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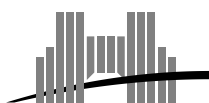
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*Timed continuous Petri nets and
optimization via linear programming*

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Timed continuous Petri nets and optimization via linear programming

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Abstract

In this paper, we consider a deterministic timed continuous Petri net model where conflicts at places are solved by using stationary routing parameters. We show how to compute the stationary firing rate for all transitions via linear programming, so as to determine the optimal routing parameters that maximize the firing rates. Finally, we discuss the relations with discrete Petri nets

Keywords: Continuous Petri nets, optimal routing, linear programming

Résumé

Dans cet article, nous considérons les réseaux de Petri continus temporisés déterministes dans lesquels les conflits dans les places sont résolus en utilisant des routages stationnaires. Nous montrons comment calculer les taux de tirs en régime stationnaire pour toutes les transitions via un programme linéaire. Nous montrons aussi comment calculer les paramètres de routage qui donnent les meilleurs taux de tirs. Enfin, nous discutons les relations qui existent avec les réseaux de Petri discrets

Mots-clés: Réseaux de Petri continus, routage optimal, programmation linéaire

1 Introduction

This paper deals with the problem of determining an optimal stationary mode of operation for a system described by a continuous timed Petri net.

The model we consider is taken from an article [4] by Cohen *et al.* It consists of a place-transition net where the arc weight, the marking of a place and the firing counter of a transition are not restricted to integer values but may take non negative real values, i.e., we have fluid quantities moving from place to place rather than discrete tokens. A release delay δ_i is associated to each place p_i , representing the time that the fluid that enters will spend in the place before it can be available to fire the output transitions. On the contrary, no delay is associated to the transitions. (Note that an equivalent model could be given associating a release delay to each transition.)

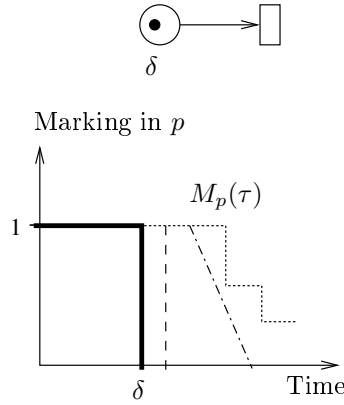


Figure 1: Evolution of the marking in a continuous Petri Net

Figure 1 displays a very simple continuous timed Petri net with a delay δ associated with the place and an initial marking equal to one: $M(0) = 1$ which is assumed to arrive at time 0 in the place.

The evolution equations (4)-(5) given in Section 2 describe all possible evolutions of such a net in terms of firing counters: a firing counter $Z_j(\tau)$ expresses the cumulative quantity of firing of transition t_j in the time interval $[0, \tau]$. For the single transition of the net in figure 1 it holds $Z(\tau) = 0$ if $\tau < \delta$, and $Z(\tau) \leq M(0) = 1$ if $\tau \geq \delta$. The value of the marking in place p can also be written as $M(\tau) = M(0) - Z(\tau)$.

Thus, the evolution of the marking in place p after time 0 is not uniquely described in this model. It could follow any of the dashed lines displayed in the figure. The "as soon as possible" (*asap*) firing policy requires that $Z^{asap}(\delta) = 1$, i.e., after a delay δ the marking in the place is null $M(\delta) = 0$ (bold line). All other firing policies have longer firing delays (all other dashed lines).

Assume now that the firing policy u is such that after a transient the net reaches a periodic mode of operation, i.e., for τ sufficiently large there exist a period Υ and a rate λ_j such that for all

transitions t_j it holds $Z_j^u(\tau + \Upsilon) = Z_j^u(\tau) + \lambda_j \Upsilon$. In this case it holds

$$\lim_{\tau \rightarrow \infty} \frac{Z_j^u(\tau)}{\tau} = \lambda_j,$$

and we say that the net asymptotically reaches a *weakly stationary behavior*. Generalizing, we define a firing policy (not necessarily periodic) as *weakly stationary* if the previous limit exists and is finite.

Note that while a weakly stationary behavior is characterized by the existence of finite firing rates λ_j for all transitions, the marking does not necessarily reaches an average constant value: as an example, it may continually increase. If for all places p_i the limit

$$\lim_{\tau \rightarrow \infty} \frac{\int_0^\tau M_i(\varrho) d\varrho}{\tau}$$

also exists and is finite, then we say that the net reaches a *strongly stationary behavior*. When it is not necessary to distinguish between the two cases, we will briefly use the adjective *stationary* without any additional qualification.

1.1 A different continuous Petri net model

A rather different continuous Petri net model has been presented in [10] by Recalde and Silva⁵. In the RS model, the timing of the net is given under the form of “speed parameters” for all transitions. Once again, the infinite server semantics is assumed, which is modeled by an evolution equation of the marking under the form of differential equations, where the decrease rate of the marking depends on its size. Although our paper is dedicated to the study of the CGQ model, it is useful to introduce at least informally the RS model because, beyond the superficial differences, there are some similarities that deserve being commented upon. Let us first give an example of how the RS model evolves. Consider the net in figure 2, where the transition has a speed parameter λ : for this system the marking evolution is given by the equation $\frac{dM(\tau)}{d\tau} = -\lambda M(\tau)$ with initial condition $M(0) = 1$ whose solution is $M(\tau) = e^{-\lambda\tau}$. Note that unlike in the previous model, it takes an infinite amount of time to empty place p .

In both models an additional assumption (that we call STAR) was that of solving conflicts at places by assigning a STationary Routing (which is described in detail in the following). Furthermore, two additional restrictions of Recalde and Silva were that of considering equal conflict nets and of allowing conflicts only at immediate transitions: such a restriction is not used here.

The evolution equations of the CGQ model (assuming that each transition fires as soon as it is enabled) can be written as the coupling of a conventional linear system with a (\min, \times) linear system, i.e., it is very easy to implement. The evolution equations of the RS model, on the contrary, can be written as the coupling of a conventional system of linear differential equations with a (\min, \times) linear system, and are more difficult to solve. On the contrary, when the net is live

⁵In the rest of the paper when comparing the two models we will denote CGQ model the one of Cohen *et al.* and RS model the one of Recalde and Silva.

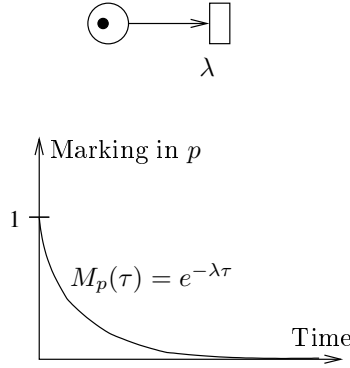


Figure 2: Marking evolution in the model of Recalde and Silva

and bounded, the RS model leads to an easy computation of a stationary behavior by means of a linear programming problem (LPP), while the CGQ model requires a more complex procedure based on dynamic programming.

1.2 Results of this paper

The use of linear or integer programming for analysis and performance evaluation of Petri nets is a rich field that is known under the name of *structural analysis* [9]. In the case of timed discrete nets this approach gives good bounds on several performance measures for interesting classes of nets [3]. In the case of continuous nets, this approach is even more useful, often providing the exact value of performance measure [10, 11].

The first result of this paper, is showing that the problem of computing the stationary behavior for the CGQ model can also be solved by a linear program. The LPP we write is slightly different from the one used to compute the strongly stationary behavior of the RS model: the first can in effect be seen as a generalization of the latter. In fact in our LPP we consider, among others, as many equations as there are places in the nets, each one corresponding to Little’s law for the fluid entering the place. On the contrary, in the LPP of Recalde and Silva this information is “compacted” in a single Little’s law associated to each P-invariant.

All the solutions of the LPP of Recalde and Silva can also be found by our LPP. In the case of structurally live and bounded net, furthermore, the two programs find the same optimal solution. This allows us to conclude that although the two models have a very different transient behavior, they both reach the same stationary behavior, in the sense that the average firing rate of each transition must satisfy the same type of equations (see section 3).

Note that this result is not yet another version of the “folk theorem” that says that for each a model where the time is associated to the transitions there exists an equivalent model where the time is associated to the places. In fact, the difference between the two models is deeper than this and, as we have already shown, their transient behavior is very different. The fact that the

stationary behavior of both models can be described by exactly the same equations came to us as a surprise.

We also show that the above mentioned difference between the LPP we write and the one of Recalde and Silva has several other implications that are developed in the paper.

Firstly, our approach can also be applied to nets that are not structurally bounded and do not admit T-invariants: in this sense we can also compute a weak stationary behavior. Secondly, it can also be directly used to determine an optimal stationary firing vector among several: being able to solve efficiently this problem of practical relevance is, we believe, the main contribution of this paper. Thirdly, we show that while an optimal stationary firing vector always corresponds to a particular STAR, in general it is not reached from the initial marking using the same STAR during the transient. The optimal transient, i.e., the transient that leads as soon as possible to the optimal stationary behavior can also be easily computed using our approach.

This paper is structured as follows. Section 2 presents the definitions and notations as well as the semantics of Continuous Petri nets (CPN) and Deterministic Timed Continuous Petri nets (DTCPN) and several properties, mainly concerning liveness issues. In particular, Subsection 2.5 shows how such nets, under a STAR policy are equivalent to structurally conflict-free nets. The material presented in this preliminary section is rather similar to the results presented in [10]. Section 3 gives the linear program LPP (6) that computes the stationary firing rates of the net. Section 4 shows how to compute the optimal routing coefficients in a DTCPN when they are not given beforehand. It also presents several possible extensions. Finally Section 5 shows how the previous results could be adapted for heuristic optimization in discrete Petri nets.

A short version of this paper was presented in [6]. The current paper provides a deeper insight on the conditions under which the linear programming approach gives the asymptotic performances of the net. The assumptions used here are more general than in the short paper. This paper also provides more detailed proofs of the main results.

2 Preliminaries

2.1 The untimed model

A *continuous Petri net* (CPN) is a structure $N = (P, T, \mathbf{Pre}, \mathbf{Post})$, where

- P is a set of m places;
- T is a set of n transitions;
- $\mathbf{Pre} : P \times T \rightarrow \mathbb{R}_0^+$ is the *pre* incidence function that specifies the weights of the arcs from places to transitions;

- **Post** : $P \times T \rightarrow \mathbb{R}_0^+$ is the *post* incidence function that specifies the weighted arcs from transitions to places.

We denote $\mathbf{C} = \mathbf{Post} - \mathbf{Pre}$ the incidence matrix of a net. The preset and postset of a node $x \in P \cup T$ are denoted $\bullet x$ and x^\bullet , respectively.

A *marking* is a vector $\mathbf{M} : P \rightarrow \mathbb{R}_0^+$ that assigns to each place a non-negative real quantity called *mark*; the marking of place p_i is denoted M_i . In discrete Petri nets, the marking is required to be integer valued. The generalization to real values yields many simplifications, in particular concerning the reachability issues which become simple linear algebraic conditions (see proposition 1 below). A *CPN system* $\langle N, \mathbf{M}(0) \rangle$ is a net N with an initial marking $\mathbf{M}(0)$.

Given a CPN N with incidence matrix \mathbf{C} : a P-invariant is a vector $\mathbf{x} \in \mathbb{R}^m$ such that $\mathbf{x} \succeq \mathbf{0}$ and $\mathbf{x}^T \mathbf{C} = \mathbf{0}$; a T-invariant is a vector $\mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{y} \succeq \mathbf{0}$ and $\mathbf{C}\mathbf{y} = \mathbf{0}$. The support of a P-invariant \mathbf{x} (resp., T-invariant \mathbf{y}) is the set $||\mathbf{x}|| = \{p_i \in P \mid x_i > 0\}$ (resp., $||\mathbf{y}|| = \{t_j \in T \mid y_j > 0\}$).

We also define a T-increase (resp., T-decrease) as a vector $\mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{y} \succeq \mathbf{0}$ and $\mathbf{C}\mathbf{y} \succeq \mathbf{0}$ (resp., $\mathbf{C}\mathbf{y} \preceq \mathbf{0}$).

A vector \mathbf{y} (in particular an invariant) is said to be *positive* if all its components are positive, i.e., if $\mathbf{y} > \mathbf{0}$.

A net is *strongly connected* if there exists a directed path from any node in $P \cup T$ to every other node.

2.2 Firing semantics

In this subsection we recall some basic results about autonomous (i.e., untimed) continuous Petri nets following [11]. These results will be used in the following, especially those characterizing the marking reachability property. Note that the CGQ model and the RS model can be seen as different timed extensions of the same underlying autonomous model.

In a CPN the firing of a transition t_j is possible under marking \mathbf{M} if $M_i > 0$ for all $p_i \in \bullet t_j$. If this is the case the transition may fire a quantity z where

$$z \leq \min_{p_i \in \bullet t_j} M_i / \text{Pre}(p_i, t_j). \quad (1)$$

This firing yields a new marking \mathbf{M}' such that

$$\mathbf{M}' = \mathbf{M} + \mathbf{C}(ze_j) \quad (2)$$

where e_j is a vector whose j -th component is equal to 1 while all other components are null. This is denoted $\mathbf{M} \xrightarrow{(j,z)} \mathbf{M}'$.

If a sequence of transitions $t_{j_1} \cdots t_{j_k}$ are fired for a quantity $z_1 \cdots z_k$, then $\mathbf{M} \xrightarrow{(j_1, z_1)} \mathbf{M}_1 \xrightarrow{(j_2, z_2)} \mathbf{M}_2 \cdots \xrightarrow{(j_k, z_k)} \mathbf{M}_k$. We say that marking \mathbf{M}_k is reachable and denote this by $\mathbf{M} \xrightarrow{(j_1, z_1) \cdots (j_k, z_k)} \mathbf{M}_k$. The marking \mathbf{M}_k satisfies $\mathbf{M}_k = \mathbf{M} + \mathbf{C}\mathbf{z}$ where

$$\mathbf{z} = \sum_{i=1}^k z_i \mathbf{e}_{j_i}$$

is a vector whose j -th component is the sum of the firing quantity associated to transition t_j in the sequence.

A CPN is *bounded* if all markings reachable from $\mathbf{M}(0)$ are bounded. Note that if a net is unbounded it must have a T-increase.

A CPN is *live* if from all reachable markings, one can find a firing sequence containing all the transitions. Liveness of CPN is remotely related with liveness of discrete nets (see [11] for more details). Consider the case displayed in figure 3.

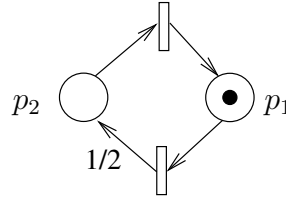


Figure 3: A live CPN.

In this net, both transitions may fire infinitely often: the CPN is live. However the associated discrete net is deadlocked. Note however that the n -th firing of each transition fires less than $(1/2)^n$ marks.

Proposition 1 ([11]). *In a CPN (N, \mathbf{M}_0) , the set of reachable markings $\mathcal{R}(\mathbf{M}_0)$ is a convex set contained in $(\mathbb{R}_0^+)^m$.*

Note that this convex set is not always a closed set even when the CPN is bounded. The set $\mathcal{R}(\mathbf{M}_0)$ of the CPN displayed in figure 3 is given in figure 4.

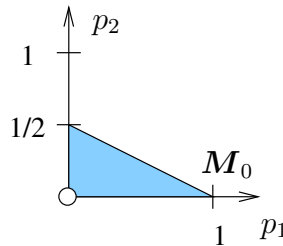


Figure 4: The set of all reachable firings

It is made of the grey triangle except the point $(0, 0)$, which is not reachable (only reachable using an infinite number of firings).

We extend the definition of reachability by saying that all markings in the adherence of $\mathcal{R}(\mathbf{M}_0)$ are reachable. This closed set is denoted $\mathcal{R}^\infty(\mathbf{M}_0)$. We will also extend the definition of liveness. A CPN is *lim-live* if starting from any marking in $\mathcal{R}^\infty(\mathbf{M}_0)$, one can fire all the transitions.

The net displayed in figure 3 is live but not lim-live since marking $(0, 0)$ belongs to $\mathcal{R}^\infty(\mathbf{M}_0)$ (see figure 4).

Unlike in discrete nets, the state equation of a net often gives necessary and sufficient conditions for reachability as shown in the following propositions.

Proposition 2 ([11]). *If a CPN is such that all transitions can be fired and if the net has a positive T-invariant then $\mathbf{M} = \mathbf{M}(0) + \mathbf{C}\mathbf{z}$ for $\mathbf{z} \in (\mathbb{R}_0^+)^n$ if and only if $\mathbf{M} \in \mathcal{R}^\infty(\mathbf{M}_0)$.*

The main restriction of the previous proposition is given by the requirement that the net admits a positive T-invariant. A weaker but more general result is the following.

Proposition 3 ([5]). *If a CPN is such that all transitions can be fired, and if there exist a vector \mathbf{z} such that $\mathbf{M} = \mathbf{M}_0 + \mathbf{C}\mathbf{z}$ then a marking \mathbf{M}' as close to \mathbf{M} as desired can be reached.*

Finally we also recall one result that will be used in the paper.

Proposition 4 ([11]). *If a CPN is such that all transitions can be fired, then a positive marking can be reached from which any T-invariants or T-increases can be fired.*

2.3 Deterministic timed CPN

A *deterministic timed CPN* (DTCPN) is a pair (N, δ) , where N is a CPN, and $\delta : P \rightarrow \mathbb{N}$, called release delay⁶, assigns a non-negative fixed duration to each place. The marks that enter place p_i at time τ must remain within the place for a time greater or equal to δ_i before they become available for firing an output transition.

Remark 5. *In this paper we consider two minor restrictions on the class of DTCPN considered.*

- A1) *A place with a release delay equal to 0 is said to be immediate and we assume there are no cycles in the net whose places are all immediate. This condition implies that there won't be an infinite number of firings within a finite time interval.*
- A2) *Without loss of generality we assume that all transitions of the net can be fired. If this is not the case, then the dead transitions can be removed from the net. Note that checking if all transitions of a continuous net can be fired is a simple procedure whose complexity is linear in the number transitions [11].*

The mark $M_i(0)$ assigned by the initial marking, is assumed to enter place p_i at time $\tau = 0$, hence it will be available at time $\tau = \delta_i$. For timed nets, we denote the marking of place p_i at time τ as $M_i(\tau)$.

⁶If the delays are rational number, everything in this paper work the same by changing the time unit.

In a DTCPN, a transition t_j may fire for a quantity z at time τ only if

$$z \leq \min_{p_i \in \bullet t_j} M_i(\tau') / \text{Pre}(p_i, t_j) \quad \forall \tau' \in [\tau - \delta_i, \tau].$$

Firing t_j for a quantity z at time τ modifies the marking as for the untimed net in Equations (2).

This timed model assumes an infinite server semantics: in fact the delay is the same no matter how many tokens are actually present in the place. To model finite server semantics, one needs to add constraints in the net by allowing a maximal quantity of token in a place (for example by recycling the places with new places containing an adequate number of tokens).

The evolution of timed net is often described in terms of the *firing counters* of the transitions. If we assign to each transition t_j a counter $Z_j(\tau)$ that represents the cumulative quantity of marks fired by the transition from time 0 to τ , and if we let $\mathbf{Z} = [Z_1 \cdots Z_n]^T$, then the marking at time τ can be determined by the state equation

$$\mathbf{M}(\tau) = \mathbf{M}(0) + \mathbf{C}\mathbf{Z}(\tau). \quad (3)$$

We are interested in timed nets that reach (after a finite transient or asymptotically) a final behavior.

Definition 6. *Let (N, δ) be a timed net.*

- *The net reaches a periodic behavior if there exist a period Υ and a finite rate λ_j such that for τ sufficiently large and for all transitions t_j it holds*

$$Z_j(\tau + \Upsilon) = Z_j(\tau) + \lambda_j \Upsilon.$$

Additionally, if for all places p_i it holds

$$M_i(\tau + \Upsilon) = M_i(\tau),$$

then the behavior is called strongly periodic else it is called weakly periodic.

- *The net reaches a stationary behavior if the average firing rates reach a limit as time goes to infinity, i.e., for all transitions t_j it holds*

$$\lim_{\tau \rightarrow \infty} \frac{Z_j(\tau)}{\tau} = \lambda_j < +\infty.$$

Additionally, if also the average marking reaches a limit as time goes to infinity, i.e., for all places p_i it holds

$$\lim_{\tau \rightarrow \infty} \frac{\int_0^\tau M_i(\varrho) d\varrho}{\tau} < +\infty,$$

then the behavior is called strongly stationary else it is called weakly stationary.

Note that a stationary behavior is a generalization of a periodic behavior.

Finally we make the following obvious observation.

Proposition 7. Assume that a timed net (N, δ) reaches a stationary behavior with

$$\lim_{\tau \rightarrow \infty} \frac{Z(\tau)}{\tau} = \lambda.$$

and let C be the incidence matrix of N .

- a) The asymptotic firing rate vector is such that $C\lambda \geq 0$.
- b) If λ is a T -increase, i.e., if $C\lambda \gneq 0$, then the behavior is weakly stationary.
- c) If the behavior is strongly stationary, then λ is a T -invariant, i.e., $C\lambda = 0$.

Proof. If the net reaches a stationary behavior, then from the state equation (3) we have that for all places p_i

$$\lim_{\tau \rightarrow \infty} \frac{M_i(\tau)}{\tau} = C(p_i, \cdot) \lim_{\tau \rightarrow \infty} \frac{Z(\tau)}{\tau} = C(p_i, \cdot)\lambda.$$

a) Assume, by contradiction, that there exists place p_i such that $C(p_i, \cdot)\lambda < 0$. Then $\lim_{\tau \rightarrow \infty} M_i(\tau) < 0$, that is clearly not possible.

b) Let p_i be a place such that $C(p_i, \cdot)\lambda > 0$. Then

$$\lim_{\tau \rightarrow \infty} \frac{M_i(\tau)}{\tau} > 0 \implies \lim_{\tau \rightarrow \infty} M_i(\tau) = +\infty \implies \lim_{\tau \rightarrow \infty} \int_0^\tau M_i(\varrho) d\varrho = +\infty$$

and using the rule of l'Hopital

$$\lim_{\tau \rightarrow \infty} \frac{\int_0^\tau M_i(\varrho) d\varrho}{\tau} = \lim_{\tau \rightarrow \infty} M_i(\tau) = +\infty.$$

c) Follows from b).

2.4 STAR deterministic timed CPN

A *stationary routing* (STAR) is a function $\rho : P \times T \rightarrow [0, 1]$ such that $\mathbf{Pre}(p, t) = 0 \implies \rho(p, t) = 0$ (i.e., $\rho(p, t) > 0$ only if $t \in p^\bullet$) and such that for all $p \in P$: $\sum_{t \in T} \rho(p, t) = 1$. If $\rho(p_i, t_j) = \rho_{i,j} > 0$, then a fraction $\rho_{i,j}$ of the mark entering place p_i is reserved only for the firing of its output transition t_j and cannot be "used" to enable and fire any other output transition. Thus, a STAR solves all conflicts that may arise in the net with a preassigned fixed policy.

In a STAR-CPN (untimed) the firing equation (1) is modified into

$$z \leq \min_{p_i \in \bullet t_j} \rho_{i,j} M_i / \text{Pre}(p_i, t_j).$$

Using the transitions firing counters $Z_j(\tau)$ described in the previous subsections and assigning to each place p_i a counter $H_i(\tau)$ that represent the quantity of marks that has entered the place

from time 0 to τ (including the initial marking), the evolution of a timed STAR-DTCPN can be written as:

$$Z_j(\tau) \leq \min_{p_i \in \bullet t_j} \left\{ \frac{\rho_{i,j} H_i(\tau - \delta_i)}{Pre(p_i, t_j)} \right\} \quad \forall t_j \in T \quad (4)$$

$$H_i(\tau) = M_i(0) + \sum_{t_j \in \bullet p_i} Post(p_i, t_j) Z_j(\tau) \quad \forall p_i \in P. \quad (5)$$

As in the general case discussed in previous subsection, the marking at time τ can be determined by the state equation (3).

The evolution that corresponds to choosing the equality in the Equation (4) will be called *asap* (as soon as possible) because it corresponds to firing a transition as soon as the fluid that has entered the input places becomes available for the firing. The corresponding counters are denoted $\mathbf{Z}^{asap} = [Z_1^{asap} \dots Z_n^{asap}]^T$.

The timed version of proposition 2 is given below.

Proposition 8. *Consider a STAR-CPN N which has a positive T -invariant. Consider a marking $\mathbf{M} = \mathbf{M}(0) + \mathbf{C}\mathbf{z}$. For any holding times δ consider the associated STAR-DTCPN (N, δ) . There exists a time τ (possibly infinite) and an evolution of the counter vector $\mathbf{Z}(\tau)$ up to time τ such that $\mathbf{M}(\tau) = \mathbf{M}$ and $\mathbf{Z}(\tau) = \mathbf{z}$.*

Proof. By assumption A1 (see remark 5) all transitions can be fired. Using proposition 2, we know that there exists a sequence $(j_1, z_1), \dots (j_k, z_k), \dots$ such that $\mathbf{M}(0) \xrightarrow{(j_1, z_1)} \mathbf{M}_1 \xrightarrow{(j_2, z_2)} \mathbf{M}_2 \dots \xrightarrow{(j_k, z_k)} \dots \mathbf{M}$.

By waiting long enough (depending on the value of δ), all marks present at time 0 become available, allowing t_{j_1} to fire a quantity z_1 . Once marking \mathbf{M}_1 is reached, one needs to waiting some more time until all marks in \mathbf{M}_1 become available. Then t_{j_2} may fire a quantity z_2 and so forth. Note that this evolution is not necessarily *asap*. \square

We are interested in finding out if a DTCPN reaches a weak stationary behavior, i.e., a condition in which the mean number of firing of each transition t_j per time unit reaches asymptotically a constant value:

$$\lambda_j = \lim_{\tau \rightarrow \infty} \frac{Z_j(\tau)}{\tau}.$$

The only known result concerning the existence of a stationary behavior for STAR-DTCPN is given in the following theorem.

Theorem 9 ([4]). *In a strongly connected STAR-DTCPN that admits a positive T -invariant and such that the sum of the delays over each circuit is not null, the limit $\lim_{\tau \rightarrow \infty} Z_j^{asap}(\tau)/\tau$ exists and is finite.*

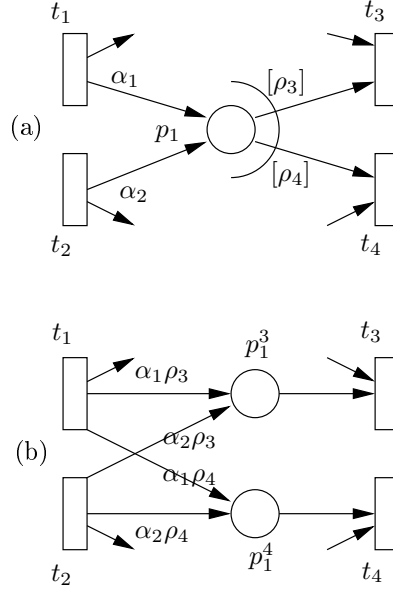


Figure 5: (a) A place with conflict; (b) transformation to solve the conflict using a STAR.

2.5 Equivalent structurally conflict-free net

Any DTCPN (N, δ) whose conflicts are solved using a STAR ρ can be transformed into an equivalent *structurally conflict-free* net ([3]) $(\overline{N}, \overline{\delta})$, i.e., a net in which each place has at most one output transition. Each place p_i in the original net is split into h places p_i^1, \dots, p_i^h (where h is the size of p_i^\bullet). The number of places in $(\overline{N}, \overline{\delta})$ is denoted \overline{m} . All the places p_i^k have the same holding time as the original place p_i : $\overline{\delta}_i^k = \delta_i$, for all $1 \leq k \leq h$. The initial marking in place p_i^k is $\overline{M}_i^k(0) = \rho(p_i, t_k) M_i(0)$. The new preset and postset are such that $\overline{\mathbf{Post}}(p_i^k, t_u) = \mathbf{Post}(p_i, t_u) \times \rho(p_i, t_k)$, $\overline{\mathbf{Pre}}(p_i^k, t_k) = \mathbf{Pre}(p_i, t_k)$ and $\overline{\mathbf{Pre}}(p_i^k, t_j) = 0$ if $j \neq k$. An example of such a construction is shown in figure 5 where the routing coefficients are displayed between brackets to be distinguished from the weights of the arcs.

Proposition 10. *The STAR-DTCPN $\langle N, \delta, \mathbf{M}(0), \rho \rangle$ and the above defined structurally conflict-free DTCPN $\langle \overline{N}, \overline{\delta}, \overline{\mathbf{M}}(0) \rangle$ are equivalent in the following sense: for all $\tau \in \overline{\mathbb{R}}_+$, $\mathbf{Z}(\tau)$ is a valid counter vector for N if and only if $\mathbf{Z}(\tau)$ is a valid counter vector for \overline{N} .*

Proof. The proof is based on the evolution equations of $\mathbf{Z}(\tau)$. We denote by $\overline{\mathbf{Z}}$ and $\overline{\mathbf{H}}$ the counter vectors of the net \overline{N} . Using the previous notations, one has for all transition t_j and all place p_i , $\overline{\delta}_i^k = \delta_i$ and $\overline{\mathbf{Post}}(p_i^k, t_j) = \mathbf{Post}(p_i, t_j) \times \rho(p_i, t_k)$ imply that $\overline{H}_i^k(\tau - \delta_i) = \rho(p_i, t_k) H_i(\tau - \delta_i)$. On the other hand, $\overline{\rho}_{i,j}^k = 1$. Therefore, $\overline{\mathbf{Z}}(\tau) = \mathbf{Z}(\tau)$. \square

This means that the properties of liveness, boundedness are the same for both nets. Moreover, if N is strongly connected and the STAR coefficients are all positive, then \overline{N} is also strongly connected.

At this point, one may notice that we could have considered a more general routing, which takes into account the origin of the marks. In place p_i , the routing coefficient to transition $t_k \in p_i^\bullet$ could also depend on the origin (a transition $t_j \in {}^\bullet p_i$) of the marks, denoted $\rho(t_j, p_i, t_k)$. In the conflict-free net, this means that the weight becomes: $\overline{\mathbf{Post}}(p_i^k, t_j) = \mathbf{Post}(p_i, t_j) \times \rho(t_j, p_i, t_k)$. In the following, we will keep the routing independent of the origin. The additional complexity introduced by origin dependent routings come from the fact that it is more difficult to write the programs using the original net. They can be overcome by minor changes in the programs.

In the following, we will mainly use the net \overline{N} instead of N since every property of N can be seen on \overline{N} and vice-versa.

Let us now consider a structurally conflict-free net. In the case of discrete nets, it was shown in [3] that if such a net is live and bounded, then it admits a single T-invariant whose support contains all the transitions.

A similar result was presented for CPNs.

Proposition 11 ([11]). *A structurally conflict-free CPN is lim-live and bounded if and only if it admits a single T-invariant whose support contains all the transitions.*

We prove three similar results.

Proposition 12. *A CPN N that is structurally conflict-free and strongly connected admits at most one T-invariant (up to a multiplicative constant). Moreover, its support contains all the transitions.*

Proof. We first show that if a CPN admits two different T-invariants \mathbf{y} and \mathbf{y}' , with $\mathbf{y} \neq \beta \mathbf{y}'$ for all $\beta \in \mathbb{R}^+$, then it also admits a T-invariant $\tilde{\mathbf{y}}$ with at least a null component. This is clearly true if either \mathbf{y} or \mathbf{y}' admits a null component. Assume now $\mathbf{y}, \mathbf{y}' > \mathbf{0}$. Then let $\beta = \min_i \{y_i/y'_i\} = y_l/y'_l$; we have that $\tilde{\mathbf{y}} = \mathbf{y} - \beta \mathbf{y}'$ is a T-invariant whose l -th component is null.

Assume now a CPN admits a T-invariant $\tilde{\mathbf{y}}$ whose l -th component is null. If the net is structurally conflict-free, for each place $p \in {}^\bullet t_l$: $C(p, t_l)$ is the only negative element of $\mathbf{C}(p, \cdot)$. Hence $\mathbf{C}(p, \cdot)\tilde{\mathbf{y}} = \mathbf{0}$ only if all transitions in ${}^\bullet p$ do not belong to the support of $\tilde{\mathbf{y}}$. Since the net is strongly connected, ${}^\bullet p$ is not empty and we can repeat this reasoning until we find that $\tilde{\mathbf{y}} = \mathbf{0}$, clearly a contradiction. \square

Proposition 13. *A CPN N that is structurally conflict-free and strongly connected, and that admits a T-invariant cannot admit a T-increase.*

Proof. We first show that if a CPN admits a T-invariant \mathbf{y} and a T-increase \mathbf{y}' , then it also admits a T-decrease $\tilde{\mathbf{y}}$ with at least a null component. In fact, by the previous proposition the T-invariant \mathbf{y} is positive and thus chosen $\eta = \min_i \{y'_i/y_i\} = y'_l/y_l > 0$ one obtains a vector $\tilde{\mathbf{y}} = \eta \mathbf{y} - \mathbf{y}' \geq \mathbf{0}$ with $\tilde{y}_l = 0$. Now one sees that $\mathbf{C}\tilde{\mathbf{y}} = \eta \mathbf{C}\mathbf{y} - \mathbf{C}\mathbf{y}' \preceq \mathbf{0}$, hence $\tilde{\mathbf{y}}$ is a T-decrease.

Now consider a place p such that $\mathbf{C}(p, \cdot)\tilde{\mathbf{y}} < \mathbf{0}$ and let $t_j = p^\bullet$ be its only output transition (it exists

because the net is strongly connected and is unique because the net is structurally conflict-free). The transition t_j must belong to the support of $\tilde{\mathbf{y}}$.

We now show that all transitions $t_{j'} \in t_j^{\bullet\bullet}$ also belong to the support of $\tilde{\mathbf{y}}$. In fact, assume there exists a place $p' \in t_j^\bullet$ with $t_{j'} = p'^\bullet$ and such that $\tilde{y}_{j'} = 0$. Then $\mathbf{C}(p', \cdot)\tilde{\mathbf{y}} \geq \mathbf{C}(p', t_j)\tilde{y}_j > 0$, because $\mathbf{C}(p', t_{j'})$ is the only negative element of $\mathbf{C}(p', \cdot)$. By induction, one can show that all transitions reachable with a direct path from t_j must belong to the support of $\tilde{\mathbf{y}}$, and because the net is strongly connected we conclude that $\tilde{\mathbf{y}} > \mathbf{0}$, clearly a contradiction. \square

Proposition 14. *Consider a CPN N that does not admit a T -invariant or a T -increase. Then all possible firing vectors are bounded.*

Proof. First consider the set \mathcal{N} of all vectors $\mathbf{y} \in \mathbb{R}^n$, with $\mathbf{y} \geq \mathbf{0}$ and $\|\mathbf{y}\| = 1$. For all $\mathbf{y} \in \mathcal{N}$, we define $d(\mathbf{y}) \stackrel{\text{def}}{=} \min_{p \in P} (\mathbf{C}(p, \cdot)\mathbf{y})$ and $\pi(\mathbf{y}) \stackrel{\text{def}}{=} \operatorname{argmin}_{p \in P} (\mathbf{C}(p, \cdot)\mathbf{y})$. By assumption (no T -invariant and no T -increase) then $d(\mathbf{y}) < 0$. Since the set \mathcal{N} is closed and bounded, the quantity $d \stackrel{\text{def}}{=} \inf_{\mathbf{y} \in \mathcal{N}} d(\mathbf{y}) < 0$.

We now prove that all firing vectors \mathbf{z} satisfy $\|\mathbf{z}\| \leq B \stackrel{\text{def}}{=} -\max_{p \in P} \mathbf{M}_0(p)/d$.

Let \mathbf{z} be any firing vector, then \mathbf{z} can be written as $\alpha\mathbf{y}$ with $\mathbf{y} \in \mathcal{N}$ and $\alpha = \|\mathbf{z}\|$. Let $p = \pi(\mathbf{y})$. After firing \mathbf{z} we get

$$\begin{aligned} \mathbf{M}(p) &= \mathbf{M}_0(p) + \mathbf{C}(p, \cdot)\mathbf{z} \\ &= \mathbf{M}_0(p) + \mathbf{C}(p, \cdot)\alpha\mathbf{y} \\ &= \mathbf{M}_0(p) + \alpha d(\mathbf{y}) \\ &\leq \max_{p \in P} \mathbf{M}_0(p) + \alpha d. \end{aligned}$$

Since $\mathbf{M}(p) \geq 0$, this implies $\alpha \leq -\max_{p \in P} \mathbf{M}_0(p)/d$. \square

3 Stationary behavior computation of STAR-DTCPN

In this section we present a linear program based on Little's law to compute the stationary transition firing rates of the model by Cohen *et al.*

Theorem 15. *Consider a STAR-DTCPN $\langle N, \delta, \mathbf{M}(0), \rho \rangle$. The structurally conflict-free net equivalent to $\langle N, \delta, \mathbf{M}(0), \rho \rangle$ is denoted $\langle \overline{N}, \overline{\delta}, \overline{\mathbf{M}}(0) \rangle$ and its incidence matrix is denoted $\overline{\mathbf{C}}$. Consider the following linear programming problem LPP (6)*

$$\begin{aligned} \max \quad & \mathbf{1}^T \mathbf{y} \\ \text{s.t.} \quad & \begin{cases} \overline{\mathbf{C}}\mathbf{y} \geq \mathbf{0} \\ \overline{\mathbf{M}}_i \geq \overline{\mathbf{Pre}}(p_i, t_j)y_j\overline{\delta}_i \quad \forall p_i \in P \text{ and } p_i^\bullet = \{t_j\} \\ \overline{\mathbf{M}} = \overline{\mathbf{M}}(0) + \overline{\mathbf{C}}\mathbf{z}, \end{cases} \end{aligned} \tag{6}$$

where the variables are $\mathbf{y} \in (\mathbb{R}_0^+)^n$, $\overline{\mathbf{M}} \in (\mathbb{R}_0^+)^{\overline{m}}$, and $\mathbf{z} \in (\mathbb{R}_0^+)^n$.

- (a) Let $(\mathbf{y}, \overline{\mathbf{M}}, \mathbf{z})$ be an admissible solution of (6). Then there exists a firing policy u and a firing rate vector $\mathbf{y}' \stackrel{\text{def}}{=} (1 - \varepsilon)\mathbf{y}$ arbitrarily close to \mathbf{y} such that the net reaches a periodic mode of operation with

$$\lim_{\tau \rightarrow \infty} \frac{Z_j^u(\tau)}{\tau} = y'_j.$$

- (b) Let $(\mathbf{y}^*, \overline{\mathbf{M}}^*, \mathbf{z}^*)$ be an optimal solution⁷ of (6). Then

i) $\mathbf{y}^* \leq \liminf_{\tau \rightarrow \infty} \mathbf{Z}^{asap}(\tau)/\tau$.

ii) If $\lim_{\tau \rightarrow \infty} \mathbf{Z}^{asap}(\tau)/\tau$ exists, then $\lim_{\tau \rightarrow \infty} \mathbf{Z}^{asap}(\tau)/\tau = \mathbf{y}^*$.

iii) If $\mathbf{y}^* = \mathbf{0}$ then $\lim_{\tau \rightarrow \infty} \mathbf{Z}^{asap}(\tau)/\tau = \mathbf{0}$.

Before proving the theorem, let us informally describe the constraints in LPP (6). The first constraint gives the set of all possible repetitive behaviors, \mathbf{y} . The second constraint is Little's law for each places and defines which repetitive behaviors, \mathbf{y} are admissible given the time delays for a given marking. The third constraint gives a set of markings that are arbitrarily close to the reachable markings. Thus for each $\overline{\mathbf{M}}$ solution of the constraint there exists a reachable marking that is reachable and such that both marking have performances which are arbitrarily close.

Proof. First of all, note that the constraint set (6) always admits a solution: indeed $\mathbf{y} = \mathbf{0}, \overline{\mathbf{M}} = \overline{\mathbf{M}}(0), \mathbf{z} = \mathbf{0}$ is an admissible solution.

Part (a)

Note that if $(\mathbf{y}, \overline{\mathbf{M}}, \mathbf{z})$ is a solution (not necessarily optimal), by proposition 3 there exists a firing sequence of the timed net such that from $\overline{\mathbf{M}}(0)$ one reaches a positive marking $\overline{\mathbf{M}}'$ such that for all places p_i , $|\overline{M}_i - \overline{M}'_i| \leq \eta$ for η arbitrarily small. Hence there also exists an arbitrarily small ε such that $\mathbf{y}' = (1 - \varepsilon)\mathbf{y}$ satisfies for all i $\overline{M}'_i \geq \overline{\text{Pre}}(p_i, t_j)y'_j\delta_i$ with $t_j = p_i^*$. The marking \mathbf{M}' with all tokens available for firing can be reached by the timed net in finite time τ_0 . We now prove that there exists a periodic mode of operation such that from $\overline{\mathbf{M}}'$ a sequence \mathbf{y}' can be fired every time unit.

We first give the proof when all delays are positive. Since $\overline{M}'_i \geq \overline{\text{Pre}}(p_i, t_j)y'_j\delta_i \geq \overline{\text{Pre}}(p_i, t_j)y'_j$, transition t_j can fire a quantity y'_j immediately, at time τ_0 . At time τ_0^+ , the marking is $\overline{M}''_i \geq \overline{M}'_i$ because $\mathbf{C}\mathbf{y}' \geq \mathbf{0}$. This marking is divided in two parts; one already present in the places before the firing which is available ($\overline{\text{Pre}}(p_i, t_j)y'_j(\delta_i - 1)$) and the new arrivals which are not yet available for firing (this part being larger or equal than $\overline{\text{Pre}}(p_i, t_j)y'_j$). The later part will be available at time $\tau_0 + \delta_i$. Therefore, we can continue to fire y'_j each time unit between τ_0 and $\tau_0 + \delta_i$. From that time on, we can continue firing y'_j each time unit because the quantity that entered δ_i times unit before has become available.

Now, if some places have null holding times, they can be ranked topologically since they do not form circuits. For p_i of rank one (immediate places following timed places), the quantity y'_j can

⁷We use the convention that if the objective function is unbounded, then all the unbounded components of \mathbf{y} are set to $+\infty$

be fired at time τ_0^+ , and the marking available for future firings is larger than \overline{M}_i' . The same reasoning is done for places of rank 2 at time τ_0^{++} . By induction on the rank, all transitions j will fire y_j' every time unit.

This timed behavior has an asymptotic firing rate \mathbf{y}' .

Part (b)

i) In part (a) we have shown that for any $(\mathbf{y}, \overline{\mathbf{M}}, \mathbf{z})$ admissible solution of (6) there exists an ultimately periodic timed behavior called u with asymptotic firing rate \mathbf{y}' arbitrarily closed to \mathbf{y} . On the other hand, for a STAR-DTCPN, using the fact that the *asap* behavior corresponds to choosing the equality in Equation (4) it is obvious that $Z_j^{asap}(\tau) \geq Z_j^u(\tau)$.

Thus, the *asap* behavior is such that for each component j , it holds

$$\liminf_{\tau \rightarrow \infty} Z_j^{asap}(\tau)/\tau \geq \sup_{\mathbf{y}} y_j \quad (7)$$

where the sup is taken over all possible solutions of the constraint set (6).

ii) If the net admits stationary rates $\boldsymbol{\lambda}$ under *asap*, then these stationary rates $\boldsymbol{\lambda}$ must satisfy $C\boldsymbol{\lambda} \geq \mathbf{0}$.

Let $\overline{\mathbf{M}}^\infty = \lim_{\tau \rightarrow \infty} \overline{\mathbf{M}}(\tau)$ under the *asap* policy. The rates $\boldsymbol{\lambda}$ also satisfy the constraint $\overline{\mathbf{M}}_i^\infty \geq \overline{\mathbf{Pre}}(p_i, t_j) \lambda_j \bar{\delta}_i$ for all $p_i \in P, t_j = p_i \bullet$ which comes from Little's law giving the relations between the stationary marking and the stationary rates for all transitions.

As for the last constraint, since $\overline{\mathbf{M}}^\infty \in \mathcal{R}^\infty(\overline{\mathbf{M}}(0))$ then there exists \mathbf{z}^∞ such that $\overline{\mathbf{M}}^\infty = \overline{\mathbf{M}}(0) + \overline{\mathbf{C}}\mathbf{z}^\infty$. Finally $(\boldsymbol{\lambda}, \overline{\mathbf{M}}^\infty, \mathbf{z}^\infty)$ is a possible solution of the program.

An optimal solution of LPP (6) $(\mathbf{y}^*, \overline{\mathbf{M}}^*, \mathbf{z}^*)$ is such $\mathbf{y}^* \geq \boldsymbol{\lambda}$ while item (b).i shows that $\boldsymbol{\lambda} \geq \mathbf{y}^*$. This implies $\boldsymbol{\lambda} = \mathbf{y}^*$.

iii) If the optimal solution is such that $\mathbf{y}^* = \mathbf{0}$, then the net does not have a T-invariant or a T-increase. To prove this, assume that there exists a vector \mathbf{y}' such that $\mathbf{y}' \succeq \mathbf{0}$ and $C\mathbf{y}' \geq \mathbf{0}$. By proposition 4, it is possible with a firing vector \mathbf{z} to reach a strictly positive marking $\overline{\mathbf{M}}$. From this marking there exists an $\varepsilon > 0$ such that $(\varepsilon\mathbf{y}', \overline{\mathbf{M}}, \mathbf{z})$ is a solution of (6) with a better cost, and thus contradicting the assumption that a solution with $\mathbf{y}^* = \mathbf{0}$ is optimal.

From proposition 11 it follows that the net is either not lim-live or unbounded. Since the net does not admit a T-increase it is certainly bounded and can only be not lim-live. Moreover, by using proposition 14, all firing vectors are bounded (by say, B). In particular, for all τ , $Z_j^{asap}(\tau)$ is bounded by B for all τ and all j . This means that $\lim_{\tau \rightarrow \infty} Z_j^{asap}(\tau)/\tau = 0$ for all j .

□

This theorem calls for several comments.

1. Note that the theorem does not need any assumption of the STAR-CPN. The only minor assumptions were those in remark 5.
2. The result of theorem 15 stating that the *asap* behavior is the "best" for continuous nets is not as obvious as it may appear. For instance, this result does not hold for discrete nets, as shown in Section 5.
3. The most interesting result is item (b).ii which gives a way to compute the rates of the *asap* policy if they exist. However a complete characterization of nets in which asymptotic firing rates exist is still not known. The following corollary gives conditions which ensure their existence.

Corollary 16. *In a strongly connected STAR-DTCPN with a optimal solution of LPP (6) such that $C\mathbf{y}^* = \mathbf{0}$, it holds that \mathbf{y}^* is finite and $\lim_{\tau \rightarrow \infty} \mathbf{Z}^{asap}(\tau)/\tau = \mathbf{y}^*$.*

Proof. If \mathbf{y}^* is finite, then by proposition 12, $\mathbf{y}^* > \mathbf{0}$. In this case, theorem 9 ensures the existence of asymptotic rates and the result follows from point (b).ii of the previous theorem. It remains to show that \mathbf{y}^* cannot be unbounded. To prove this assume that \mathbf{y}^* is unbounded, then there also exists finite solutions with $C\mathbf{y}' = \mathbf{0}$. Again, \mathbf{y}' is a T-invariant and theorem 9 ensures the existence of finite asymptotic rates. Clearly a contradiction using point (b).ii of the theorem. \square

4. We cannot exclude so far that the asymptotic regime which is computed by LPP (6) is only reached asymptotically, in the sense that it may take an infinite amount of time for the rates in the net under *asap* firing to reach \mathbf{y}^* .
5. The places p_i where the equality holds, $\overline{M}_i^* = \overline{Pre}(p_i, t_j) y_j^* \overline{\delta}_i$ can be viewed as bottlenecks. Increasing the holding time in any one of these places will result in a decrease of the asymptotic firing rates \mathbf{y}^* .
6. Cohen *et al.* provide a computation of the asymptotic firing rates by means of a policy iteration over a dynamic programming problem. On the contrary we use linear programming. The worst case complexity of both approaches is polynomial, and in practice they both convergence very fast to the solution.

The real advantage of the LPP approach is the fact that it is more general. In fact, first of all it can be applied to nets that do not admit T-invariants (e.g., weakly stationary but not stationary). Secondly, with minor extensions the LPP approach can also be applied to nets with conflicts, as shown in the next section.

7. A linear program similar to LPP (6) can be used to compute the stationary behavior for the model of Recalde and Silva by replacing δ_i (the holding time of place p_i) by s_j (the mean service time of transition t_j). This is rather surprising (at least for us) since the transient behavior is very different in both models.

The LPP used in [10] to compute the stationary behavior is similar to ours but a single Little law is written for each P-invariant (i.e., not for each place), and the LP takes the slowest one.

Our method, unlike the approach of Recalde and Silva does not require that the net is lim-live and bounded. It does not even require that it has a positive T-invariant.

8. The solution of LPP (6) can also be used to compute second order asymptotic values (at least partially) when the solution is not null $\mathbf{y}^* > \mathbf{0}$, which is the most interesting case. Indeed, it may be used to partially determine the average stationary marking (i.e., some of its components).

We know that the earliest firing policy provides an optimal solution of LPP (6) (see the proof of theorem 15). However, it could be that many other transient behaviors (leading to different markings) also yield the same value for \mathbf{y}^* . Since we have no control on the solution provided by a LPP solver, we may not know whether the marking provided as a solution is the marking corresponding to the *asap* policy.

We are still exploring a characterization of all those cases where the stationary marking found by LPP (6) is the stationary marking reached using the *asap* firing policy.

As an immediate consequence of the form of LPP (6), the case where the net does not contain any synchronization (this class includes state machines) is a case where the stationary marking is unique because equality is reached in all constraints over $\overline{\mathbf{M}}^*$.

3.1 Equivalent linear program

A linear program equivalent to LPP (6) can also be written using the original STAR-DTCPN $\langle N, \delta, \mathbf{M}(0), \rho \rangle$ with incidence matrix \mathbf{C} .

We get

$$\begin{aligned} \max \quad & \mathbf{1}^T \mathbf{y} \\ \text{s.t.} \quad & \begin{cases} \mathbf{C}\mathbf{y} \geq \mathbf{0} \\ \mathbf{Pre}(p_i, t_j)y_j \leq \rho(p_i, t_j)\mathbf{Post}(p_i, \cdot)\mathbf{y} & \text{for all } p_i \in P : |p_i^\bullet| > 1, t_j \in p_i^\bullet \\ M_i \geq \mathbf{Pre}(p_i, \cdot)\mathbf{y}\delta_i & \text{for all } p_i \in P \\ \mathbf{M} = \mathbf{M}(0) + \mathbf{C}\mathbf{z}, \\ \mathbf{Pre}(p_i, t_j)z_j \leq \rho(p_i, t_j)[\mathbf{Post}(p_i, \cdot)\mathbf{z} + M_i(0)] & \text{for all } p_i \in P : |p_i^\bullet| > 1, t_j \in p_i^\bullet \end{cases} \end{aligned} \quad (8)$$

where the variables are $\mathbf{y} \in (\mathbb{R}_0^+)^n$, $\mathbf{M} \in (\mathbb{R}_0^+)^m$, and $\mathbf{z} \in (\mathbb{R}_0^+)^n$.

We believe that formal proof that LPP (6) and LPP (8) are completely equivalent is not necessary. We simply will comment the main differences.

Firstly we observe that constraints of the form

$$\mathbf{Pre}(p_i, t_j)y_j \leq \rho(p_i, t_j)\mathbf{Post}(p_i, \cdot)\mathbf{y}$$

impose that the output flow from a conflict place p_i due to the firing of transition t_j , i.e., $\mathbf{Pre}(p_i, t_j)y_j$, cannot be greater than the fraction reserved for t_j of the flow entering place p_i , i.e., the fraction $\rho(p_i, t_j)$ of the total input flow $\mathbf{Post}(p_i, \cdot)\mathbf{y}$. These constraints were implemented in the structure of the equivalent structurally conflict free net \overline{N} .

Secondly, constraints of the form

$$M_i \geq \mathbf{Pre}(p_i, \cdot)\mathbf{y}\delta_i$$

are slightly different from the corresponding ones in LPP (6) because now each place may have more than one output transition and we need to use the complete vector \mathbf{y} .

Finally, constraints of the form

$$\mathbf{Pre}(p_i, t_j)z_j \leq \rho(p_i, t_j)[\mathbf{Post}(p_i, \cdot)\mathbf{z} + M_i(0)]$$

ensure that also during the transient only the mark reserved for transition t_j may be used to fire t_j . Again, these constraints were implemented in the structure of the equivalent structurally conflict free net \overline{N} .

3.2 Development of an example

This following example describes a manufacturing system where parts that enter may be routed to three different processing lines. The first line (path $t_1 - t_3$) only uses resources from unit A (place p_2), the second line (path $t_7 - t_9$) only uses resources from unit B (place p_8), the last one (path $t_4 - t_6$) uses resources from unit A and B.

We assume the routing is such that all three lines receive the same number of parts (routing at p_6 is 1/3 for all output transitions) and correspondingly each resource is equally shared between the two lines that use it (routing at p_2 and p_8 is 1/2 for all output transitions).

The vector of place delays is the following: $\delta = [2 \ 1 \ 5 \ 20 \ 1 \ 2 \ 7 \ 2 \ 3]^T$.

If we compute the equivalent conflict-free net and solve LPP (6) (or equivalently if we solve LPP (8)) we obtain an optimal solution with $\mathbf{y}^* = \frac{1}{11}[1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1]^T$. This vector is a T-invariant. By corollary 16 the net reaches under *asap* a (strongly) stationary behavior with rates $\lambda = \mathbf{y}^*$.

4 Optimal routing policy in DTCPN

Here is the main objective of the section: given a deterministic timed continuous Petri net, find the best routing parameters $\rho(p, t)$ which give the optimal stationary firing rates \mathbf{y} according to a given objective function $\mathbf{a}^T\mathbf{y}$ with $\mathbf{a} \geq 0$. We will show that this problem can be solved with a slight modification of LPP (6).

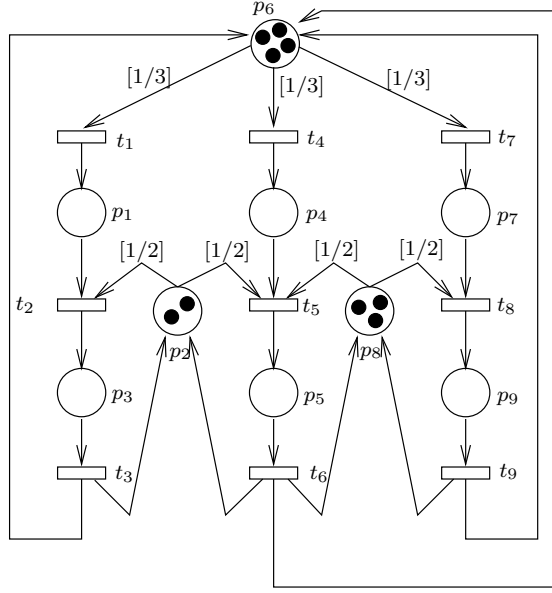


Figure 6: A STAR-DTCPN

In a DTCPN where the routing parameters are not fixed beforehand, the transformation into a structurally conflict-free net is not possible because the matrix \overline{C} would depend on the routing parameters.

One could think of the following approach for solving the optimization problem:

1. Choose initial values for the routing parameters arbitrarily and solve LPP (6).
2. Use sensitivity analysis techniques on the routing parameters to evaluate how they should vary in order to increase the rates [12].
3. Iterate this procedure until convergence is reached.

However, this technique has several drawbacks. Firstly, it may not work in the sense that the objective function may not be a continuous function of the routing parameters. Secondly, it may not converge to the optimal solution. Thirdly, it has a rather high computational complexity.

Another approach is to come up with a new linear program having additional degrees of freedom where the routing constraints are removed. This can be done by using the version LPP (8) of the program where the objective function is $\mathbf{a}^T \mathbf{y}$ and the routing constraints are removed. Next theorem shows that this approach is correct.

Theorem 17. Consider a DTCPN $\langle N, \delta, \mathbf{M}(0) \rangle$ with incidence matrix \mathbf{C} . Consider a linear

programming problem

$$\begin{aligned} \max \quad & \mathbf{a}^T \mathbf{y} \\ \text{s.t.} \quad & \begin{cases} \mathbf{C}\mathbf{y} \geq \mathbf{0} \\ M_i \geq \mathbf{Pre}(p_i, \cdot) \mathbf{y} \delta_i \quad \text{for all } p_i \in P \\ \mathbf{M} = \mathbf{M}(0) + \mathbf{C}\mathbf{z}, \end{cases} \end{aligned} \quad (9)$$

where the variables are $\mathbf{y} \in (\mathbb{R}_0^+)^n$, $\mathbf{M} \in (\mathbb{R}_0^+)^m$, and $\mathbf{z} \in (\mathbb{R}_0^+)^n$, with optimal solution $(\mathbf{y}^*, \mathbf{M}^*, \mathbf{z}^*)$

Define a STAR-DTCPN $\langle N, \delta, \mathbf{M}^*, \rho^* \rangle$ with

$$\rho^*(p_i, t_j) = \frac{\mathbf{Pre}(t_i, t_j) y_j}{\sum_{t_k \in p_i^\bullet} y_k^* \mathbf{Pre}(t_i, t_k)} \quad \forall t_j \in p_i^\bullet.$$

This net has a stationary behavior \mathbf{y}^* such that $\mathbf{a}^T \mathbf{y}^*$ is larger than $\mathbf{a}^T \mathbf{y}$ where \mathbf{y} is the stationary behavior of a STAR-DTCPN $\langle N, \delta, \mathbf{M}, \rho \rangle$ where \mathbf{M} is any reachable marking from $\mathbf{M}(0)$, and ρ are arbitrary routings.

Proof. Firstly note that if $(\mathbf{y}^*, \mathbf{M}^*, \mathbf{z}^*)$ is a solution of LPP (9) and ρ^* is chosen accordingly, then $(\mathbf{y}^*, \mathbf{M}^*, \mathbf{z}^*)$ is also a solution of LPP (8) with routing ρ^* . By using the same reasoning as in the proof of theorem 15 part (a), we conclude that there exists a stationary behavior u of the STAR-DTCPN with firing rates $\boldsymbol{\lambda} = \mathbf{y}^*$.

Secondly, for any other routing ρ , consider the associated STAR-DTCPN $\langle N, \delta, \mathbf{M}, \rho \rangle$. By using the same argument as in the proof of theorem 15 part (b).ii, every stationary behavior of this net is an admissible solution of LPP (8) and *a fortiori* an admissible solution of LPP (9) but not necessarily optimal with respect to the performance index $\mathbf{a}^T \mathbf{y}$. \square

The choice of the cost function $\mathbf{a}^T \mathbf{y}$ used to define the optimal behavior depends on the context. The transformation of a given objective function into the coefficients a_j may sometimes be tricky. Such problems are detailed in [2].

One main drawback of theorem 17 is that it does not say anything about the behavior of the net under the *asap* policy. In particular, it does not prove that the best routing under *asap* is ρ^* because we do not know if *asap* reaches asymptotic rates. The possibility shown in the following diagram cannot be excluded *a priori*

$$\begin{array}{ccc} \text{asap}(\rho^*) & \implies & u^*(\rho^*) \\ \uparrow & & \Downarrow \\ \text{asap}(\rho) & \longrightarrow & u^*(\rho) \end{array}$$

where $u^*(\rho)$ is an optimal ultimately periodic behavior under routing ρ and $v \rightarrow u$ means that

$$\liminf_{\tau \rightarrow \infty} \mathbf{a}^T \mathbf{Z}^v(\tau) / \tau > \liminf_{\tau \rightarrow \infty} \mathbf{a}^T \mathbf{Z}^u(\tau) / \tau,$$

while $v \implies u$ means that

$$\liminf_{\tau \rightarrow \infty} \mathbf{a}^T \mathbf{Z}^v(\tau)/\tau \geq \liminf_{\tau \rightarrow \infty} \mathbf{a}^T \mathbf{Z}^u(\tau)/\tau.$$

However, there are cases where such a situation may be ruled out.

Theorem 18. *If the optimal solution of LPP (9) is such that $\mathbf{a}^T \mathbf{y}^* = 0$ then for all routing ρ , $\liminf_{\tau \rightarrow \infty} \mathbf{a}^T \mathbf{Z}^{asap(\rho)}(\tau)/\tau = 0$.*

Proof. If $\mathbf{a}^T \mathbf{y}^* = 0$ then it is easy to show using the same reasoning as in the proof of part (b).iii of theorem 15 that the net does not admit any T-increase or T-invariant whose support intersects the support of \mathbf{a} . It is possible to generalize proposition 14 and show that a firing vector \mathbf{z} satisfies $z_j < B$ if $a_j > 0$. In particular for all τ and all ρ , $Z_j^{asap(\rho)}(\tau) \leq B$. \square

Theorem 19. *If the net N is strongly connected and bounded, then the optimal behavior under asap is obtained by choosing ρ^* as defined in theorem 17. Furthermore, $\lim_{\tau \rightarrow \infty} \mathbf{a}^T \mathbf{Z}^{asap(\rho^*)}(\tau)/\tau = \mathbf{a}^T \mathbf{y}^*$.*

Proof. If the net N is strongly connected and bounded then when the STAR is enforced the STAR-DTCPN admits at most one T-invariant by proposition 12 and no T-increase. Now, for all routing ρ , the asap policy under ρ has a stationary behavior (see corollary 16). This implies that

$$\begin{aligned} asap(\rho^*) &= u^*(\rho^*) \\ \Downarrow \quad \Downarrow \\ asap(\rho) &= u^*(\rho). \end{aligned}$$

\square

Remark 20. *We conclude this section with an important observation. Theorem 17 shows that an optimal periodic mode of operation can always be enforced by a STAR ρ^* . However, it may well be the case that marking \mathbf{M}^* that must be reached to start this periodic mode cannot be reached using STAR ρ^* .*

In fact, the reachability condition $\mathbf{M} = \mathbf{M}(0) + \mathbf{C}\mathbf{z}$ does not take into account the routing parameters (which are computed a posteriori).

Consider the case of a net displayed in figure 7 where the delays for all the places are 1, where the transient does not follow ρ^ .*

The optimal solution with objective function $\mathbf{1}^T \mathbf{y}$ is marking is $\mathbf{y}^ = \frac{1}{2}[1 \ 1 \ 2]^T$, $\mathbf{M}^* = \frac{1}{2}[3 \ 1 \ 2]^T$ and $\mathbf{z}^* = \frac{1}{2}[1 \ 0 \ 4]^T$. This yields a routing ρ^* such that $\alpha = 2/3$ and the objective is $\mathbf{1}^T \mathbf{y}^* = 2$.*

Since the net does not contain any synchronization, this is the unique optimal solution. Note that the optimal marking \mathbf{M}^ is reached from \mathbf{M}_0 by firing a vector \mathbf{z}^* incompatible with the routing ρ^* .*

If the routing ρ^ is used from the start, the optimal firing policy leads to a total firing rate of $4/3$.*

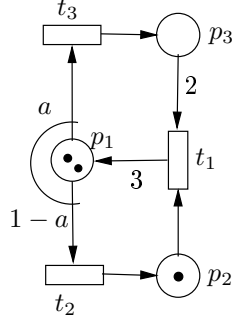


Figure 7: A CPN where the transient regime should not follow the optimal routing ρ^* .

4.1 Development of an example

Let us continue with the example discussed in subsection 3.2. Assume that the routing is not fixed (see net in figure 8).

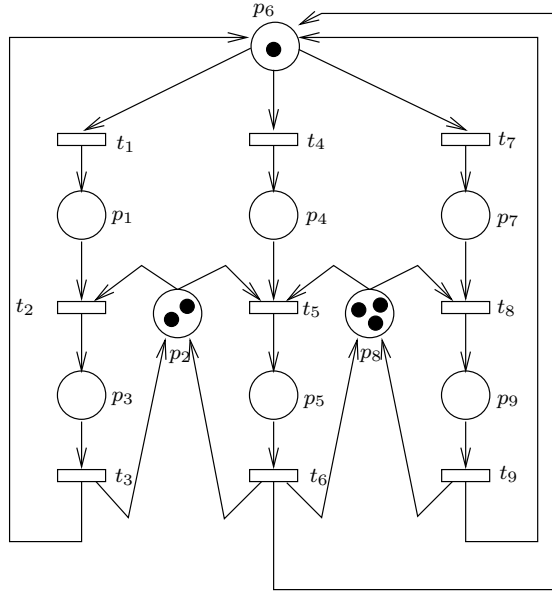


Figure 8: Example with no routing fixed beforehand

We want to maximize the throughput of the net and choose the objective function $\mathbf{a}^T \mathbf{y} = y_3 + y_6 + y_9$. Solving LPP (9) we obtain the optimal rates $\mathbf{y}^* = \frac{1}{12}[4 \ 4 \ 4 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1]^T$ and the corresponding optimal routing is $\rho^*(6, 1) = 4/5$, $\rho^*(6, 4) = 0$, $\rho^*(6, 7) = 1/5$, $\rho^*(2, 2) = 1$, $\rho^*(2, 5) = 0$, $\rho^*(8, 5) = 0$, $\rho^*(8, 8) = 1$.

Note that this net is strongly connected and bounded hence by theorem 19 the optimal behavior under *asap* is obtained by choosing ρ^* and $\lim_{\tau \rightarrow \infty} \mathbf{a}^T \mathbf{Z}^{asap(\rho^*)}(\tau)/\tau = \mathbf{a}^T \mathbf{y}^*$.

5 Heuristic optimization of discrete nets

One of the major drawbacks of the fluidification of Petri nets is that the results obtained for the continuous net do not hold for the discrete one. This is in particular the case when we are faced with the problem of enforcing the routing policy found with the optimization of the continuous model.

The first difficulty is that in a discrete net the stationary routing parameters of the continuous nets are replaced by routing functions: each token entering a place p is assigned an output transition according to a routing function $r_p : \mathbb{N} \rightarrow p^\bullet$. The link with the routing parameters of the continuous net is done by assuming that the function r_p sends a proportion of $\rho(p, t)$ tokens to transition t asymptotically:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mathbf{1}_{\{r_p(n)=t\}} = \rho(p, t). \quad (10)$$

This is done, for example, by choosing a *stochastic iid Bernoulli function* r_p where $Pr(r_p(n) = t) = \rho(p, t)$. Another (deterministic) possibility is to choose a *stair case routing function* r_p such that for all K ,

$$\sum_{k=1}^K \mathbf{1}_{\{r_p(k)=t\}} = \lfloor \rho(p, t)K \rfloor.$$

The evolution equations in the discrete case become

$$Z_j(\tau) \leq \min_{p_i \in \bullet t_j} \left\lfloor \frac{A_{i,j}(H_i(\tau - \delta_i))}{Pre(p_i, t_j)} \right\rfloor \quad \forall t_j \in T$$

$$H_i(\tau) = M_i(0) + \sum_{t_j \in \bullet p_i} Post(p_i, t_j) Z_j(\tau) \quad \forall p_i \in P,$$

where $A_{i,j}(K) = \sum_{k=1}^K \mathbf{1}_{\{r_i(k)=t_j\}}$.

When the net is live and bounded, the firing rate of the transitions, $\phi_j = \lim_{\tau \rightarrow \infty} Z_j^{asap}(\tau)/\tau$ is shown to exist as soon as the limit $\lim_{K \rightarrow \infty} A(K)/K$ exists using transition blocking techniques in [7]. However, the computation of ϕ_j seems to be rather difficult. There are several reasons for that.

Firstly, the firing rate in the discrete case depends on the choice of the routing functions. In particular, the stair case and the Bernoulli routings may not have the same rates (there are strong evidences to claim that for live nets, the stair case is always the best routing function satisfying the asymptotic condition (10)).

Secondly, the reachability condition in the discrete case is known to be difficult to check (this is the main reason why LPP (6) does not work here).

Another difficulty comes from the fact that, unlike what we have proved in this paper for continuous nets, the optimal stationary regime in the discrete case may not be reached by the asap firing policy. The example in figure 9 is a straightforward translation of a Tetris heap taken from [8] into a Petri net.

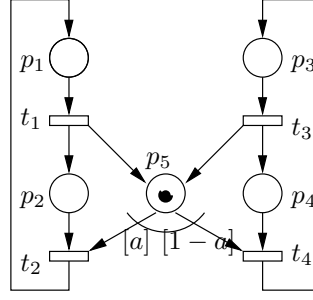


Figure 9: A Petri net with routing

If the delays in the places are chosen such that $\delta_1 = 8/89$, $\delta_2 = 32/89$, $\delta_3 = 81/89$, $\delta_4 = 8/89$, $\delta_5 = 0$ then the stationary optimal routing under *asap* is $(4, 3, 3, 3)^\infty$ with a total rate of $89/32$ while the optimal routing policy is $(4, 3, 3)^\infty$ with a higher rate: 3.

It is not difficult to prove (just looking at the evolution equations) that the firing rates in the discrete case are not larger than in the continuous case. Therefore, LPP (6) provides an upper bound for a discrete net with a fixed routing policy. It is also easy to construct examples where the rate in the discrete case is strictly smaller than in the continuous case.

Finally, one can note that any routing function in the discrete case can be implemented by using supervisory control (as it is done in [1]).

6 Conclusion

In this paper we have used structural analysis for determining optimal stationary control policies for Continuous Petri Nets.

We have shown that for decision-free nets, where the only control policy consists in determining how fast should a transition consume the tokens that have been reserved for its firing, the problem of computing the stationary behavior can be solved by a linear program.

We have also considered the case in which the decision policy also requires solving a routing problem and we have shown how to compute an optimal routing using a different linear programming problem.

Among all admissible control policies, we have characterized important properties of the "as soon as possible" policy that is particularly relevant in many optimization problems.

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