Machine Learning

Tran Kha Uyen - 11207413

12th August, 2022

1

a. The marginal distribution p(x) and p(y)

(*) The marginal distribution p(x)

$$
\begin{aligned}
p(X = x_1) &= p(X = x_1, Y = y_1) + p(X = x_1, Y = y_2) + p(X = x_1, Y = y_3) \\
&= 0.01 + 0.05 + 0.1 = 0.16 \\
p(X = x_2) &= p(X = x_2, Y = y_1) + p(X = x_2, Y = y_2) + p(X = x_2, Y = y_3) \\
&= 0.02 + 0.1 + 0.05 = 0.17 \\
p(X = x_3) &= p(X = x_3, Y = y_1) + p(X = x_3, Y = y_2) + p(X = x_3, Y = y_3) \\
&= 0.03 + 0.05 + 0.03 = 0.11 \\
p(X = x_4) &= p(X = x_4, Y = y_1) + p(X = x_4, Y = y_2) + p(X = x_4, Y = y_3) \\
&= 0.1 + 0.07 + 0.05 = 0.22 \\
p(X = x_5) &= p(X = x_5, Y = y_1) + p(X = x_5, Y = y_2) + p(X = x_5, Y = y_3) \\
&= 0.1 + 0.2 + 0.04 = 0.34
\end{aligned}
$$

(*) The marginal distribution p(y)

$$
\begin{aligned}
p(Y = y_1) &= p(Y = y_1, X = x_1) + p(Y = y_1, X = x_2) + p(Y = y_1, X = x_3) + p(Y = y_1, X = x_4) + p(Y = y_1, X = x_5) \\
&= 0.01 + 0.02 + 0.03 + 0.1 + 0.1 = 0.26 \\
p(Y = y_2) &= p(Y = y_2, X = x_1) + p(Y = y_2, X = x_2) + p(Y = y_2, X = x_3) + p(Y = y_2, X = x_4) + p(Y = y_2, X = x_5) \\
&= 0.05 + 0.1 + 0.05 + 0.07 + 0.02 = 0.47 \\
p(Y = y_3) &= p(Y = y_3, X = x_1) + p(Y = y_3, X = x_2) + p(Y = y_3, X = x_3) + p(Y = y_3, X = x_4) + p(Y = y_3, X = x_5) \\
&= 0.1 + 0.05 + 0.03 + 0.05 + 0.04 = 0.27
\end{aligned}
$$

b. The conditional distributions $p(x|Y = y_1)$ and $p(x|Y = y_3)$

(*) The conditional distributions $p(x|Y=y_1)$

$$
\begin{aligned}
p(X=x_1|Y=y_1) &= \frac{p(X=x_1,Y=y_1)}{p(Y=y_1)} = \frac{0.01}{0.26} = \frac{1}{26} \\
p(X=x_2|Y=y_1) &= \frac{p(X=x_2,Y=y_1)}{p(Y=y_1)} = \frac{0.02}{0.26} = \frac{1}{13} \\
p(X=x_3|Y=y_1) &= \frac{p(X=x_3,Y=y_1)}{p(Y=y_1)} = \frac{0.03}{0.26} = \frac{3}{26} \\
p(X=x_4|Y=y_1) &= \frac{p(X=x_4,Y=y_1)}{p(Y=y_1)} = \frac{0.1}{0.26} = \frac{5}{13} \\
p(X=x_5|Y=y_1) &= \frac{p(X=x_5,Y=y_1)}{p(Y=y_1)} = \frac{0.1}{0.26} = \frac{5}{13}
\end{aligned}
$$

(*) The conditional distributions $p(x|Y=y_3)$

$$
\begin{aligned}
p(X=x_1|Y=y_3) &= \frac{p(X=x_1,Y=y_3)}{p(Y=y_3)} = \frac{0.1}{0.27} = \frac{10}{27} \\
p(X=x_2|Y=y_3) &= \frac{p(X=x_2,Y=y_3)}{p(Y=y_3)} = \frac{0.05}{0.27} = \frac{5}{27} \\
p(X=x_3|Y=y_3) &= \frac{p(X=x_3,Y=y_3)}{p(Y=y_3)} = \frac{0.03}{0.27} = \frac{1}{9} \\
p(X=x_4|Y=y_3) &= \frac{p(X=x_4,Y=y_3)}{p(Y=y_3)} = \frac{0.05}{0.27} = \frac{5}{27} \\
p(X=x_5|Y=y_3) &= \frac{p(X=x_5,Y=y31)}{p(Y=y_3)} = \frac{0.04}{0.27} = \frac{4}{27}
\end{aligned}
$$

2

$$
\begin{aligned}
E_Y[E_X[x|y]] &= E_Y[\sum_x x \cdot p(X=x|Y)] = \sum_y [\sum_x x \cdot p(X=x|Y=y)] \cdot p(Y=y) \\
&= \sum_x \sum_y x \cdot p(X=x,Y=y) = \sum_x \sum_y x \cdot P(X=x,Y=y) \\
&= \sum_x x \sum_y p(X=x,Y=y) = \sum_x x \cdot p(X=x) = E_X[X]
\end{aligned}
$$

3

$$
\begin{aligned}
p(X) &= 0.207 \\
p(Y) &= 0.5 \\
p(X|Y) &= 0.365 \\
\Rightarrow p(\bar{X}) &= 1 - 0.207 = 0.793 \\
\Rightarrow p(\bar{X}|Y) &= 1 - 0.365 = 0.635
\end{aligned}
$$

a. Use both X and Y.

$$p(X,Y) = p(X|Y) \cdot P(Y) = 0.365 \cdot 0.5 = 0.1825$$

b. Use Y, given that use X

$$p(Y|\bar{X}) = \frac{p(\bar{X}|Y) \cdot p(Y)}{p(\bar{X})} = \frac{0.635 \cdot 0.5}{0.793} = 0.4004$$

4

According to definition:

$$\begin{aligned}
V_X(X) &= E_X[(X - E_X[X])^2] \\
&= E_X[X^2 - 2XE_X[X] + E[X]^2] \\
&= E_X[X^2] - E[2XE_X[X]] + E[E[X]^2] \\
&= E_X[X^2] - 2E_X[X]E_X[X] + E_X[X]^2 \\
&= E_X[X^2] - 2E_X[X]^2 + E_X[X]^2 = E_X[X^2] - E_X[X]^2
\end{aligned}$$

5

Assume I choose the 1st door. Let X be the event when the car is behind the 1st door and Y be the event when Monty opens the 2nd door.
We have

$$p(X|Y) = \frac{p(Y|X) \cdot p(X)}{p(Y)}$$

where $p(Y|X)$ is the probability that Monty opens the 2nd door given that the car is behind the 1st door.
Assume the car is really behind the 1st door, the probability that Monty opens the 2nd door is $\frac{1}{2}$ because he's already known that the goats is behind both 2nd and 3rd door.Hence, $p(Y|X) = \frac{1}{2}$
Because Monty's already known 2 doors the goats are really behind, he only chooses one of those. So $p(Y) = \frac{1}{2}$

Given that, $p(X) = \frac{1}{3}$

So, the probability that the car is behind the 1st door given that Monty opened the 2nd door is

$$p(X|Y) = \frac{p(Y|X) \cdot p(X)}{p(Y)} = \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{2}} = \frac{1}{3}$$

Then we have the probability that the car is behind the 3rd door given that Monty opened the 2nd door is

$$p(\bar{X}|Y) = 1 - p(X|Y) = 1 - \frac{1}{3} = \frac{2}{3}$$

Therefore, if I have another chance, I will switch to the 3rd door.