Shixin Wang
MATH 400-01
2/15/2022

Homework 02

1. $f(x) = e^{1-x^2}$, $x_0 = 1$, 4P

Sol$^{\underline{n}}$: Let $a = x_0 = 1$

$$f(x) = e^{1-x^2}$$

$$f'(x) = -2e^{1-x^2}x$$

$$f''(x) = (-2)(e^{1-x^2} - 2x^2e^{1-x^2})$$

$$f'''(x) = (-2)(4x^3e^{1-x^2} - 6xe^{1-x^2})$$

$$f^{(4)}(x) = (-2)(-8x^4e^{1-x^2} + 24x^2e^{1-x^2} - 6e^{1-x^2})$$

Hence: $P_0(x) = f(a) = 1$

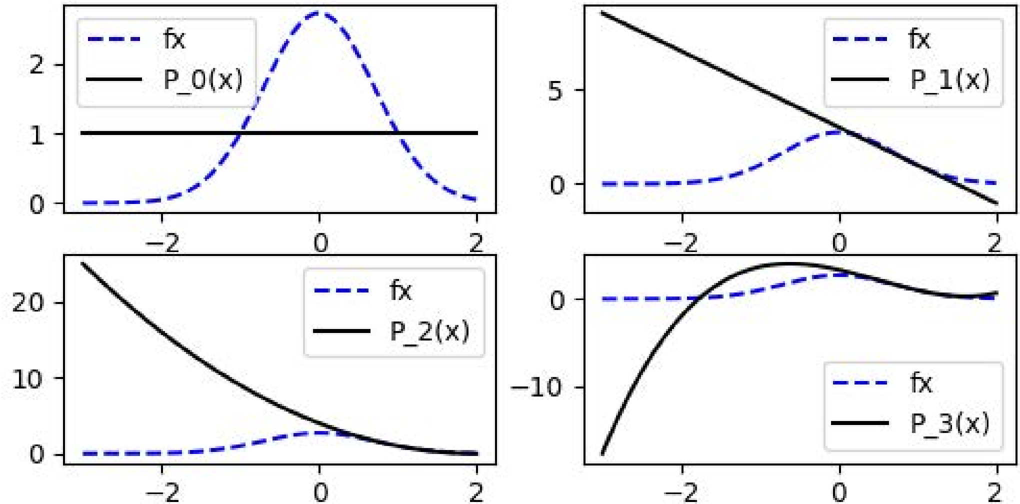
$$P_1(x) = P_0(x) - 2(x-1)$$

$$P_2(x) = P_1(x) + (x-1)^2$$

$$P_3(x) = P_2(x) + \frac{2}{3}(x-1)^3$$

$$P_4(x) = P_3(x) + (x-1)^4$$

Graph: Graph is in "shixin_[illegible]_homework_question_01.pdf"


fx
P_0(x)
fx
P_1(x)
fx
P_2(x)
fx
P_3(x)

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2. a) $\tan^{-1}(x)$, $x=0$

$\Rightarrow f(x)=\tan^{-1}(x)$

$f'(x)=\dfrac{1}{x^2+1}$, $f''(x)=-\dfrac{2x}{(x^2+1)^2}$

$f'''(x)=-\dfrac{2(-3x^2+1)}{(x^2+1)^3}$

$f^{(4)}(x)=\dfrac{24x(-x^2+1)}{(x^2+1)^4}$

$f^{(5)}(x)=\dfrac{24(5x^4-10x^2+1)}{(x^2+1)^5}$

Maclaurin Series:

$$P(x)=f(0)+f'(0)x+f''(0)\cdot\frac{1}{2}x^2+f'''(0)\frac{1}{3!}\cdot x^3+\cdots+f^{(n)}(0)\frac{x^n}{n!}$$

$\Rightarrow f(0)=0$, $f'(0)=1$, $f''(0)=0$, $f'''(0)=-2$
$f^{(4)}(0)=0$, $f^{(5)}(0)=24$

$$\Rightarrow P(x)=0+x+0+\frac{x^3}{3}+0+\frac{x^5}{5}+\cdots$$

~~$\therefore \tan^{-1}(x)=\sum_{n \text{ is odd}}^{\infty}\frac{x^n}{n}$~~

$$\therefore \tan^{-1}(x)=\sum_{n/2}^{\infty}\frac{x^{n-1}}{n-1}$$

2) $\because$ Centered at $x=0$, arctan x, same as (a)

$$\therefore P_n(x)=P_{n-1}(x)+\frac{x^{n-1}}{n-1}$$

c)

$$\pi \approx T_n = 16 P_n\left(\frac{1}{5}\right) - 4P_n\left(\frac{1}{239}\right)$$

From Question b):

$$P_n(x) = P_{n-1}(x) + \frac{x^{n-1}}{n-1}$$

$$\Rightarrow P_n\left(\frac{1}{5}\right) = P_{n-1}\left(\frac{1}{5}\right) - \frac{5}{4x^{4/5}}$$

$$P_n\left(\frac{1}{239}\right) = P_{n-1}\left(\frac{1}{239}\right) - \frac{239}{238x^{238/239}}$$

$$\Rightarrow T_n = 16\left(P_{n-1}\left(\frac{1}{5}\right) - \frac{5}{4x^{4/5}}\right) - 4\left(P_{n-1}\left(\frac{1}{239}\right) - \frac{239}{238x^{238/239}}\right)$$

Absolute Error:

$$T_1 = 16\left(P_0\left(\frac{1}{5}\right) - \frac{5}{4x^{4/5}}\right) - 4\left(P_0\left(\frac{1}{239}\right) - \frac{239}{238x^{238/239}}\right)$$

$$T_1 = 16\left(-5 - \frac{5}{4x^{4/5}}\right) - 4\left(-239 - \frac{239}{238x^{238/239}}\right)$$

$$\left|T_n - \pi\right|_{n=1} = \left|\left[876 - \frac{5}{4x^{4/5}} + \frac{239}{238x^{238/239}}\right] - \pi\right|$$

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3. $x_0 = 4$, $f(x_0) = 1$, $x_1 = 3$

Sol^{n}: According to Newton's Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

We can know:

$$f'(x) = \frac{0 - f(x_n)}{x_{n+1} - x_n}$$

Hence:

$$f'(x_0) = \frac{0 - f(x_0)}{x_1 - x_0} = \frac{0-1}{3-4} = 1$$

$\therefore$ The derivative of f at x_0 is 1.

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4. $x^p = N$, $x_0 = 2$, 10^{-6}

Sol$^{\underline{n}}$: $x^p = N$

$\Rightarrow x^p - N = 0$

$\Rightarrow f(x) = x^p - N$

Hence: $f'(x) = p x^{p-1}$

Apply Newton's method:

$$\Rightarrow x_{n+1} = x_n - \frac{x_n^p - N}{p x_n^{p-1}}$$

Apply them ~~into~~ code "question_04.py", we get 5 iterations.

$\therefore$ Therefore, ~~we~~ I need 5 steps.

```
PS D:\Math\MATH 400> python -u "d:\Math\MATH 400\02_Homework\homework_02\question_04.py"
Number of iterations =  5
An estimate of the root is  1.4953487812212707
PS D:\Math\MATH 400> 
```

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5. $f(x) = x^4 - 5$, $[1,2]$, 10^{-6}

Apply Secant Method:

$$x_{n+1} = x_n - f(x_n)\frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

We have Python code "question_05.py".

Let $f = x^4 - 5$,
$x0 = 1$,
$x1 = 2$,
$Delta = 10^{-6}$
$Nmax = 100$

$\Rightarrow$ we got the root $= 1.4953487812075685$ with tolerance 10^{-6}.

The program is in "question_05.py"
It will output both number of iterations and the estimate of the root with tolerance 10^{-6}.

```
PS D:\Math\MATH 400> python -u "d:\Math\MATH 400\02_Homework\homework_02\question_05.py"
Number of iterations =  7
An estimate of the root is  1.4953487812075685
```