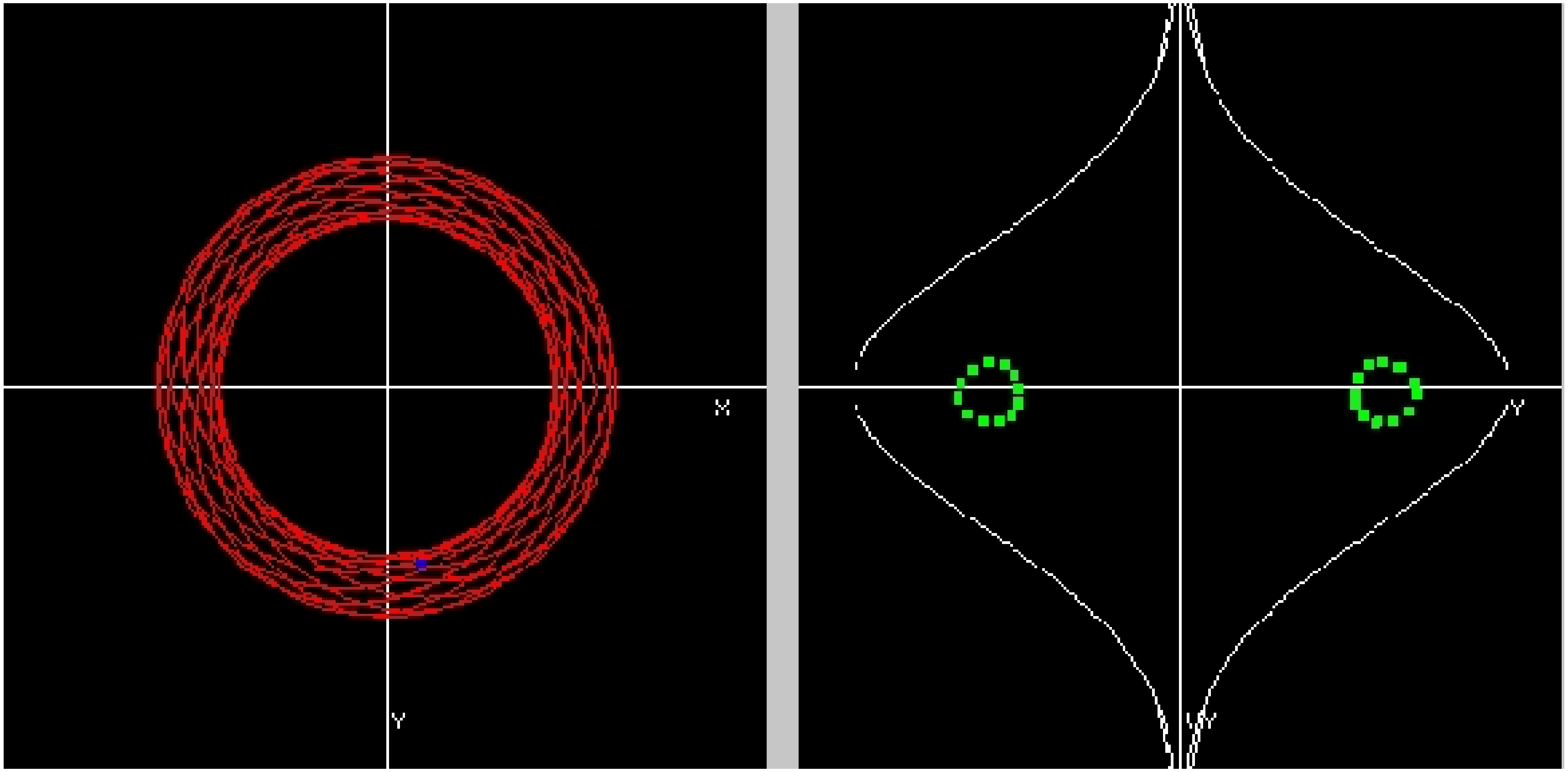
A good approximative model to illustrate the properties of orbits in axisymmetric potentials is one of the form:

$$\Phi = \frac{1}{2} v_0^2 \ln \left(R^2 + \frac{z^2}{q^2} \right) \tag{37}$$

(e.g., see Binney & Tremaine, figs. 3-2, 3-3 and 3-4). This potential asymptotes to a 2D harmonic oscillator (a homogeneous sphere) close to the center, and for $q = 1$ it is equivalent to the spherical logarithmic potential which yields constant rotation curves.

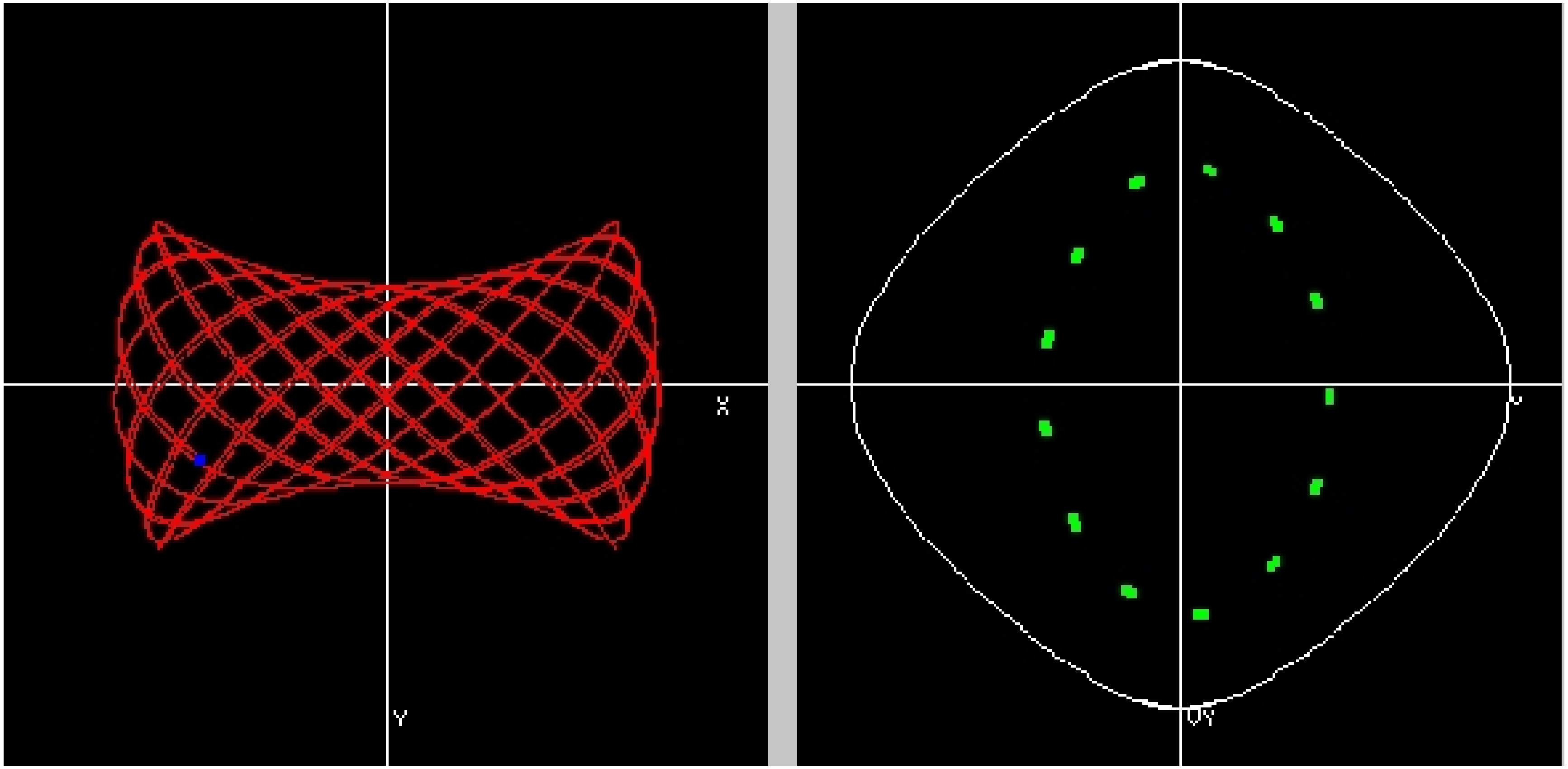
What kinds of orbits do we find in this potential?

Loop Orbit



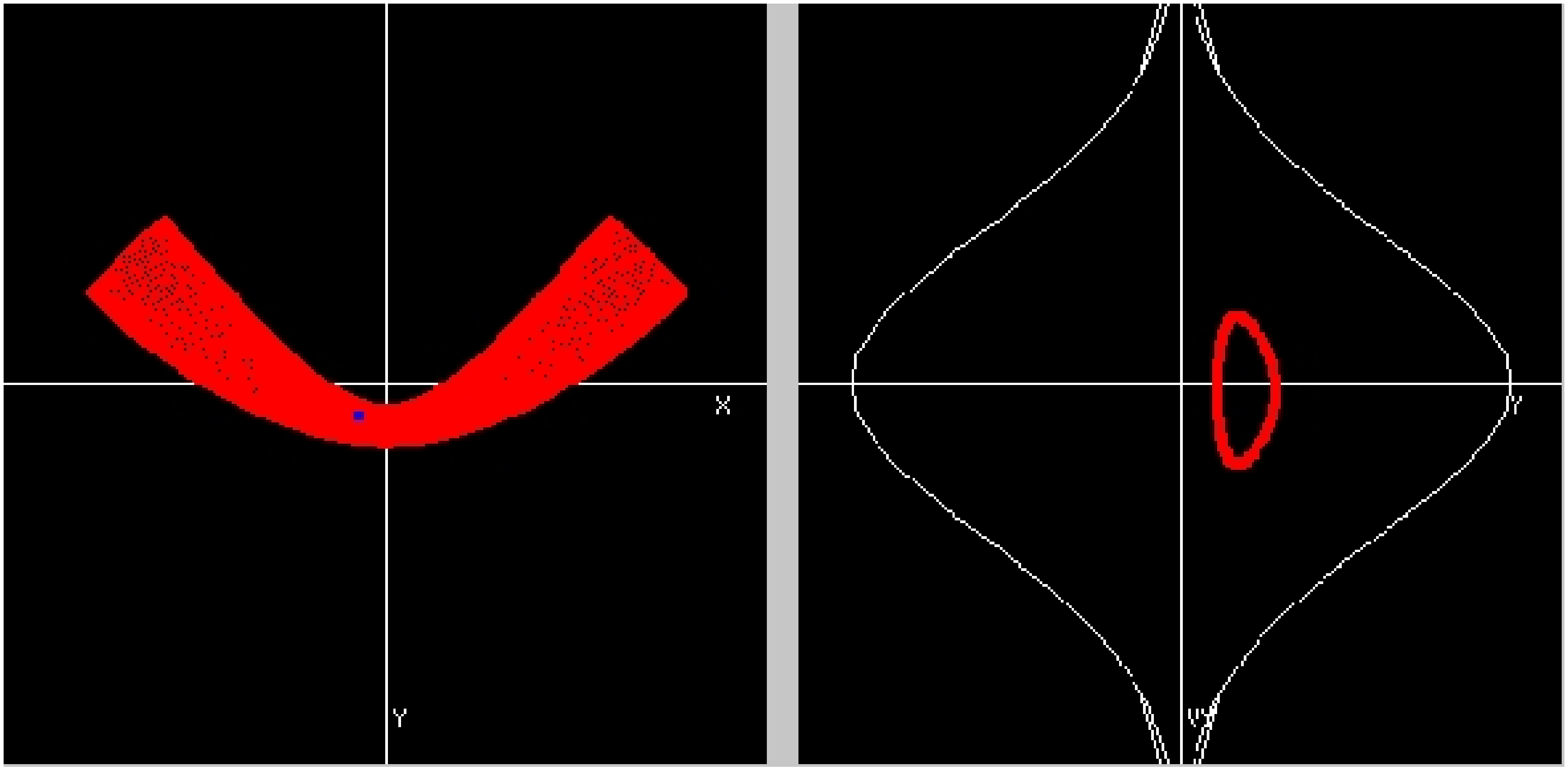
As mentioned earlier, the radial and azimuthal frequencies of an orbit depend on the potential and generally aren't commensurate. In general, the trajectory of a particle forms a *rosette*. If the particle moves so that there's a sense of net rotation, the orbit is called a *loop* orbit. (Above: a member of the 1:1 resonance family; more later).

Box Orbit



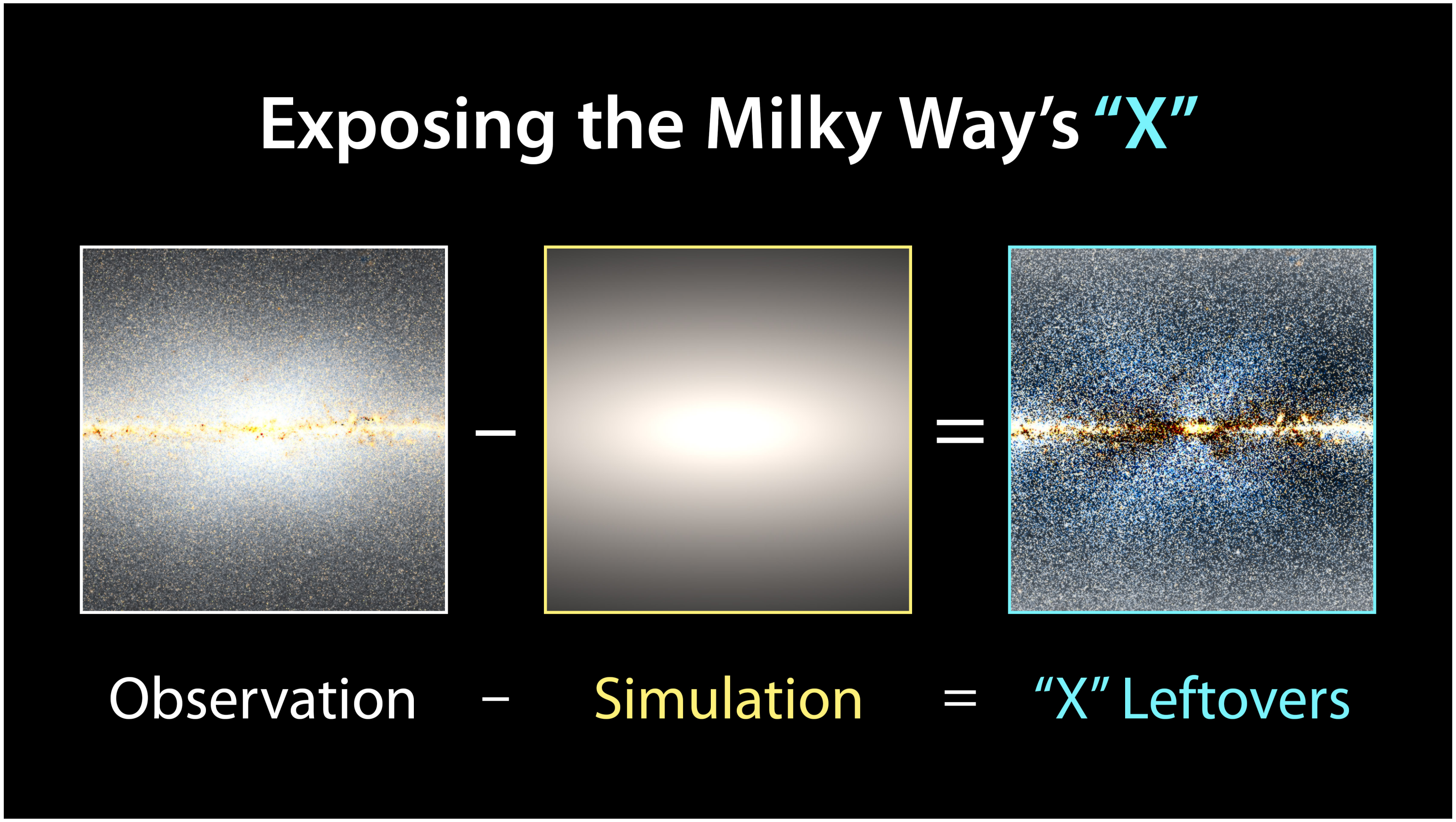
Orbits with no net sense of rotation belong to a class of *box orbits*. Particles in these orbits can get arbitrarily close to the center; over time, they will completely fill the "box" in the $x - y$ plot. A large fraction of stars in **galactic bars** are on box orbits (e.g. Valluri 2016).

Banana Orbit



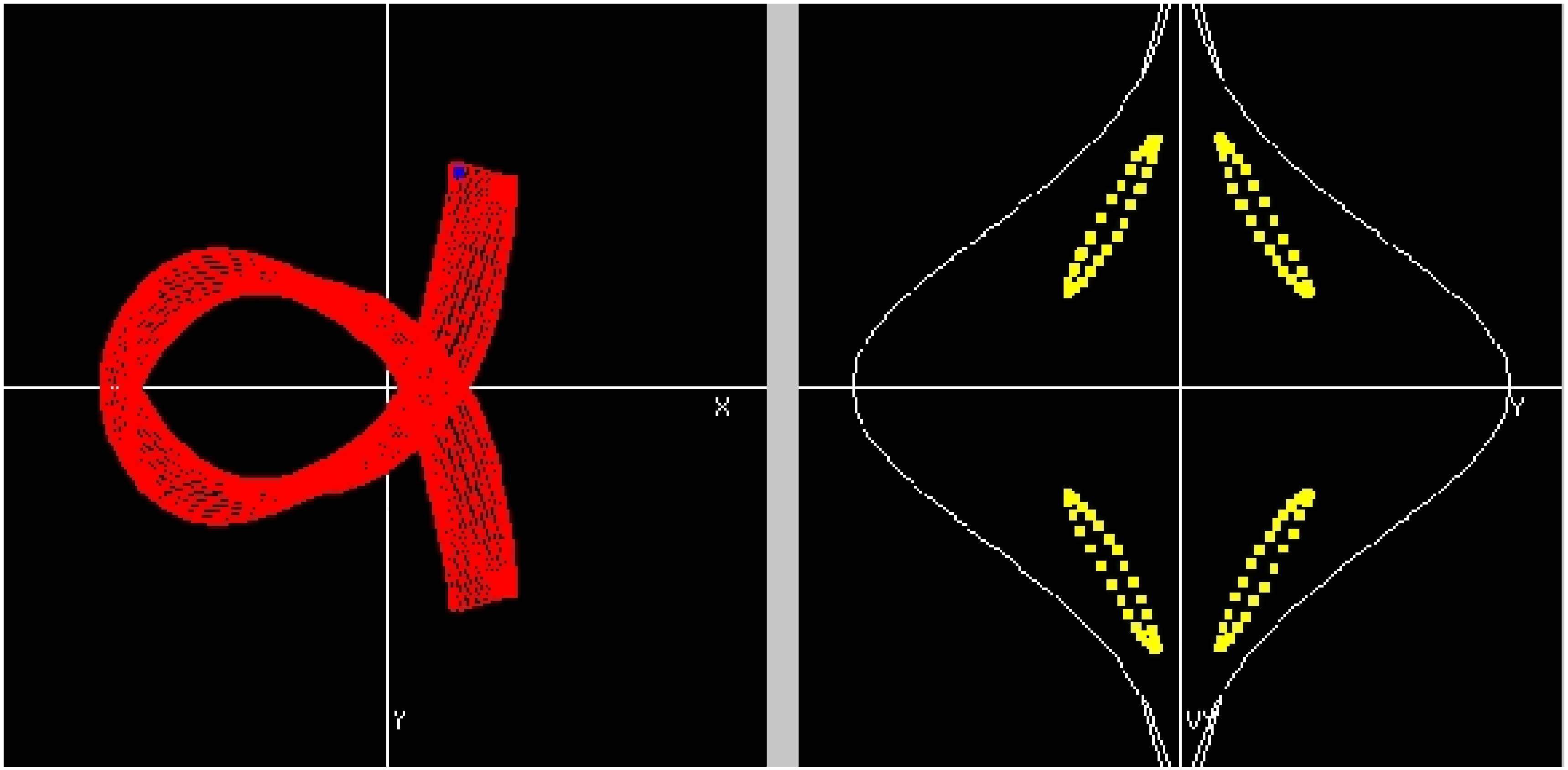
If oscillation frequencies are nearly commensurate we find orbits that are more isolated in $x - y$ space. Above: a member of the "bannana orbit" family (associated with the 2:1 resonance).

X-shaped Bulge of the Milky Way



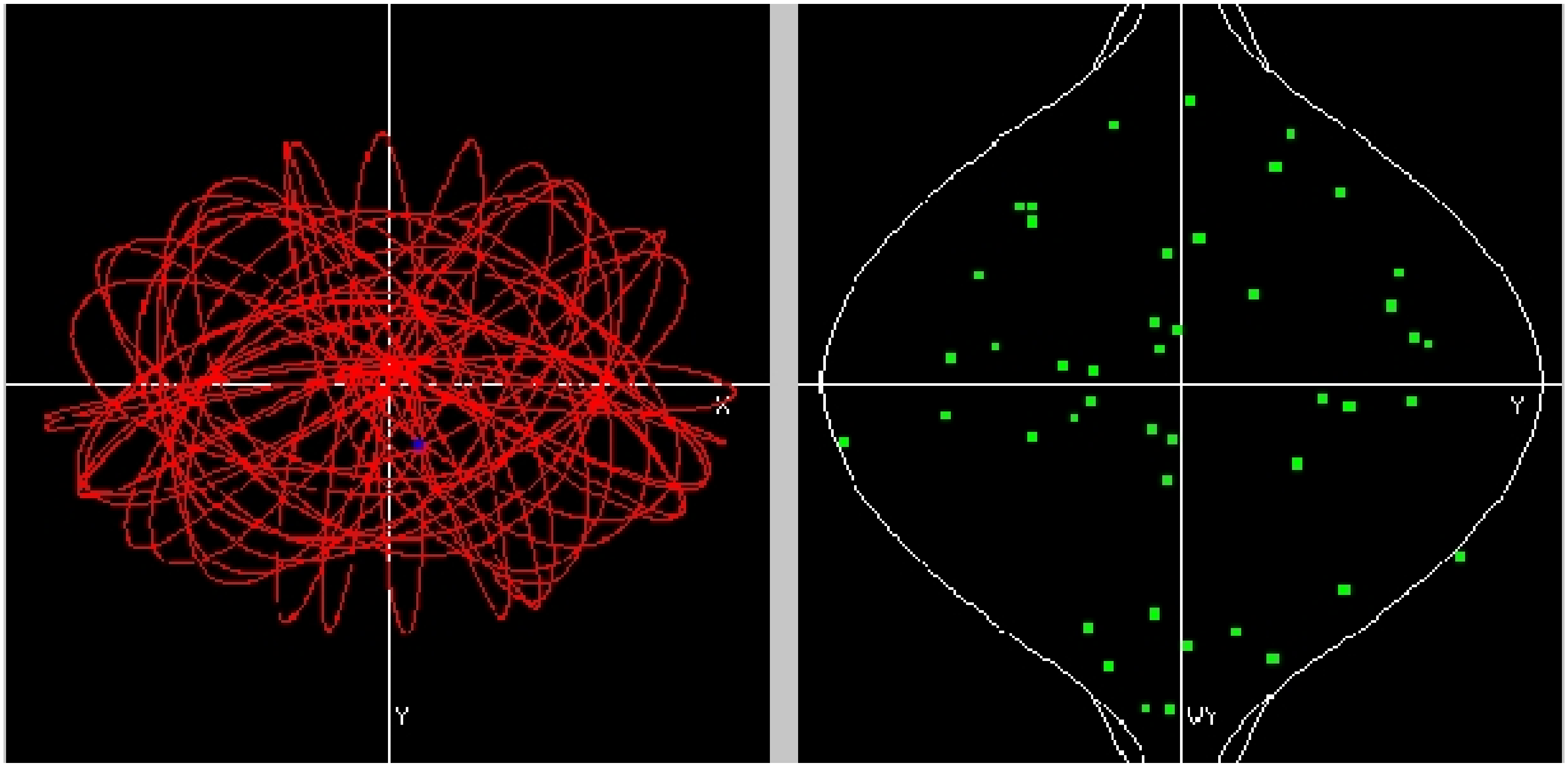
X-shaped Bulge of the Milky Way as revealed by WISE (Ness & Lang 2006; https://arxiv.org/pdf/1603.00026.pdf)

Fish Orbit



Above: a member of the "fish orbit" family (associated with the 3:2 resonance).

Box Orbit Scattered by a Point Mass



Adding a point mass into the center (e.g., a black hole), destroys the order: a particle can get arbitrarily close to the center and be scattered by the BH to a different orbit every time it does. An extreme example: hypervelocity stars (Brown et al. 2005)

Axisymmetric Potentials: Insight into galaxy shapes

Studies of orbits in axisymmetric potentials give us insight into the origin of structures observed in galaxies (rings, bulges, bars, etc.). This is a rich area of theory to explore!

For an interesting recent article on how all this applies to the structure of the Galactic bulge, see Portail, Wegg and Gerhard (2015; arXiv:1503.07203).

For a modern approach, see Thomas et al. 2004, MNRAS 353, 391: orbit libraries, a Voronoi tessellation of the surface of section, the reconstruction of phase-space distribution function

For a more classic orbital analysis (and if you are interested in finding out what is an "antipretzel"), see Miralda-Escudé & Schwarzschild 1989 (ApJ 339, 752): Another classic paper is de Zeeuw 1985 (MNRAS 216, 272) (interested in "unstable butterflies"?)

Epicycle Approximation for Orbits

Assume an axisymmetric potential $\Phi_{\rm eff}$ and **nearly circular orbits**; expand $\Phi_{\rm eff}$ in a Taylor series about its minimum:

$$\Phi_{\rm eff} = \text{const} + \frac{1}{2}\kappa^2 x^2 + \frac{1}{2}\nu^2 z^2 + \cdots , \tag{38}$$

where

$$x \equiv R - R_g, \quad \kappa^2 \equiv \left.\frac{\partial^2 \Phi_{\rm eff}}{\partial R^2}\right|_{(R_g,0)}, \quad \nu^2 \equiv \left.\frac{\partial^2 \Phi_{\rm eff}}{\partial z^2}\right|_{(R_g,0)}. \tag{39}$$

The equations of motion decouple and we have two integrals:

$$x = X\cos(\kappa t + \phi_0) \qquad z = Z\cos(\nu t + \zeta)$$
$$E_R \equiv \frac{1}{2}[v_R^2 + \kappa^2 (R - R_g)^2] \qquad E_z \equiv \frac{1}{2}[v_z^2 + \nu^2 z^2].$$

Epicycle Approximation

Now compare the **epicycle frequency**, κ, with the angular frequency, Ω.

$$\Omega^2 \equiv \frac{v_c^2}{R^2} = \frac{1}{R}\frac{\partial \Phi}{\partial R} = \frac{1}{R}\frac{\partial \Phi_{\text{eff}}}{\partial R} + \frac{L_z^2}{R^4}, \tag{40}$$

$$\kappa^2 = \frac{\partial (R^2 \Omega^2)}{\partial R} + \frac{3L_z^2}{R^4} = R\frac{\partial \Omega^2}{\partial R} + 4\Omega^2. \tag{41}$$

Since Ω always decreases, but rarely faster than Keplerian ($\Omega \propto R^{-\frac{3}{2}}$):

$$\Omega \leq \kappa \leq 2\Omega. \tag{42}$$

Note: These are not the infamous epicycles of Ptolemy's!

Epicycle Approximation

The epicycle approximation also makes a prediction for the ϕ-motion since $L_z = R^2\dot{\phi}$ is conserved. Let

$$y \equiv R_g[\phi - (\phi_0 + \Omega t)] \tag{43}$$

be the displacement in the ϕ direction from the "guiding center". If we expand L_z to first order in displacements from the guiding center, we obtain

$$\phi = \phi_0 + \Omega t - \frac{2\Omega X}{\kappa R_g} \sin(\kappa t + \phi_0). \tag{44}$$

Therefore

$$y = -Y \sin(\kappa t + \phi_0) \quad \text{where} \quad \frac{Y}{X} = \frac{2\Omega}{\kappa} \equiv \gamma \geq 1. \tag{45}$$

$\Rightarrow$ The epicycles are elongated tangentially (for Keplerian motion $\gamma = 2$: **epicycles are not circles** as assumed by Hipparchus and Ptolomey!)

The epicycle frequency (κ) is related to **Oort's constants**:

$$A \equiv \frac{1}{2}\left(\frac{v_c}{R} - \frac{dv_c}{dR}\right)_{R_\odot} = -\frac{1}{2}\left(R\frac{d\Omega}{dR}\right)_{R_\odot} \tag{46}$$

$$B \equiv -\frac{1}{2}\left(\frac{v_c}{R} + \frac{dv_c}{dR}\right)_{R_\odot} = -\left(\frac{1}{2}R\frac{d\Omega}{dR} + \Omega\right)_{R_\odot} = A - \Omega_\odot \tag{47}$$

Then

$$\kappa_\odot^2 = -4B(A-B) = -4B\Omega_\odot \tag{48}$$

In the solar neighborhood,

$$A = 14.5 \pm 1.5 \text{ km/s/kpc}, \quad B = -12 \pm 3 \text{ km/s/kpc}, \tag{49}$$

and so

$$\kappa_\odot = 36 \pm 10 \text{ km/s/kpc}, \tag{50}$$

and

$$\frac{\kappa_\odot}{\Omega_\odot} = 1.3 \pm 0.2 \quad (> 1 \text{ and } < 2!) \tag{51}$$

For improvements to epicycle approximation see Dehnen 1999 (AJ 118, 1190)