cheatsheet

December 9, 2019

1 First Order Linear ODEs

Integrating Factor:

$$\frac{dy}{dt} + p(t)y = g(t)$$

$$\mu(t)y(t) = \int \mu(t)g(t)dt + c \text{ where } \mu(t) = e^{\int p(t)dt}$$

Theorems of Validity

$$y'(t) + p(t)y = g(t), y(t) = y_0$$

If p(t) and g(t) are continuous in $\alpha < t < \beta$ and interval contains t_0, then there is a unique solution on that interval.

$$y' = f(t, y)$$

If f(t,y) and $\frac{\partial f}{\partial y}$ are continuous in some rectangle containing (t_0, y_0) then there is a unique solution.

2 Non-linear First Order ODEs

Separable

$$N(y)\frac{dy}{dx} = M(x)$$

Exact

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \text{ where } \frac{\partial f}{\partial x} = M(x,y) \text{ and } \frac{\partial f}{\partial y} = N(x,y)$$

$$f(x,y) = c$$

Bernoulli

$$y' + p(x)y = q(x)y^n \text{ substitute } v = y^{1-n} \text{ to get}$$

$$\frac{1}{1-n}v' + p(x)v = q(x)$$

Solve by integrating factor method.

Other substitutions

$$y' = F\left(\frac{y}{x}\right)$$

$$\text{Substitute } v(x) = \frac{y}{x} \text{ to get } \frac{dv}{F(v) - v} = \frac{dx}{x}$$

$$y' = G(ax + by)$$

$$\text{Substitute } v(x) = ax + by \text{ to get } \frac{dv}{a + bG(v)} = dx$$

3 Homogeneous 2nd Order ODEs

Characteristic equation

$$ay'' + by' + cy = 0$$

real, distinct roots (\$r_1 $\neq r\_2$\$) : $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ complex roots ($r_{1,2} = \lambda \pm \mu i$): $y(t) = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t$ double root ($r_1 = r_2 = r$): $y(t) = c_1 e^{rt} + c_2 t e^{rt}$

4 Non-homogeneous 2nd Order ODEs with constant coefs

$$y''(t) + p(t)y'(t) + q(t)y = g(t)$$

$$y(t) = t_c(t) + y_p(t)$$

Particular Solutions

Undetermined Coefficients Guess $y_p(t)$ based on the form of g(t) then plug into complementary solution to solve.

	$g(t) \quad y_p(t) guess$
$ae^{\beta t}$	$Ae^{\beta t}$
$a \cos(\beta t)$	$A \cos \beta t + B \sin \beta t$
$a \sin(\beta t)$	$A \cos \beta t + B \sin \beta t$
n^{th} polynomial	$A_n t^n + A_{n-1} t^{n-1} ... A_1 t + A_0$

Variation of Parameters Given,

$$y_c(t) = c_1 y_1(t) + c_2 y_2(t)$$

$$y_p(t) = y_1 u_1 + y_2 u_2, \text{ where}$$

$$u_1 = -\int \frac{y_2 g(t)}{W(y_1, y_2)} dt, u_2 = \int \frac{y_1 g(t)}{W(y_1, y_2)}, \text{ where}$$

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$$

5 System of Linear First-Order ODEs

$$\overrightarrow{X}' = A\overrightarrow{X} + \overrightarrow{b}(t)$$

Eigenvalues method real, distinct eigenvalues: $\overrightarrow{X}(t) = c_1 e^{\lambda_1 t}\overrightarrow{\eta}^{(1)} + c_2 e^{\lambda_2 t}\overrightarrow{\eta}^{(2)}$

complex eigenvalues: Plug into the euqation above and simplify with $e^{i\theta} = \cos\theta + i\sin\theta$

double eigenvalues: $\overrightarrow{X} = c_1 e^{\lambda t}\overrightarrow{\eta} + c_2(te_{\lambda t}\overrightarrow{\eta} + e^{\lambda t}\overrightarrow{\rho})$ where $(A - \lambda I)\overrightarrow{\rho} = \overrightarrow{\eta}$

Undetermined coefficients Coefficients are vectors now.

Variation of parameters Given $\overrightarrow{X}$ is a solution:

$$\overrightarrow{X_P} = X\int X^{-1}\overrightarrow{b}\,dt$$

Matrix exponential method

$$\overrightarrow{X} = e^{At}(Y_0 + A^{-1}b) - A^{-1}b \text{ where}$$

$$e^{At} = SDS^{-1} \text{ where}$$

S is the matrix of eigenvectors and D is the diagonal exponential matrix of eigenvalues.

6 Homogeneous 2nd Order ODEs with non-constant coefs

Euler equations

$$ax^2y'' + bxy' + cy = 0, \text{ solve}$$

$$ar(r-1) + b(r) + c = 0$$

real, distinct roots: $y(x) = c_1x^{r_1} + c_2x^{r_2}$ double roots: $y(x) = x^r(c_1 + c_2\ln x)$ complex roots: $y(x) = c_1x^\lambda\cos\mu\ln x + c_2x^\lambda\sin\mu\ln x$

Bessel Functions

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0$$

$$y(x) = c_1 J_\alpha(x) + c_2 Y_\alpha(x)$$

Spherical Bessel Functions

$$x^2y'' + 2xy' + (x^2 - n(n+1))y = 0$$

$$y(x) = c_1 j_n(x) + c_2 y_n(x)$$

Modified Bessel Functions

$$x^2y'' + xy' + (x^2 + \alpha^2)y = 0$$

$$y(x) = c_1 I_\alpha(x) + c_2 K_\alpha(x)$$

7 First Order PDEs

Kinematic Wave Equations

$$\frac{\partial y}{\partial t} + \frac{\partial \Phi}{\partial x} = 0$$

Substitute $\Phi = \Phi(y)$

$$\frac{\partial y}{\partial t} + c\frac{\partial y}{\partial x} = 0, c = \frac{d\Phi}{dy}$$

Characteristic Curves: Homogeneous Quasilinear Equations with Constant Coefficients

$$P(z,x,y)\frac{\partial z}{\partial x} + Q(z,x,y)\frac{\partial z}{\partial y} = R(z,x,y)$$

with IC $z = z^0(x,y)$ along the curve I^0: $y = y^0(x)$
Express IC as a parametric equation:

$$x = \xi, y = y^0(\xi), z = z^0(\xi)$$

The solution surface is represented in parametric form:

$$\frac{dz}{ds} = \frac{\partial z}{\partial x}\frac{dx}{ds} + \frac{\partial z}{\partial y}\frac{dy}{ds}$$

with

$$\frac{dx}{ds} = P(z,x,y), \frac{dy}{ds} = Q(z,x,y), \frac{dz}{ds} = R(z,x,y)$$

8 Second Order PDE Examples

Heat Equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, u(x,0) = f(x), u(0,t) = 0, u(L,t) = 0$$

Separation of variables, reduces PDE to two ODEs

$$u(x,t) = \Phi(x)G(t)$$

$$\frac{1}{kG}\frac{dG}{dt} = \frac{1}{\Phi}\frac{d^2\Phi}{dx^2}$$

Solve and sum over all POSSIBLE solutions:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-k(\frac{n\pi}{L})^2 t} \text{ where}$$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

Substitution

$$A(x,y)\frac{\partial^2 z}{\partial x^2} + B(x,y)\frac{\partial^2 z}{\partial x \partial y} + C(x,y)\frac{\partial^2 z}{\partial y^2} = 0$$

Find a solution with the non-linear change of variables:

$$\xi = \xi(x,y), \eta = \eta(x,y), \text{ such that}$$

$$D(x,y)\frac{\partial^2 z}{\partial \xi \partial \eta} = 0$$

In order for extra terms to vanish, the following must be satisfied:

$$A\left(\frac{dy}{dx}\right)_\xi^2 - B\left(\frac{dy}{dx}\right)_\xi^2 + C = 0$$

$$A\left(\frac{dy}{dx}\right)_\eta^2 - B\left(\frac{dy}{dx}\right)_\eta^2 + C = 0$$

If $B^2 - 4AC > 0$, then the PDE is *hyperbolic*:

$$\frac{\partial^2 z}{\partial \xi \partial \eta} = 0$$

If $B^2 - 4AC = 0$, then the PDE is *parabolic*:

$$\frac{\partial^2 z}{\partial \eta^2} = 0$$

If $B^2 - 4AC < 0$, then the PDE is *elliptic*:

$$\frac{\partial^2 z}{\partial \xi^2} + \frac{\partial^2 z}{\partial \eta^2} = 0$$

9 Bessel Function Equations

If $(1-a^2) \geq 4c$ *and if neither* d, p, *nor* q *is zero, then except in the special case when it reduces to Euler's equation, the ODE*

$$x^2\frac{d^2y}{dx^2} + x(a+2bx^p)\frac{dy}{dx} + [c + dx^{2q} + b(a+p-1)x^p + b^2x^{2p}]y = 0$$

has the general solution

$$y(x) = x^{\alpha}e^{-\beta x^p}[c_1 J_v(\lambda x^q) + c_2 Y_v(\lambda x^q)]$$

where

$$\alpha = \frac{1-a}{2}, \beta = \frac{b}{p}, \lambda = \frac{\sqrt{|\, d \,|}}{q}, v = \frac{[(1-a)^2 - 4c]^{1/2}}{2q}$$

If d<0, J_v and Y_v should be replace with I_v and K_v respectively.

$$E_{\alpha-1}(x) + E_{\alpha+1}(x) = \frac{2\alpha}{x}E_{\alpha}(x) \qquad (J, Y)$$

$$E_{\alpha-1}(x) - E_{\alpha+1}(x) = 2E'_{\alpha}(x) \qquad (J, Y)$$

$$E'_{\alpha}(x) = E_{\alpha-1}(x) - \frac{\alpha}{x}E_{\alpha}(x) \qquad (J, Y, I)$$

$$E'_{\alpha}(x) = -E_{\alpha+1}(x) + \frac{\alpha}{x}E_{\alpha}(x) \qquad (J, Y, K)$$

$$\frac{d}{dx}\left[x^{\alpha}E_{\alpha}(x)\right] = x^{\alpha}E_{\alpha-1}(x) \qquad (J, Y, I)$$

$$\frac{d}{dx}\left[x^{-\alpha}E_{\alpha}(x)\right] = -x^{-\alpha}E_{\alpha+1}(x) \qquad (J, Y, K)$$

$$I_{\alpha-1}(x) - I_{\alpha+1}(x) = \frac{2\alpha}{x}I_{\alpha}(x)$$

$$K_{\alpha-1}(x) - K_{\alpha+1}(x) = -\frac{2\alpha}{x}K_{\alpha}(x)$$

$$I_{\alpha-1}(x) + I_{\alpha+1}(x) = 2\Gamma_{\alpha}(x)$$

$$-K_{\alpha-1}(x) - K_{\alpha+1}(x) = 2K'_{\alpha}(x)$$

$$K'_{\alpha}(x) = -K_{\alpha+1}(x) - \frac{\alpha}{x}K_{\alpha}(x)$$

$$I'_{\alpha}(x) = I_{\alpha+1}(x) + \frac{\alpha}{x}I_{\alpha}(x)$$

$$\frac{d}{dx}\left[x^{\alpha}K_{\alpha}(x)\right] = -x^{\alpha}K_{\alpha-1}(x)$$

$$\frac{d}{dx}\left[x^{-\alpha}I_{\alpha}(x)\right] = x^{-\alpha}I_{\alpha+1}(x)$$

[ ]: