Variable-Rate Anytime Transmission with Feedback

Invited Paper

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***Abstract*—A generalization of the ensemble of non-terminated systematic LDPC convolutional codes developed in our previous work is proposed that allows us to design codes with lower rates than the original structure. We show that over the BEC, the modified codes have improved asymptotic and finite-length behavior and we determine the operational anytime exponent. Having shown the advantages of lowering the rate of the code, we propose a feedback protocol that permits encoder and decoder to operate at a variable rate. The rate is set on-the-fly and depends on the decoding success of the decoder. We describe the construction of the variable rate code structure and demonstrate by simulations the superiority of the variable rate scheme as compared to a scheme using a fixed rate.**

I. Introduction

A major trend in modern industrial systems involving controllers, sensors and actuators, is to replace the point-to-point communication between the different components by connections via a shared network. The use of so-called Networked Control Systems (NCSs) has many advantages, such as enhanced resource utilization, reduced wiring, easier diagnosis and maintenance and re-configurability. A fundamental characteristic of such a cyber-physical system is however that the communication between the different nodes of the system takes place over communication channels of limited bandwidth and is subject to interference and noise. Moreover due to the low-complexity constraints of e.g. the sensing devices, the resolution of the observed values might be quite coarse. Such issues challenge the standard assumption of classical control theory, that communication is performed instantaneously, reliably and with infinite precision; and it makes the underlying communication problem fundamentally different from classic point-to-point transmission. When designing error control schemes for NCSs to protect the transmitted data against transmission errors, we now have to take into account strict delay requirements, reliability constraints and the fact that only limited resources are available. Sahai [1] and Mitter [2] developed a fundamental framework, termed *anytime information theory*, for such reliable causal communication of bit streams through noisy channels. *Anytime reliability* is a key property within this framework. If the decoding error probability of block j decays exponentially with the decoding delay $d(t, j) = t - j$ to the present block at time t, then the error-correcting code is called anytime reliable, or an anytime code. As the decoding delay increases, the reliability improves.

Anytime codes are tree codes since the stream of information is causally encoded. Schulman [3] showed that a class of nonlinear tree codes exists for interactive communication. Building on his work *trajectory codes* were developed in [4] and *potent tree codes* in [5]. In general, however, no deterministic constructions are known for these codes. A structured class of anytime random linear codes was proposed in [6] together with a simple constructive maximum-likelihood decoding algorithm for transmission over the binary erasure channel (BEC). In [7] it is shown that in general random linear tree codes are anytime reliable. A first result on practical anytime codes is presented in [8], showing that protograph-based LDPC-Convolutional Codes (LDPC-CCs) have anytime properties asymptotically under message-passing decoding. The results were generalized in [9] and shown to be valid for transmission over the AWGN channel as well. In [10] a combinatorial approach was presented that allows the finite-length analysis of the codes for transmission over the BEC. A different approach towards the development of practical anytime codes is taken in [11]. Here a class of spatially coupled codes is shown to have anytime properties.

In this paper we build on the protograph-based LDPC-CCs developed in our earlier work [8], [10]. The contributions of the paper are twofold:

- We generalize the design and analysis of the LDPC-CCs proposed in [10] to a broader class of codes with rates equal to $R = 1/r$, where $r \in \mathbb{Z} \geq 2$.
- We propose a feedback protocol with which the codes can be operated at variable rate and which allows reliable transmission even on low quality channels.

II. Anytime Channel Coding over the BEC

The system of interest is shown in Fig. 1. A source produces an information sequence of increasing size

$$\boldsymbol{u}_{[1,t]} = [\boldsymbol{u}_1, \boldsymbol{u}_2 ..., \boldsymbol{u}_t], \tag{1}$$

where $\boldsymbol{u}_j$ is a binary vector of size k. Given the information sequence, for every new information block $\boldsymbol{u}_t$ the encoder emits a code block $\boldsymbol{v}_t$ of size n that is a function of all information blocks seen so far:

$$\boldsymbol{v}_t = \mathcal{E}(\boldsymbol{u}_1, ..., \boldsymbol{u}_t) \tag{2}$$

The code block $\boldsymbol{v}_t$ is transmitted over the BEC and a corrupted version $\tilde{\boldsymbol{v}}_t$ is observed at the receiver. The decoder then produces an estimate of all transmitted information blocks seen so far

$$\hat{\boldsymbol{u}}_{[1,t]} = [\hat{\boldsymbol{u}}_1, ..., \hat{\boldsymbol{u}}_t] = \mathcal{D}(\tilde{\boldsymbol{v}}_1, ..., \tilde{\boldsymbol{v}}_t). \tag{3}$$

based on all received code blocks $\tilde{\boldsymbol{v}}_1, ..., \tilde{\boldsymbol{v}}_t$. The receiver can decide to start decoding at any time.

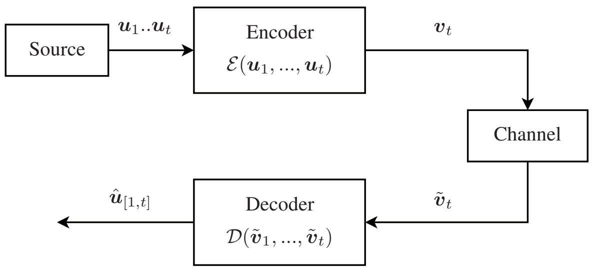

Fig. 1: Anytime system model.

Anytime reliability [1] is formally defined at block level as:

$$P(\hat{\boldsymbol{u}}_j \neq \boldsymbol{u}_j | \boldsymbol{u}_{[1,t]} \text{was transmitted}) \leq \beta 2^{-\alpha d(t,j)} \quad (4)$$

where $d(t, j) = t - j$ denotes the decoding delay between information block j and the most recent block t and β is a constant. For a particular code, the largest α such that (4) is fulfilled is referred to as the *anytime exponent* of the code. As in [10] we are however interested in the *operational anytime exponent* α_o, which is the achievable exponent of a particular code ensemble.

III. LDPC Convolutional Coding Scheme

Following [10], we consider a channel coding scheme based on systematic LDPC convolutional codes. We now briefly review its structure. Encoder and decoder operate at time t on a parity-check matrix given as

$$\boldsymbol{H}_{[1,t]} = \begin{bmatrix} \boldsymbol{H}_0(1) & & & & \\ \boldsymbol{H}_1(1) & \boldsymbol{H}_0(1) & & & \\ \boldsymbol{H}_2(2) & \boldsymbol{H}_1(2) & \boldsymbol{H}_0(2) & & \\ \vdots & \vdots & \vdots & \ddots & \\ \boldsymbol{H}_{t-1}(t) & \boldsymbol{H}_{t-2}(t) & \dots & \boldsymbol{H}_1(t) & \boldsymbol{H}_0(t) \end{bmatrix}, \quad (5)$$

where the elements $\boldsymbol{H}_i(t)$ are binary matrices of size $(n - k) \times n$ that satisfy the properties: $\boldsymbol{H}_i(t) = 0,\ i < 0, \forall t$. Moreover $\boldsymbol{H}_0(t) \neq 0$, $\forall t$, has full rank. At time t the parity-check matrix is then of size $t(n - k) \times tn$. Thanks to the lower-triangular structure of the parity-check matrix the coding scheme is causal and can be efficiently encoded using shift registers [12]. The parity check matrices are built based on a protograph $\boldsymbol{B}_{[1,t]}$, given at time t by

$$\boldsymbol{B}_{[1,t]} = \begin{bmatrix} \boldsymbol{B}_0(1) & & & & \\ \boldsymbol{B}_1(1) & \boldsymbol{B}_0(1) & & & \\ \boldsymbol{B}_2(2) & \boldsymbol{B}_1(2) & \boldsymbol{B}_0(2) & & \\ \vdots & \vdots & \vdots & \ddots & \\ \boldsymbol{B}_{t-1}(t) & \boldsymbol{B}_{t-2}(t) & \dots & \boldsymbol{B}_1(t) & \boldsymbol{B}_0(t) \end{bmatrix}. \quad (6)$$

The protograph considered in [10] is time-invariant, that is $\boldsymbol{B}_i(t_1) = \boldsymbol{B}_i(t_2)\ \forall t_1, t_2 \geq 1$. The matrices $\boldsymbol{B}_i$ are then of size $k_0 \times n_0$, where $k_0, n_0 \in \mathbb{Z}^+$. $\boldsymbol{B}_0$ is full rank. As in [10] we only consider the matrices $\boldsymbol{B}_i$ to have binary entries but in general positive integer entries are possible. The parity check matrix $\boldsymbol{H}_{[1,t]}$ is obtained by *lifting* $\boldsymbol{B}_{[1,t]}$ to the order of M, namely

- each zero of $\boldsymbol{B}_{[1,t]}$ is replaced by a $M \times M$ all-zero matrix; and
- each one of $\boldsymbol{B}_{[1,t]}$ is replaced by a $M \times M$ permutation matrix.

The relation between the size of the parity check matrix at time t and the size of the protograph matrix at time t is then given as $t(n-k) = tM(n_0-k_0)$ and $tn = tMn_0$. The *lifting factor* M determines the sparsity of the parity check matrix. The LDPC-CCs proposed in [10] are rate-$1/2$ codes with a protograph structure compactly written as $\boldsymbol{B}_0 = [1\ 1]$ and $\boldsymbol{B}_i = [1\ 0]$ for all $i \geq 1$, meaning that $k_0 = 1$ and $n_0 = 2$. The lifting factor is chosen equal to the message length k. In order to analyze the behavior of the codes the decoding erasure performance is determined either through an asymptotic analysis, a finite-length analysis or through simulations. The probability of interest is here the decoding erasure probability $P_e^i(t)$ of all messages at time $i \leq t$. By analysing $P_e^i(t)$ over time, we effectively determine the decoding erasure probability over delay $d = t - i$. Since we consider systematic codes, we term the variable nodes corresponding to the systematic part of the codeword as information variable node and the variable nodes corresponding to the non-systematic part of the codeword as parity variable nodes.

A. Finite-Length Behavior

Fig. 2 shows the simulated decoding erasure probability over time when transmitting over BECs with different erasure rates. The code has slightly better protection capabilities for the very first messages, we therefore depict the performance corresponding to a message with a larger index, here $i = 45$. We can see that for a channel with $\epsilon = 0.5$ the code is not capable of correcting any erasure. The erasure floor decreases with decreasing erasure probability and for $\epsilon = 0.2$ there is no visible erasure floor above 10^{-6}. The sudden change from a high erasure floor to a low erasure floor when going from $\epsilon = 0.4$ to $\epsilon = 0.3$ suggests that the codes should not be operated at high erasure rates. The reason for the erasure floors are the existence of so called *increasing erasure patterns* [10]. These erasure patterns are patterns of erasures at information bit positions in the received codeword stream, that the decoder fails to decode. As we can see the rate-1/2 codes are not suitable for low quality channels. Therefore we now propose an extended code structure that allows more reliable transmission over these channels. This is done by first designing codes with a similar structure as the original one but with lower rates and then by the use of a feedback protocol resulting in a variable rate scheme.

IV. Lower Rate Codes

We now extend the code structure to accommodate for other rates than $R = 1/2$. This is done by adding additional parity-

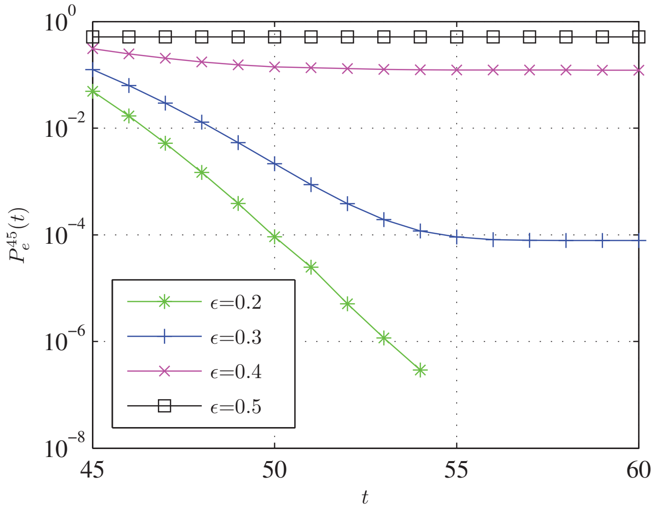

Fig. 2: $P_e(t)$ for messages x_{45} for different ϵ and $k = 10$.

check equations in every time step t in a structured way. The new code design is given by the time-invariant matrices $\boldsymbol{B}_0^r$ and $\boldsymbol{B}_i^r, i \geq 1$, with

$$\boldsymbol{B}_0^r = \begin{bmatrix} 1 & 1 & & & \\ 1 & 0 & 1 & & \\ 1 & 0 & 0 & 1 & \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix} \quad \boldsymbol{B}_i^r = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & & \\ 1 & 0 & \dots & 0 \end{bmatrix} \tag{7}$$

Here $\boldsymbol{B}_0^r$ and $\boldsymbol{B}_i^r$ are of size $k_0 \times n_0$ and $n_0 = rk_0$, where $r \in \mathbb{Z}, r \geq 2$. Thus we construct codes of rates equal to $R = 1/r$. Note that as for the rate-$1/2$ code the check node degree is always equal to 1. The reason for this design choice becomes clear later. Similar to the rate-$1/2$ codes, the rate-$1/r$ codes can be decoded using the expanding window decoder presented in [10].

A. Asymptotic Analysis

In the asymptotic limit of infinite block lengths ($k \to \infty$) the performance of protograph-based LDPC-CCs can be evaluated using Protograph-based EXIT analysis (P-EXIT) [13] and it was shown in [10] that the rate-1/2 codes have anytime properties. The P-EXIT algorithm works on matrices $\boldsymbol{I_{Av,t}}$ and $\boldsymbol{I_{Ev,t}}$ containing the mutual information of the messages sent between variable nodes and check nodes. The entries of $\boldsymbol{I_{Ev,t}}$ and $\boldsymbol{I_{Av,t}}$ in position (i, j) at time t when ℓ iterations are performed are denoted as $I_{Ev,t}^{\ell}(i, j)$ and $I_{Av,t}^{\ell}(i, j)$. The vector of a-posteriori decoding erasure probabilities at time t for a code of rate $1/r$ is given as $\boldsymbol{P}_{\text{APP},\boldsymbol{t}} = [P_{\text{APP},t}(1), \dots, P_{\text{APP},t}(rt)]$ where $P_{\text{APP},t}(j)$ denotes the decoding erasure probability of variable node j for the protograph at time t. For the BEC $\boldsymbol{P}_{\text{APP},\boldsymbol{t}} = 1 - \boldsymbol{I}_{\text{APP},\boldsymbol{t}}$. For a particular variable node (index j) the behavior of the decoding erasure probability over time is described with the vector $\boldsymbol{P}^{\boldsymbol{j}}_{\text{APP},\boldsymbol{[1,t]}} = [P_{\text{APP},1}(j), P_{\text{APP},2}(j), \dots, P_{\text{APP},t}(j)]$. The P-EXIT update equations at time t are described as follows: For pairs (i, j), where $B^r(i, j) = 0$, the entries of $I_{Ev,t}(i, j)$ and $I_{Av,t}(i, j)$ are equal to 0, for all other pairs (i, j), the entries of $I_{Ev,t}(i, j)$ and $I_{Av,t}(i, j)$ are calculated as follows:

- **Initialization:** For all (i, j) if $B^r(i, j) \neq 0$

$$I_{Ev,t}^{0}(i, j) = 1 - \epsilon \tag{8}$$

- **Check to variable update**: For all (i, j) if $B^r(i, j) \neq 0$

$$I_{Av,t}^{\ell+1}(i, j) = \prod_{s=1, s\neq j}^{2i} I_{Ev,t}^{\ell}(i, s)^{B^r(i,s)} I_{Ev,t}^{\ell}(i, j)^{B^r(i,j)-1} \tag{9}$$

- **Variable to check update**: For all (i, j) if $B^r(i, j) \neq 0$

$$I_{Ev,t}^{\ell+1}(i, j) = 1 - \epsilon \prod_{s=\lceil j/2 \rceil, s\neq i}^{t(1/R-1)} (1 - I_{Av,t}^{\ell+1}(s, j))^{B^r(s,j)} (1 - I_{Av,t}^{\ell+1}(i, j))^{B^r(i,j)-1} \tag{10}$$

- **A-posteriori decoding erasure probabilities**:

$$P_{APP,t}(j) = 1 - I_{\text{APP},t}(j) = \epsilon \prod_{s=\lceil j/2 \rceil}^{t} (1 - I_{Av}^{\infty}(s, j))^{B^r(s,j)} \tag{11}$$

The P-EXIT update equations described above have most of the product indices adopted to match with rate-1/2 codes. For the lower rate codes, there are however many more entries $B^r(i, j)$ equal to 0. Instead of further modifying the indices in the products above in order to take into account the structure of the lower rate codes, it suffices to do this during the fixed point discussion. Due to the choice of $\boldsymbol{B}_0^r$ and $\boldsymbol{B}_i^r$, by analyzing the above equations for different indices (i, j), we can see that the mutual information $I_{Ev,t}^{\ell}(i, j)$ and $I_{Av,t}^{\ell}(i, j)$ are the same for all $(r-1)$ check nodes i introduced together in the same time step. That is,

$$I_{Ev,t}^{\ell}(i, j) = I_{Ev,t}^{\ell}(i', j) \tag{12}$$

$$I_{Av,t}^{\ell}(i, j) = I_{Av,t}^{\ell}(i', j) \tag{13}$$

if $(i, i') \in [(u_p - 1)(r - 1) + 1, u_p(r - 1)]$ for some integer $u_p \in [1, t]$. This is because, as for the original structure, in the lower rate protograph structure all variable nodes corresponding to parity check bits are degree-1 nodes. But for all degree-1 nodes in the protograph we have $I_{Ev,t}^{\ell}(i, j) = (1 - \epsilon)$ for all ℓ, t. We can therefore for the fixed point discussion focus on the relation between the mutual information corresponding to different information blocks, that is variable nodes with indices $j = (c_q - 1)r + 1$, where $c_q = 1, \dots, t$. Denote by $I_{Ev,t}^{\ell}(u_p, c_q)$ the value of the mutual information entry equal to the entries $I_{Ev,t}^{\ell}(i, j)$ in $\boldsymbol{I}_{Ev,t}$ with $i \in [(u_p - 1)(r - 1) + 1, \dots, u_p(r - 1)]$ and $j = (c_q - 1)r + 1$. Denote $I_{Av,t}^{\ell}(u_p, c_q)$ similarly. We can then in the same way as in [10] first establish a number of inequalities and approximations and then evaluate the steady-state behavior. The approach is very closely related to the one in [10] and details are therefore largely omitted. It can be verified by induction that the following inequalities hold for the lower-rate code structure:

- $I_{Av,t}^{\ell}(u_p + 1, c_q) \leq I_{Av,t}^{\ell}(u_p, c_q)$
- $I_{Ev,t}^{\ell}(u_p + 1, c_q) \geq I_{Ev,t}^{\ell}(u_p, c_q)$

- $I^{\ell}_{Ev,t}(u_p, c_q) \geq I^{\ell}_{Ev,t}(u_p, c_q + j')$
- $I^{\ell}_{Av,t}(u_p, c_q) \leq I^{\ell}_{Av,t}(u_p, c_q + j')$

Due to the degree-1 nodes, we have $I^{\infty}_{Av,t}(i, j) \leq (1 - \epsilon)$ for all (i, j) and therefore as well $I^{\infty}_{Av,t}(u_p, c_q) \leq (1 - \epsilon)$ for all (u_p, c_q). For the steady-state behavior we can now apply the same limits as in [10], namely $\lim_{t\to\infty} I^{\infty}_{Av,t}(u_p, c_q) = (1-\epsilon)$. The decoding erasure probability in steady state is given as

$$P_{APP,t}(c_q) = \epsilon \prod_{(c_q-1)(r-1)+1}^{t(r-1)} (1 - I^{\infty}_{Av,t}(s, c_q))^{B^r_t(s,c_q)} \quad (14)$$

$$= \epsilon \prod_{(c_q-1)(r-1)+1}^{t(r-1)} (1 - (1 - \epsilon))^{B^r_t(s,c_q)} \quad (15)$$

Since $B_t(s, c_q) = 1$ for $s = (c_q - 1)(r - 1) + 1, \ldots, t(r - 1)$, we have

$$P_{APP,t}(c_q) = \epsilon^{(t-c_q+1)(r-1)}. \quad (16)$$

It follows directly that

$$P_{APP,t}(c_q + 1) = \epsilon^{(t-c_q+2)(r-1)} = P_{APP,t}(c_q)\epsilon^{r-1} \quad (17)$$

$$P_{APP,t+1}(c_q) = \epsilon^{(t+1-c_q+1)(r-1)} = P_{APP,t}(c_q)\epsilon^{r-1}. \quad (18)$$

The operational anytime exponent is therefore given as

$$\alpha_o = (r - 1)\log(\epsilon) = (1/R - 1)\log(\epsilon). \quad (19)$$

We can see that by decreasing the rate we can achieve a larger operational anytime exponent and therefore a steeper decay of the decoding erasure probability over delay.

B. Numerical Examples

To support the derivation of the anytime exponent, we compare the slopes of the asymptotic performance for different erasure rates obtained by P-EXIT analysis with the slopes determined by the corresponding anytime exponents. Fig. 3 shows the asymptotic performance as a function of the delay for the first information block. We can verify that the slopes of the decoding erasure probability obtained with P-EXIT analysis correspond perfectly with the theoretically determined anytime exponents $\alpha_{o,R=1/2} = -\log(\epsilon), \alpha_{o,R=1/3} = 2\log(\epsilon)$ and $\alpha_{o,R=1/4} = 3\log(\epsilon)$.

C. Finite-Length Analysis

In [10] a combinatorial approach is proposed to determine the finite-length behavior of the rate-1/2 codes. We now generalize the approach to codes with a structure given in (7) with rates $R = 1/r$. The analysis assumes transmission over a so called *static* erasure channel. The static erasure channel, with erasure rate ϵ_s is a channel, where the number of erased bits E per block of k bits is *exactly* equal to $E = \epsilon_s k$, where ϵ_s and k are chosen such that E is an integer. With the finite-length analysis it is then possible to calculate the probability $P(t)$ that an increasing erasure pattern exists involving tE erasures in information bits at time t. In other words, $P(t)$ is the worst-case probability that *none* of the erasures introduced by the channel into the systematic part of the entire received codeword stream up to time t is resolved by the decoder. A related probability is $P(t|t-1)$, which is the probability that the erasure pattern increases from containing $(t-1)E$ erasures at time $(t-1)$ to containing tE erasures at time t. The relation between $P(t)$ and $P(t|t-1)$ is then given as

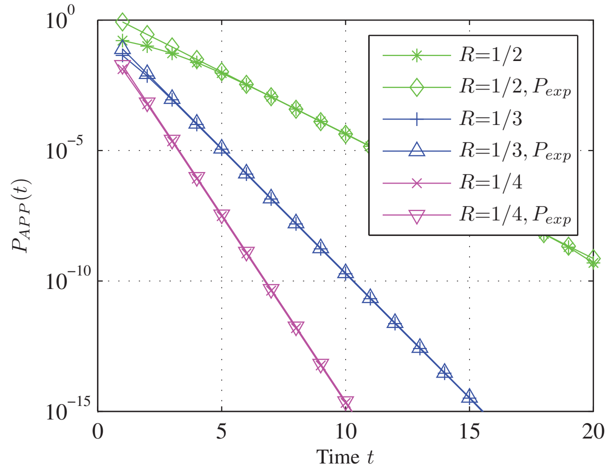

Fig. 3: P-EXIT analysis and corresponding curve P_{exp} decaying with the calculated anytime exponent for different rates R, when $\epsilon = 1/3$.

$$P(t) = P(t|t-1)P(t-1). \quad (20)$$

By noticing that for a code of rate R, in every time step a set of $k(1/R - 1)$ check equations is added, we can see that the approximation of $P(t|t-1)$ given in [10] is generalized to

$$P(t|t-1) \approx \left(1 - t\epsilon_s (1 - \epsilon_s)^{t-1}\right)^{(k-E)(1/R-1)}. \quad (21)$$

Similarly,

$$P(t = 1) = \left(\frac{(\epsilon_s k)!(k(1-\epsilon_s))!}{k!}\right)^{1/R-1}. \quad (22)$$

The probability $P(t)$ can then be found recursively using (20):

$$P(t) = P(t = 1)\prod_{i=2}^{t} P(i|i-1) \quad (23)$$

$$\approx P(t = 1)\left[\prod_{i=2}^{t}\left(1 - i\epsilon_s (1 - \epsilon_s)^{i-1}\right)\right]^{k(1-\epsilon_s)(\frac{1}{R}-1)}. \quad (24)$$

We can verify the accuracy of the approximation through simulations. Fig. 4 shows the simulated and calculated probability $P(t)$ when transmitting over a static BEC with $\epsilon_s = 1/3$ for codes of different rates R. A message length of $k = 3$ bits is employed.

D. Finite-Length Behavior

We now investigate the finite-length behavior of the modified codes by simulating their decoding erasure probability P_e over time when transmitting over an erasure channel with erasure rate $\epsilon = 0.5$. Fig. 5 shows the performance for different rates R when transmitting with a message length of $k = 10$. Here we can see the decoding erasure probability

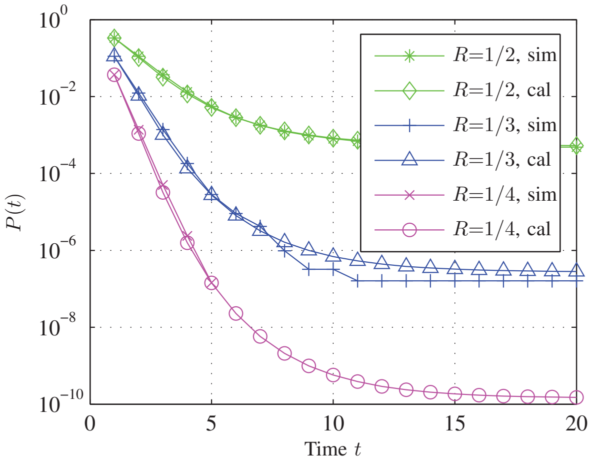

Fig. 4: Simulated and calculated probability $P(t)$ for different values of the rate R. Here $k = 3$ and $\epsilon_s = 1/3$.

$P_e^i(t)$ corresponding to message $i = 45$. As we can see the probability $P_e^i(t)$ corresponding to the code with the lower rate decays both faster over time and reaches a lower erasure floor. This makes it possible to transmit over erasure channels with high erasure rates at the expense of a more complex decoder.

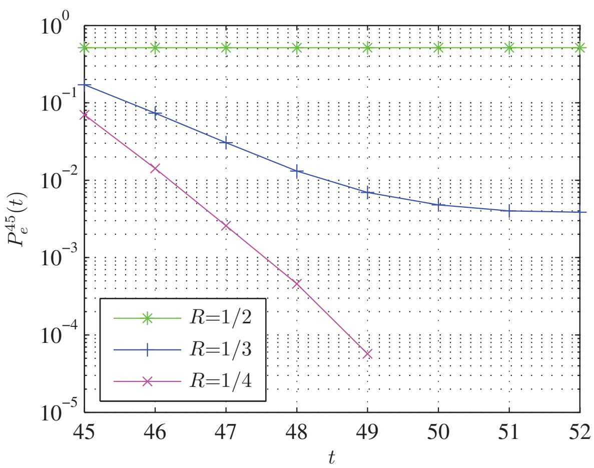

Fig. 5: Behavior $P_e(t)$ over time for $\epsilon = 0.5, k = 10$. Comparison of codes with different rates.

V. Feedback For Variable Rate Transmission

The fact that the probability $P(t)$ flattens out for low rate codes as well shows that the probability that an increasing erasure pattern of large size exists is not vanishing even though the probability of such an occurrence is very low. Since reducing the rate even further in an attempt to lower this probability is not very practical, we now investigate the possibility to change the rate of the code on-the-fly. By introducing a feedback link between the decoder and the encoder, we allow the decoder to inform the encoder about the current decoding success. The feedback link between the decoder and the encoder is assumed to be error free. The current decoding success is measured at every time step t by counting the number of erasures s_t in all information bits up to time t. An increasing erasure pattern is detected if this number exceeds a certain threshold $s^{th} = s^* E$. Using the result of this detection process we now propose a feedback protocol that allows the encoder to counteract the growth of an increasing error pattern. The action taken by the encoder is then to either lower or increase the rate. The feedback protocol is given in Algorithm 1. The way the encoding/decoding matrix is modified is explained hereafter. Note that only a single bit is transmitted over the feedback link. We now describe the way the parity-check matrix is modified when a rate change is requested. The construction is similar to the lower rate code construction described in Section IV. For improved performance we do however also randomize the code construction.

Algorithm 1 Feedback scheme for the BEC

Decoder side:

1: In every time step t a detector checks the occurrence of an increasing error event.
2: If an increasing erasure pattern is detected, the decoder sends a request to decrease the rate.
3: If no increasing erasure pattern is detected, the decoder sends a request to increase the rate.

Encoder side:

1: The encoding/decoding matrix is modified according to the request of the decoder.
2: If the request is to decrease the rate, the rate changes from $R_t = k/n_t$ to $R_{t+1} = k/(n_t + k)$.
3: If the request is to increase the rate, the rate changes from $R_t = k/n_t$ to $R_{t+1} = \min(0.5, k/(n_t - k))$.
4: Otherwise the rate is kept constant $R_t = R_{t+1}$.

A. Code Construction On-The-Fly

Denote by R_i the rate assigned to the i-th message. The corresponding codeword is then of size $n_i = k/R$. The message length of the variable rate coding scheme is constant equal to k for all messages. Due to the change of the rate the protograph matrix is now time-variant. We choose the matrices to have the following dependencies and structures: The matrix $B_0(t)$ depends on the rate R_t assigned to the current message $\boldsymbol{u}_t$ only. Its structure is equal to B_0^r of the lower rate code given in (7). The size of $B_0(t)$ is however rate dependent and at time t equal to $(1/R_t - 1) \times 1/R_t$. The size of the matrices $\boldsymbol{B}_i(t)$ depends on the rate R_i assigned to message $\boldsymbol{u}_i$ and the rate R_t assigned to the current message $\boldsymbol{u}_t$: Their basic structure is partly random given as

$$\boldsymbol{B}_i(t) = \begin{bmatrix} p_{i,t} & 0 & 0 & \dots \\ p_{i,t} & 0 & 0 & \dots \\ \vdots & \vdots & & \end{bmatrix} \tag{25}$$

where $p_{i,t}$ indicates that the corresponding entry is equal to 1 with probability $p_{i,t}$. The size of $\boldsymbol{B}_i(t)$ depends on R_i and

R_t and is given as $(1/R_i - 1) \times 1/R_t$. We choose to make the probability $p_{i,t}$ above, rate depended by setting

$$p_{i,t} = R_t/m, \tag{26}$$

where $m \geq 1$. With this choice of $p_{i,t}$, in the protograph the variable nodes corresponding to a message $\boldsymbol{u}_i$ encoded with a very high rate are strongly connected to the variable nodes of the current message $\boldsymbol{u}_t$. And the variable nodes corresponding to a message $\boldsymbol{u}_i$ encoded with a low rate are loosely connected to the variable nodes corresponding to the current message $\boldsymbol{u}_t$. The motivation for this choice is that given the described feedback protocol the rate of the code is high only if there are few erasures remaining in the received bitstream. The new message can therefore safely be involved in check equations with many other bits with a low risk of creating an increasing erasure pattern. If the current rate of the code is low, on the other hand, there are many erasures remaining in the received bitstream and in order to lower the probability, that the erasure pattern continues increasing, the new message should only be involved in check equations with few other bits. The parameter m allows for further modifications.

B. Simulations

We now simulate the performance of the codes over time with the feedback strategy given in Algorithm 1 when transmitting over the BEC. Fig. 6 shows the performance of the variable rate anytime code with threshold $s^{th} = 5k\epsilon$ together with the performance obtained when transmitting with a fixed rate anytime code instead. The rate of the fixed rate anytime code is chosen equal to $R = 1/3$ which is approximately equal to the average rate ($\bar{R} = 0.38$) of the variable code when measuring over $T = 100$ messages. In order to make a fair comparison the performance is shown over the number of transmitted bits. For the variable rate anytime code the average number of transmitted bits per time step is chosen. We show the performance of messages $i = 20, 22, 24$. As we can see for an erasure rate of $\epsilon = 0.5$ the fixed rate anytime code does not have anytime properties but quickly looses its error correction capabilities. The variable rate anytime code on the other hand has an exponential decay of the decoding error probability at least down to 10^{-4}. Moreover the decoding error performance is independent of the message index i.

VI. Conclusion

We have generalized the LDPC-CC ensemble proposed in our earlier work to accommodate for transmission at different rates and we have determined the corresponding operational anytime exponent in the asymptotic case of infinite block length and infinite number of iterations. For finite block lengths we have shown by analysis and simulation that the probability of increasing erasure patterns is lowered by decreasing the rate of the code. Finally, we have proposed a feedback protocol that allows the encoder and decoder to operate at variable rate. In the protocol the rate is set on-the-fly depending on the current decoding success and the parity-check matrix is modified such that decoding is more likely to be successful. We have demonstrated the excellent performance of the scheme by simulations.

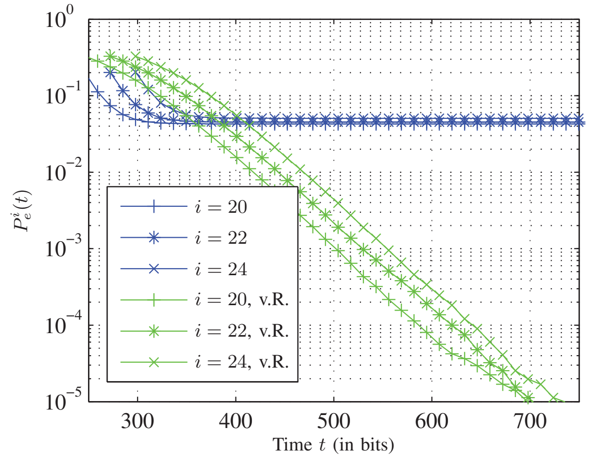

Fig. 6: Decoding erasure probability $P_e^i(t)$ over time for messages $i = 20, 22, 24$ when transmitting over a BEC with $\epsilon = 0.5$ using either a fixed rate code with $R = 1/3$ or a variable rate code with feedback strategy according to Algorithm 1 and $s^* = 5, m = 2$. Message length $k = 5$. The variable rate code has a measured rate of $\bar{R} = 0.38$.

References

[1] A. Sahai, "Why block-length and delay are not the same thing," *arXiv:cs/0610138v1 [cs.IT]*, Oct. 2006.

[2] A. Sahai and S. Mitter, "The necessity and sufficiency of anytime capacity for stabilization of a linear system over a noisy communication link - Part I: Scalar systems," *IEEE Trans. Inf. Theory*, vol. 52, no. 8, pp. 3369–3395, Aug. 2006.

[3] L. J. Schulman, "Coding for interactive communication," *IEEE Trans. Inf. Theory*, vol. 42, no. 6, pp. 1745–1756, Jun. 1996.

[4] R. Ostrovsky, Y. Rabani, and L. Schulman, "Error-correcting codes for automatic control," in *IEEE Symp. Found. Comput. Sci.*, Pittsburgh, PA, Oct. 2005, pp. 309–316.

[5] R. Gelles, A. Moitra, and A. Sahai, "Efficient and explicit coding for interactive communication," in *IEEE Ann. Symp. of Comput. Sci.*, Palm Springs, CA, Oct. 2011, pp. 768–777.

[6] R. Sukhavasi and B. Hassibi, "Anytime reliable codes for stabilizing plants over erasure channels," in *IEEE Conf. Decision Contr.*, Orlando, FL, Dec. 2011, pp. 5254–5259.

[7] G. Como, F. Fagnani, and S. Zampieri, "Anytime reliable transmission of real-valued information through digital noisy channels," *SIAM J. Control and Opt.*, vol. 48, no. 6, pp. 3903–3924, Mar. 2010.

[8] L. Dössel, L. K. Rasmussen, R. Thobaben, and M. Skoglund, "Anytime reliability of systematic LDPC convolutional codes," in *IEEE Int. Conf. Commun.*, Ottawa, ON, Jun. 2012, pp. 2171–2175.

[9] A. Tarable, A. Nordio, F. Dabbene, and R. Tempo, "Anytime reliable LDPC convolutional codes for networked control over wireless channel," in *IEEE Int. Symp. Inf. Theory*, Istanbul, Turkey, Jul. 2013, pp. 2064–2068.

[10] L. Grosjean, L. K. Rasmussen, R. Thobaben, and M. Skoglund, "Systematic LDPC convolutional codes: Asymptotic and finite-length anytime properties," *IEEE Trans, Commun.*, vol. 62, no. 12, pp. 4165–4183, Dec. 2014.

[11] M. Noor-A-Rahim, K. Nguyen, and G. Lechner, "Anytime characteristics of spatially coupled codes," in *Ann. Allerton Conf. Commun., Contr., Comput.*, Allerton, US, 2013.

[12] A. Jimenez Feltström and K. S. Zigangirov, "Time-varying periodic convolutional codes with low-density parity-check matrix," *IEEE Trans. Inf. Theory*, vol. 45, no. 6, pp. 2181–2191, Sep. 1999.

[13] G. Liva and M. Chiani, "Protograph LDPC codes design based on EXIT analysis," in *IEEE Global Telecom. Conf.*, Washington D. C., USA, Nov. 2007, pp. 3250–3254.