Simulations: Guided Homework #1

Due: Thurs Apr 19th, before class.
Submit on Canvas as a .pdf, .docx, or a scanned document. Submit your code as well. I encourage you to try typing it out rather than writing by hand. It will be good practice. Also, if you know LaTeX, I highly encourage you to use it!
This should not take more than 30 minutes to do.

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Problem 1

This is a simple recap on how ODEs work and how you can numerically solve for them.

Let's look at the Lotka-Volterra Model, also known as the predator-prey equations, which is a pair of first order, non-linear, differential equations frequently used to describe the dynamics of biological systems in which two species interact as populations, one a predator and the other its prey. The model was proposed independently by Alfred J. Lotka in 1925 and Vito Volterra in 1926, and can be described by:

$$\frac{dx}{dt} = Ax - Bxy$$
$$\frac{dy}{dt} = -Cy + Dxy$$
$$Let : A, B, C, D > 0$$

In which x denotes the number of prey and y denotes the number of predators.

For this problem, let: $A = 1$, $B = 0.2$, $C = 2$, $D = 0.25$.

1. Given $x(0) = 6$ and $y(0) = 5$, graph this system with a numerical solver from $t = 0, \cdots, 150$. MATLAB is preferred but anything else is fine too. When making a function to use an ODE solver on, it is best if the function is a separate file that you call.
(If you don't know how to do this, or it's taking too much time, it's ok. Download `hw1p1.m` and `hw1ode.m` off the simulations page on the class site and use that.)

2. At any $t = \boldsymbol{Z}^+$ (all positive integers), you can see that rarely is the value of y or x an actual integer. You can't have something like 12.45 deer! Why are these values a decimal and what do the values actually mean?

3. Try to adjust the initial starting populations at $t = 0$. In other words, change the $x(0)$ and $y(0)$ values. Can you find a situation when y or x would drop below 1? Does it make sense for the value to be less than one? How about less than 0.001 (Some very small number)? What should the model do to make it "more realistic?" What should happen if x or y drops below a very small number? Try to change your ode function to correctly show this model behavior.
This problem is called the "atto-fox" problem, and is one of the reasons why some biologists using this model do not believe it can accurately describe predator-prey relations.
(If you don't know how to do this, download `hw1ode2.m` and use this.)

4. Download `PhasePlot.m` off the class site and run it in MATLAB. Note that `PhasePlot.m` uses `hw1ode2.m` so replace where the code uses `hw1ode2` with your function file's name.
The resulting graph is called a phase plot. It is plotting the system as a bunch of vectors, made up by (dx, dy) at each value of (x, y). You can see that it looks like a spiral. Identify the value of (x, y) at the center of the spiral and use those values for the numerical model.
What do you see?

Problem 2

We will look at a partial differential equation for 2-D wave formation. The standard wave equation is:

$$\frac{\partial^2 u}{\partial t^2} = \nabla \cdot \nabla u$$

If you remember from MATH 126, you may recognize ∇ as the **gradient**. The gradient is defined as:

$$\nabla u = \frac{\partial u}{\partial x_1} + \cdots + \frac{\partial u}{\partial x_n}$$

Where $x_1 \cdots x_n$ are the dimensions in the function and $u = f(x_1, \cdots, x_n, t)$. For most cases, we will look at a 2-D physical problem, that is, $u = f(x, y, t)$.

The wave equation can be simplified to:

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u$$

∇^2 is called the **Laplacian**. In 2-D space, the Laplacian of a function u is defined as:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Note: $\frac{\partial^2 u}{\partial t^2}$ is **NOT** part of the Laplacian!
(Physicists use ∇^2 to mean Laplacian. Some mathematicians use Δ to symbolize the Laplacian but that's dumb and confusing so we will almost always use ∇^2, but you may see Δ being used when looking at stuff online. Why do mathematicians have to be so confusing?)
We will see the Laplacian many times in the future, as it is fundamental for fluid dynamics.
Most PDEs are impossible to solve analytically. The wave equation is one of the few PDEs that can be solved analytically, but it still takes a few pages of equations to solve, which no one wants to do. Especially us lol. We'll use a computer instead.

1. We will be looking at the function $u(x, y, t)$ and according to the equation, u will behave like a wave across (x, y). What kind of graph do we need to plot on for this function?
   In other words, what is/are the independent variables and what is/are the dependent variables?

Download the code `pdedemo6.m`[1] off the class site and run it in `MATLAB`. If you don't have MATLAB, use the ME department's remote computers. Instructions to connect to ME's computers are here: `https://www.me.washington.edu/computing/remotedesktop`.
You will see a couple graphs pop up. In the console, type

```
movie(M)
```

to see an animation of the function.

2. There are boundary conditions and initial conditions.
   The boundary conditions are shown in `figure 3`:
   · These are known as **Dirichlet boundary conditions**, when the function = constant. This problem sets the neumann boundary conditions at 0.

$$u|_{E2}\,, u|_{E4} = 0$$

[1] taken from PDE examples in MATLAB

· These are known as **Neumann boundary conditions**, when the derivative of the function = a constant or a function (In this case, u(x,y,t)).

$$\left.\frac{\partial u}{\partial t}\right|_{E1}, \left.\frac{\partial u}{\partial t}\right|_{E3} = 0.$$

The **initial conditions** are:

$$u(x,0) = \arctan\left(\cos\left(\frac{\pi x}{2}\right)\right)$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 3\sin(\pi x)\exp\left(\sin\left(\frac{\pi y}{2}\right)\right)$$

Try to change around the initial conditions, solution times, and switching the boundary condition edges and see what happens. Breaking the code is fine (as long as you have the original)! See what works and what doesn't.

Provide a brief explanation (less than a few sentences) of your observations.