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MATH 400

Homework 4

1. a)

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$E_1 \equiv I \to R_4 \to R_4 + 2R_2$ $\quad E_2 \equiv I \to R_3 \to 5R_3$

b)

$$[E_1 \mid I] \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[I \mid E_1^{-1}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 & 0 & 1 \end{bmatrix}$$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix}$$

$$[I \mid E_2^{-1}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

c)
$$E_2E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

d) $(E_2E_1)^{-1}$
$$\left[I \,\vdots\, (E_2E_1)^{-1}\right] = \left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 & 0 & 1 \end{array}\right]$$
$$(E_2E_1)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix}$$

2. a) I) The entry that would serve as the first pivot for GE with no pivoting is 0.2115.

II) The entry that would serve as the first pivot for GE with partial pivoting is 6.099 since it is the greatest one in first column.

III) The entry that would serve as the first pivot for GE with partial pivoting is 6.099 since the largest coefficient is 23.20 in R_3.

```
PS C:\Windows\system32> python -u "d:\Math\MATH 400\02_Homework\homework_04\question_2b_I_code.py"
============Intermediate Matrix: 1============

Coefficient Matrix:
[[ 0.2115      2.296       2.715       3.215     ]
 [ 0.         -0.82906667 -3.928      -3.79233333]
 [ 6.099       4.324      23.2         1.578     ]
 [ 4.623       0.8926     15.32        5.305     ]]

Right-hand Side Vector:
[[ 8.438     ]
 [-8.55053333]
 [35.2       ]
 [26.14      ]]

============Intermediate Matrix: 2============

Coefficient Matrix:
[[  0.2115       2.296        2.715        3.215     ]
 [  0.          -0.82906667  -3.928       -3.79233333]
 [  0.         -61.88547518 -55.09212766 -91.13256738]
 [  4.623        0.8926      15.32         5.305     ]]

Right-hand Side Vector:
[[   8.438     ]
 [  -8.55053333]
 [-208.12558865]
 [  26.14      ]]

============Intermediate Matrix: 3============

Coefficient Matrix:
[[  0.2115       2.296        2.715        3.215     ]
 [  0.          -0.82906667  -3.928       -3.79233333]
 [  0.         -61.88547518 -55.09212766 -91.13256738]
 [  0.         -49.29372624 -44.02489362 -64.96897163]]

Right-hand Side Vector:
[[   8.438     ]
 [  -8.55053333]
 [-208.12558865]
 [-158.29912057]]

============Intermediate Matrix: 4============

Coefficient Matrix:
[[ 2.11500000e-01  2.29600000e+00  2.71500000e+00  3.21500000e+00]
 [ 0.00000000e+00 -8.29066667e-01 -3.92800000e+00 -3.79233333e+00]
 [ 0.00000000e+00  0.00000000e+00  2.38112456e+02  1.91945211e+02]
 [ 0.00000000e+00 -4.92937262e+01 -4.40248936e+01 -6.49689716e+01]]

Right-hand Side Vector:
[[   8.438     ]
 [  -8.55053333]
 [ 430.12684584]
 [-158.29912057]]

============Intermediate Matrix: 5============

Coefficient Matrix:
[[ 2.11500000e-01  2.29600000e+00  2.71500000e+00  3.21500000e+00]
 [ 0.00000000e+00 -8.29066667e-01 -3.92800000e+00 -3.79233333e+00]
 [ 0.00000000e+00  0.00000000e+00  2.38112456e+02  1.91945211e+02]
 [ 0.00000000e+00  0.00000000e+00  1.89521773e+02  1.60511377e+02]]

Right-hand Side Vector:
[[  8.438     ]
 [ -8.55053333]
 [430.12684584]
 [350.08900586]]

============Intermediate Matrix: 6============

Coefficient Matrix:
[[ 2.11500000e-01  2.29600000e+00  2.71500000e+00  3.21500000e+00]
 [ 0.00000000e+00 -8.29066667e-01 -3.92800000e+00 -3.79233333e+00]
 [ 0.00000000e+00  0.00000000e+00  2.38112456e+02  1.91945211e+02]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  7.73567822e+00]]

Right-hand Side Vector:
[[  8.438     ]
 [ -8.55053333]
 [430.12684584]
 [  7.73647175]]

Result:
[0.99908054 0.99991306 1.00020784 1.00010258]
```

```
PS C:\Windows\system32> python -u "d:\Math\MATH 400\02_Homework\homework_04\question_2b_II_code.py"
============Intermediate Matrix: 1============

Coefficient Matrix:
[[6.09900000e+00 4.32400000e+00 2.32000000e+01 1.57800000e+00]
 [0.00000000e+00 3.60610979e+00 2.03143138e-02 2.73890871e+00]
 [2.11500000e-01 2.29600000e+00 2.71500000e+00 3.21500000e+00]
 [4.62300000e+00 8.92600000e-01 1.53200000e+01 5.30500000e+00]]

Right-hand Side Vector:
[[35.2       ]
 [ 6.36530448]
 [ 8.438     ]
 [26.14      ]]

============Intermediate Matrix: 2============

Coefficient Matrix:
[[6.09900000e+00 4.32400000e+00 2.32000000e+01 1.57800000e+00]
 [0.00000000e+00 3.60610979e+00 2.03143138e-02 2.73890871e+00]
 [0.00000000e+00 2.14605312e+00 1.91047467e+00 3.16027841e+00]
 [4.62300000e+00 8.92600000e-01 1.53200000e+01 5.30500000e+00]]

Right-hand Side Vector:
[[35.2       ]
 [ 6.36530448]
 [ 7.21734088]
 [26.14      ]]

============Intermediate Matrix: 3============

Coefficient Matrix:
[[ 6.09900000e+00  4.32400000e+00  2.32000000e+01  1.57800000e+00]
 [ 0.00000000e+00  3.60610979e+00  2.03143138e-02  2.73890871e+00]
 [ 0.00000000e+00  2.14605312e+00  1.91047467e+00  3.16027841e+00]
 [ 0.00000000e+00 -2.38496222e+00 -2.26544024e+00  4.10888687e+00]]

Right-hand Side Vector:
[[35.2       ]
 [ 6.36530448]
 [ 7.21734088]
 [-0.5413576 ]]

============Intermediate Matrix: 4============

Coefficient Matrix:
[[ 6.09900000e+00  4.32400000e+00  2.32000000e+01  1.57800000e+00]
 [ 0.00000000e+00  3.60610979e+00  2.03143138e-02  2.73890871e+00]
 [ 0.00000000e+00  0.00000000e+00  1.89838530e+00  1.53031040e+00]
 [ 0.00000000e+00 -2.38496222e+00 -2.26544024e+00  4.10888687e+00]]

Right-hand Side Vector:
[[35.2       ]
 [ 6.36530448]
 [ 3.42924724]
 [-0.5413576 ]]

============Intermediate Matrix: 5============

Coefficient Matrix:
[[ 6.09900000e+00  4.32400000e+00  2.32000000e+01  1.57800000e+00]
 [ 0.00000000e+00  3.60610979e+00  2.03143138e-02  2.73890871e+00]
 [ 0.00000000e+00  0.00000000e+00  1.89838530e+00  1.53031040e+00]
 [ 0.00000000e+00  0.00000000e+00 -2.25200502e+00  5.92031086e+00]]

Right-hand Side Vector:
[[35.2       ]
 [ 6.36530448]
 [ 3.42924724]
 [ 3.6684451 ]]

============Intermediate Matrix: 6============

Coefficient Matrix:
[[ 6.09900000e+00  4.32400000e+00  2.32000000e+01  1.57800000e+00]
 [ 0.00000000e+00  3.60610979e+00  2.03143138e-02  2.73890871e+00]
 [ 0.00000000e+00  0.00000000e+00 -2.25200502e+00  5.92031086e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  6.52098804e+00]]

Right-hand Side Vector:
[[35.2       ]
 [ 6.36530448]
 [ 3.6684451 ]
 [ 6.52165697]]

Result:
[0.99908054 0.99991306 1.00020784 1.00010258]
```

b) Answer is in "Question_2b" folder.

3. $A = LU$

$m_{21} = -\frac{1}{4}$, $m_{31} = -\frac{1}{4}$

$$\begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix} \xrightarrow[R_3 + \frac{1}{4}R_1]{R_2 + \frac{1}{4}R_1} \begin{bmatrix} 4 & -1 & -1 \\ 0 & \frac{15}{4} & -\frac{5}{4} \\ 0 & -\frac{5}{4} & \frac{15}{4} \end{bmatrix}$$

$m_{32} = -\frac{5}{4} / \frac{15}{4} = -\frac{5}{4} \cdot \frac{4}{15} = -\frac{1}{3}$

$$\xrightarrow{R_3 + \frac{1}{3}R_2} \begin{bmatrix} 4 & -1 & -1 \\ 0 & \frac{15}{4} & -\frac{5}{4} \\ 0 & 0 & \frac{10}{3} \end{bmatrix}$$

$$L = L_1^{-1} L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{21} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ -\frac{1}{4} & -\frac{1}{3} & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 4 & -1 & -1 \\ 0 & 15/4 & -5/4 \\ 0 & 0 & 10/3 \end{bmatrix}$$

4. Let $Ux = y$, then $Ly = Pb$

Forward Substitution:

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/3 & 1/4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 10 \\ -12 \end{bmatrix}$$

$$\rightarrow \begin{cases} 1y_1 = -12 \\ \frac{1}{2}y_1 + y_2 = 2 \\ \frac{1}{3}y_1 + \frac{1}{4}y_2 + y_3 = 10 \end{cases} \Rightarrow \begin{cases} y_1 = -12 \\ y_2 = 8 \\ y_3 = 12 \end{cases}$$

Back-Substitution: $Ux = y$

$$\Rightarrow \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -12 \\ 8 \\ 12 \end{bmatrix} \Rightarrow \begin{cases} 2x_1 + 3x_2 + x_3 = -12 \\ x_2 + 2x_3 = 8 \\ 2x_3 = 12 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = -3 \\ x_2 = -4 \\ x_3 = 6 \end{cases}$$