Safety in Numbers

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1 Problem

There are n contestants in a reality TV show. Each contestant is assigned a point value by the judges and receives votes from the audience. The point value given by the judges and the audience's votes are combined to form a final score for the contestant, in the following way:

Let s be the sum of the judge-assigned point values of all contestants. Now suppose a contestant got s_i points from the judges, and that she received a fraction α_i (between 0 and 1, inclusive) of the audience's votes (α_i might be, for example, 0.3). Then that contestant's final score is $c_i = s_i + \alpha_i s$. Note that the sum of all contestants' audience vote fractions must be 1.

The contestant with the lowest score is eliminated.

Given the points contestants got from judges, your job is to find out, for each contestant, the minimum percentage of audience votes he/she must receive in order for him/her to be guaranteed not to be eliminated, no matter how the rest of the audience's votes are distributed.

If the lowest score is shared by multiple contestants, no contestants will be eliminated. [1]

2 Solution

We have:

$$s = \sum_{i=1}^{n} s_i$$
$$c_i = s_i + \alpha_i s$$
$$\sum_{i=1}^{n} \alpha_i = 1, \alpha_i \in [0,1]$$

For c_i to be eliminated, it needs to be smaller than every $c_j, j \neq i$. Let's write this out and pursue it:

$$\begin{aligned} c_i \text{ eliminated } &\Rightarrow c_i < c_j, j \neq i \\ &\Rightarrow s_i + \alpha_i s < s_j + \alpha_j s, j \neq i \end{aligned}$$

These are $n-1$ inequalities. Adding them up, left side and right side, we get:

$$(n-1)(s_i + \alpha_i s) < s - s_i + (1 - \alpha_i)s$$

or:

$$\alpha_i < \frac{2s - ns_i}{ns}$$

This is interesting, but so far it doesn't help us keep c_i safe.

Let's try again, expanding the inequalities slightly different.

For c_i to be eliminated, it needs to be smaller than all other $c_j, j \neq i$. If $s_j > c_i$, then this is already the case, no matter the audience vote distribution. If $s_j \leq c_i$, then what part of the audience votes does c_j need to be bigger than c_i ?

$$s_j + \alpha_j s > c_i \Leftrightarrow \alpha_j > \frac{c_i - s_j}{s}$$

This means, for all c_j with $s_j \leq c_i$ to beat c_i, they need a combined audience vote bigger than

$$\sum_{j=1, j\neq i, s_j \le c_i}^{n} \frac{c_i - s_j}{s}$$

It follows that the smallest α_i that keeps c_i safe, satisfies

$$1 - \alpha_i = \sum_{j=1, j\neq i, s_j \le c_i}^{n} \frac{c_i - s_j}{s}$$

This looks promising, but the problem is that the right-hand side of the equation depends on α_i in non-trivial ways. To determine the value of α_i, we have to find better and better approximations of α_i. A straightforward way to accomplish this is by halving the interval of possibilities for α_i at each iteration. We start with interval $[0, 1]$ and some initial candidate value for α_i. If

$$1 - \alpha_i < \sum_{j=1, j\neq i, s_j \le c_i}^{n} \frac{c_i - s_j}{s}$$

then α_i is too big and the new interval is $[0, \alpha_i]$. If

$$1 - \alpha_i > \sum_{j=1, j\neq i, s_j \le c_i}^{n} \frac{c_i - s_j}{s}$$

then α_i is too small and the new interval is $[\alpha_i, 1]$. We then pick a new candidate in this new interval, for example the middle of the interval and iterate. What should the initial candidate be? We can certainly use the middle, namely 0.5. But we can do better than that. Our initial exploration gave us a good candidate:

$$\alpha_i = \frac{2s - ns_i}{ns}$$

This concludes the solution. A complete implementation can be found at the author's github account.

References

[1] Bartholomew Furrow. *Codejam 2012 Round 1B: Problem A: Safety in Numbers*. 2012. URL: https://code.google.com/codejam/contest/1836486/dashboard#s=p0&a=0.