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Homework 5

1. $A=\begin{bmatrix}5 & 6\\ 7 & 8\end{bmatrix}$

$\|A\|_1 = \max\{|5|+|7|, |6|+|8|\} = 14$

$\|A\|_\infty = \max\{|5|+|6|, |7|+|8|\} = 15$

$A^{-1} = \frac{1}{5\times8-6\times7}\begin{bmatrix}8 & -6\\ -7 & 5\end{bmatrix} = \begin{bmatrix}-4 & 3\\ 3.5 & -2.5\end{bmatrix}$

$\|A^{-1}\|_1 = \max\{|-4|+|3.5|, |3|+|-2.5|\} = 7.5$

$\|A^{-1}\|_\infty = \max\{|-4|+|3|, |3.5|+|-2.5|\} = 7$

$K_1(A) = \|A\|_1\|A^{-1}\|_1 = 14\times7.5 = 105$

$K_\infty(A) = \|A\|_\infty\|A^{-1}\|_\infty = 15\times7 = 105$

Hence: $\|A\|_1 = 14$, $\|A\|_\infty = 15$, $K_1(A) = 105$, $K_\infty(A) = 105$

2. $A=\begin{pmatrix}1.2969 & 0.8648\\ 0.2161 & 0.1441\end{pmatrix}$, $b=\begin{pmatrix}0.8642\\ 0.1440\end{pmatrix}$, $x=\begin{pmatrix}2\\ -2\end{pmatrix}$, $x_1=\begin{pmatrix}0\\ 1\end{pmatrix}$

$x_2=\begin{pmatrix}0.9911\\ -0.487\end{pmatrix}$

a) If x is the exact solution of $Ax=b$

Then, $\begin{pmatrix}1.2969 & 0.8648\\ 0.2161 & 0.1441\end{pmatrix}\begin{pmatrix}2\\ -2\end{pmatrix}$ need equal to b.

$\Rightarrow \begin{pmatrix}1.2969 & 0.8648\\ 0.2161 & 0.1441\end{pmatrix}\begin{pmatrix}2\\ -2\end{pmatrix} - \begin{pmatrix}0.8642\\ 0.1440\end{pmatrix}$

$= \begin{pmatrix}0\\ 0\end{pmatrix}$

Hence, x is the exact solution of $Ax=b$.

b) $e = x - \tilde{x}$, $r = b - A\tilde{x}$

$$e_1 = x - x_1 = \begin{pmatrix} 2 \\ -2 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$e_2 = x - x_2 = \begin{pmatrix} 2 \\ -2 \end{pmatrix} - \begin{pmatrix} 0.9911 \\ -0.487 \end{pmatrix} = \begin{pmatrix} 1.0089 \\ -1.513 \end{pmatrix}$$

$$\tilde{r}_1 = b - Ax_1 = \begin{pmatrix} 0.8642 \\ 0.144 \end{pmatrix} - \begin{pmatrix} 1.2969 & 0.8648 \\ 0.2161 & 0.1441 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -0.0006 \\ -0.0001 \end{pmatrix}$$

$$\tilde{r}_2 = b - Ax_2 = \begin{pmatrix} 0.8642 \\ 0.144 \end{pmatrix} - \begin{pmatrix} 1.2969 & 0.8648 \\ 0.2161 & 0.1441 \end{pmatrix} \begin{pmatrix} 0.9911 \\ -0.487 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \times 10^{-8} \\ -1 \times 10^{-8} \end{pmatrix}$$

c) $$A^{-1} = \frac{1}{1.2969 \times 0.1441 - 0.8648 \times 0.2161} \begin{pmatrix} 0.1441 & -0.8648 \\ -0.2161 & 1.2969 \end{pmatrix}$$

$$= \begin{pmatrix} 14410000 & -86480000 \\ -21610000 & 12969000 \end{pmatrix}$$

$$\|A\|_\infty = \max\{|1.2969| + |0.8648|, |0.2161| + |0.1441|\} = 2.1617$$

$$\|A^{-1}\|_\infty = \max\{|14410000| + |-86480000|, |-21610000| + |12969000|\}$$

$$= \max\{100890000, 15130000\}$$

$$= 151300000$$

$$K_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty = 2.1617 \times 151300000 = 327065210$$

d) $\frac{\|e\|}{\|x\|} \le K(A)\frac{\|r\|}{\|b\|}$

$K_\infty(A)\frac{\|r_1\|_\infty}{\|b\|_\infty} = 32706520 \frac{\max\{|-0.0006|, |-0.0001|\}}{\max\{|0.8642|, |0.144|\}}$

$= 227076.0542$

$K_\infty(A)\frac{\|r_2\|_\infty}{\|b\|_\infty} = 32706520 \frac{\max\{|1\times10^{-8}|, |-1\times10^{-8}|\}}{\max\{|0.8642|, |0.144|\}}$

$= 3.78246096$

$\frac{\|e_1\|_\infty}{\|x_1\|_\infty} = \frac{\max\{|2|, |-3|\}}{\max\{|0|, |1|\}} = 3 \le K_\infty(A)\frac{\|r_1\|_\infty}{\|b\|_\infty} = 227076.0542$

$\frac{\|e_2\|_\infty}{\|x_2\|_\infty} = \frac{\max\{|1.0089|, |-1.513|\}}{\max\{|0.9911|, |-0.487|\}} = 1.52658662$

$\le K_\infty(A)\frac{\|r_2\|_\infty}{\|b\|_\infty} = 3.78246096$

Hence, This result holds for the approximate solutions x_1, x_2.

3. $A = \begin{bmatrix} 2 & 7 & 5 \\ 6 & 20 & 10 \\ 4 & 3 & 0 \end{bmatrix}$

a) $L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 11 & 1 \end{bmatrix}$, $U_1 = \begin{bmatrix} 2 & 7 & 5 \\ 0 & -1 & -5 \\ 0 & 0 & 45 \end{bmatrix}$

$L_1U_1 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 11 & 1 \end{bmatrix}\begin{bmatrix} 2 & 7 & 5 \\ 0 & -1 & -5 \\ 0 & 0 & 45 \end{bmatrix} = \begin{bmatrix} 2 & 7 & 5 \\ 6 & 20 & 10 \\ 4 & 3 & 0 \end{bmatrix} = A$

Let $y = Ux \Rightarrow Ly = b$

$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 11 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -43 \end{bmatrix}$

$$Ux=y \Rightarrow \begin{bmatrix} 2 & 7 & 5 \\ 0 & -1 & -5 \\ 0 & 0 & 45 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -43 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ 7/9 \\ -43/45 \end{bmatrix}$$

b)
$$L_2 = \begin{bmatrix} 2 & 0 & 0 \\ 6 & -1 & 0 \\ 4 & -11 & 45 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 1 & \frac{7}{2} & \frac{5}{2} \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L_2U_2 = \begin{bmatrix} 2 & 0 & 0 \\ 6 & -1 & 0 \\ 4 & -11 & 45 \end{bmatrix} \begin{bmatrix} 1 & \frac{7}{2} & \frac{5}{2} \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & 5 \\ 6 & 20 & 10 \\ 4 & 3 & 0 \end{bmatrix}$$

Let $y = Ux \Rightarrow Ly = b$

$$\Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 6 & -1 & 0 \\ 4 & -11 & 45 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \\ -\frac{43}{45} \end{bmatrix}$$

$$Ux=y \Rightarrow \begin{bmatrix} 1 & \frac{7}{2} & \frac{5}{2} \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \\ -\frac{43}{45} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ \frac{7}{9} \\ -\frac{43}{45} \end{bmatrix}$$