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MATH 400
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Homework 6

1.

a) Code is in “question_01a_code.py”

```
PS D:\Math\MATH 400> python -u "d:\Math\MATH 400\02_Homework\homework_06\question_01a_code.py"
              h                        error                        error/h                                    error/h^2

5.00000000e-01          2.02221084e-01          4.04442167e-01                8.08884334e-01

2.50000000e-01          9.52716617e-02          3.81086647e-01                1.52434659e+00

1.25000000e-01          4.59766451e-02          3.67813161e-01                2.94250529e+00

6.25000000e-02          2.25501609e-02          3.60802574e-01                5.77284118e+00

3.12500000e-02          1.11627277e-02          3.57207287e-01                1.14306332e+01

1.56250000e-02          5.55293123e-03          3.55387599e-01                2.27448063e+01
```

b) Code is in “question_01b_code.py”

```
PS D:\Math\MATH 400> python -u "d:\Math\MATH 400\02_Homework\homework_06\question_01b_code.py"
              h                        error                        error/h                                    error/h^2

5.00000000e-01          2.90966823e-02          5.81933647e-02                1.16386729e-01

2.50000000e-01          7.34271206e-03          2.93708482e-02                1.17483393e-01

1.25000000e-01          1.83998583e-03          1.47198867e-02                1.17759093e-01

6.25000000e-02          4.60266072e-04          7.36425716e-03                1.17828115e-01

3.12500000e-02          1.15083375e-04          3.68266800e-03                1.17845376e-01

1.56250000e-02          2.87718974e-05          1.84140143e-03                1.17849692e-01
```

I think the approximation of b is more accurate since it has smaller error number.

2. $D_+f(x) = \dfrac{f(x+h)-f(x)}{h}$, $D_-f(x) = \dfrac{f(x)-f(x-h)}{h}$

a) $D_+D_-f(x) = D_+\left(\dfrac{f(x)-f(x-h)}{h}\right)$

$$= \frac{\frac{f(x+h)-f(x+h-h)}{h} - \frac{f(x)-f(x-h)}{h}}{h}$$

$$= \frac{f(x+h)-f(x)-f(x)+f(x-h)}{h^2}$$

$$= \frac{f(x+h)-2f(x)+f(x-h)}{h^2}$$

b) $D_+D_-f(x) = \dfrac{f(x+h)-2f(x)+f(x-h)}{h^2}$

$$= \frac{1}{h^2}\Big[\big(f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\cdots\big)-2f(x)$$

$$+\big(f(x)-hf'(x)+\frac{h^2}{2!}f''(x)-\cdots\cdots\big)\Big]$$

$$= f''(x)+\frac{2}{h^2}\left(\frac{h^4}{4!}f^{(4)}(x)+\frac{h^6}{6!}f^{(6)}(x)+\cdots\right)$$

$$= f''(x)+O(h^2)$$

3. a) $f_{xy}(x,y) = D_0^y D_0^x f(x,y)$

$$= D_0^y\left(\frac{f(x+h,y)-f(x-h,y)}{2h}\right)$$

$$= \frac{1}{2h}\left(D_0^y f(x+h,y) - D_0^y f(x-h,y)\right)$$

$$= \frac{1}{2h}\left[\frac{f(x+h,y+h)-f(x-h,y+h)}{2h} - \left(\frac{f(x+h,y-h)-f(x-h,y-h)}{2h}\right)\right]$$

$$= \frac{f(x+h,y+h)-f(x-h,y+h)-f(x+h,y-h)+f(x-h,y-h)}{4h^2}$$

b)

4. $2x_1 + x_2 = 1$, $x_1 + x_2 = -1$

5. $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $e^{(k)} = \left(\frac{1}{4}\right)^k \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$D = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $D^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, $L+U = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$B_J = -D^{-1}(L+U) = -\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$