

cheatsheet

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1 First Order Linear ODEs

Integrating Factor:

$$\frac{dy}{dt} + p(t)y = g(t)$$

$$\mu(t)y(t) = \int \mu(t)g(t)dt + c \text{ where } \mu(t) = e^{\int p(t)dt}$$

Theorems of Validity

$$y'(t) + p(t)y = g(t), y(t) = y_0$$

If $p(t)$ and $g(t)$ are continuous in $\alpha < t < \beta$ and interval contains t_0 , then there is a unique solution on that interval.

$$y' = f(t, y)$$

If $f(t, y)$ and $\frac{\partial f}{\partial y}$ are continuous in some rectangle containing (t_0, y_0) then there is a unique solution.

2 Non-linear First Order ODEs

Separable

$$N(y)\frac{dy}{dx} = M(x)$$

Exact

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \text{ where } \frac{\partial f}{\partial x} = M(x, y) \text{ and } \frac{\partial f}{\partial y} = N(x, y)$$

$$f(x, y) = c$$

Bernoulli

$$y' + p(x)y = q(x)y^n \text{ substitute } v = y^{1-n} \text{ to get}$$

$$\frac{1}{1-n}v' + p(x)v = q(x)$$

Solve by integrating factor method.

Other substitutions

$$y' = F\left(\frac{y}{x}\right)$$

$$\text{Substitute } v(x) = \frac{y}{x} \text{ to get } \frac{dv}{F(v) - v} = \frac{dx}{x}$$

$$y' = G(ax + by)$$

$$\text{Substitute } v(x) = ax + by \text{ to get } \frac{dv}{a + bG(v)} = dx$$

3 Homogeneous 2nd Order ODEs

Characteristic equation

$$ay'' + by' + cy = 0$$

real, distinct roots ($r_1 \neq r_2$): $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ complex roots ($r_{1,2} = \lambda \pm \mu i$): $y(t) = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t$ double root ($r_1 = r_2 = r$): $y(t) = c_1 e^{rt} + c_2 t e^{rt}$

4 Non-homogeneous 2nd Order ODEs with constant coefs

$$y''(t) + p(t)y'(t) + q(t)y = g(t)$$

$$y(t) = t_c(t) + y_p(t)$$

Particular Solutions

Undetermined Coefficients Guess $y_p(t)$ based on the form of $g(t)$ then plug into complementary solution to solve.

$g(t)$	$y_p(t)$ guess
$a e^{\beta t}$	$A e^{\beta t}$
$a \cos(\beta t)$	$A \cos \beta t + B \sin \beta t$
$a \sin(\beta t)$	$A \cos \beta t + B \sin \beta t$
n^{th} polynomial	$A_n t^n + A_{n-1} t^{n-1} \dots A_1 t + A_0$

Variation of Parameters Given,

$$y_c(t) = c_1 y_1(t) + c_2 y_2(t)$$

$$y_p(t) = y_1 u_1 + y_2 u_2, \text{ where}$$

$$u_1 = - \int \frac{y_2 g(t)}{W(y_1, y_2)} dt, u_2 = \int \frac{y_1 g(t)}{W(y_1, y_2)} dt, \text{ where}$$

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$$

5 System of Linear First-Order ODEs

$$\vec{X}' = A \vec{X} + \vec{b}(t)$$

Eigenvalues **method** real, distinct eigenvalues: $\vec{X}(t) =$
 $c_1 e^{\lambda_1 t} \vec{\eta}(1) + c_2 e^{\lambda_2 t} \vec{\eta}(2)$
 complex eigenvalues: Plug into the equation above and simplify with $e^{i\theta} = \cos \theta + i \sin \theta$
 double eigenvalues: $\vec{X} = c_1 e^{\lambda t} \vec{\eta} + c_2 (t e^{\lambda t} \vec{\eta} + e^{\lambda t} \vec{\rho})$ where $(A - \lambda I) \vec{\rho} = -\vec{\eta}$

Undetermined coefficients Coefficients are vectors now.

Variation of parameters Given \vec{X} is a solution:

$$\vec{X}_p = X \int X^{-1} \vec{b} dt$$

Matrix exponential method

$$\vec{X} = e^{At} (Y_0 + A^{-1}b) - A^{-1}b \text{ where}$$

$$e^{At} = S D S^{-1} \text{ where}$$

S is the matrix of eigenvectors and D is the diagonal exponential matrix of eigenvalues.

6 Homogeneous 2nd Order ODEs with non-constant coeffs

Euler equations

$$ax^2y'' + bxy' + cy = 0, \text{ solve}$$

$$ar(r-1) + b(r) + c = 0$$

real, distinct roots: $y(x) = c_1 x^{r_1} + c_2 x^{r_2}$ double roots: $y(x) = x^r (c_1 + c_2 \ln x)$
 complex roots: $y(x) = c_1 x^\lambda \cos \mu \ln x + c_2 x^\lambda \sin \mu \ln x$

Bessel Functions

$$x^2 y'' + x y' + (x^2 - \alpha^2) y = 0$$

$$y(x) = c_1 J_\alpha(x) + c_2 Y_\alpha(x)$$

Spherical Bessel Functions

$$x^2 y'' + 2xy' + (x^2 - n(n+1))y = 0$$

$$y(x) = c_1 j_n(x) + c_2 y_n(x)$$

Modified Bessel Functions

$$x^2 y'' + xy' + (x^2 + \alpha^2)y = 0$$

$$y(x) = c_1 I_\alpha(x) + c_2 K_\alpha(x)$$

7 First Order PDEs

Kinematic Wave Equations

$$\frac{\partial y}{\partial t} + \frac{\partial \Phi}{\partial x} = 0$$

Substitute $\Phi = \Phi(y)$

$$\frac{\partial y}{\partial t} + c \frac{\partial y}{\partial x} = 0, c = \frac{d\Phi}{dy}$$

Characteristic Curves: Homogeneous Quasilinear Equations with Constant Coefficients

$$P(z, x, y) \frac{\partial z}{\partial x} + Q(z, x, y) \frac{\partial z}{\partial y} = R(z, x, y)$$

with IC $z = z^0(x, y)$ along the curve $I^0: y = y^0(x)$

Express IC as a parametric equation:

$$x = \xi, y = y^0(\xi), z = z^0(\xi)$$

The solution surface is represented in parametric form:

$$\frac{dz}{ds} = \frac{\partial z}{\partial x} \frac{dx}{ds} + \frac{\partial z}{\partial y} \frac{dy}{ds}$$

with

$$\frac{dx}{ds} = P(z, x, y), \frac{dy}{ds} = Q(z, x, y), \frac{dz}{ds} = R(z, x, y)$$

8 Second Order PDE Examples

Heat Equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, u(x, 0) = f(x), u(0, t) = 0, u(L, t) = 0$$

Separation of variables, reduces PDE to two ODEs

$$u(x, t) = \Phi(x)G(t)$$

$$\frac{1}{kG} \frac{dG}{dt} = \frac{1}{\Phi} \frac{d^2\Phi}{dx^2}$$

Solve and sum over all POSSIBLE solutions:

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-k(\frac{n\pi}{L})^2 t} \text{ where}$$

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

Substitution

$$A(x, y) \frac{\partial^2 z}{\partial x^2} + B(x, y) \frac{\partial^2 z}{\partial x \partial y} + C(x, y) \frac{\partial^2 z}{\partial y^2} = 0$$

Find a solution with the non-linear change of variables:

$$\xi = \xi(x, y), \eta = \eta(x, y), \text{ such that}$$

$$D(x, y) \frac{\partial^2 z}{\partial \xi \partial \eta} = 0$$

In order for extra terms to vanish, the following must be satisfied:

$$A \left(\frac{dy}{dx} \right)_{\xi}^2 - B \left(\frac{dy}{dx} \right)_{\xi} + C = 0$$

$$A \left(\frac{dy}{dx} \right)_{\eta}^2 - B \left(\frac{dy}{dx} \right)_{\eta} + C = 0$$

If $B^2 - 4AC > 0$, then the PDE is *hyperbolic*:

$$\frac{\partial^2 z}{\partial \xi \partial \eta} = 0$$

If $B^2 - 4AC = 0$, then the PDE is *parabolic*:

$$\frac{\partial^2 z}{\partial \eta^2} = 0$$

If $B^2 - 4AC < 0$, then the PDE is *elliptic*:

$$\frac{\partial^2 z}{\partial \xi^2} + \frac{\partial^2 z}{\partial \eta^2} = 0$$

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