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MATH 400-01
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Homework 02

2. a) $\tan^{-1}(x)$, $x=0$

$\Rightarrow f(x)=\tan^{-1}(x)$

$f'(x)=\frac{1}{x^2+1}$, $f''(x)=-\frac{2x}{(x^2+1)^2}$

$f'''(x)=-\frac{2(-3x^2+1)}{(x^2+1)^3}$

$f^{(4)}(x)=\frac{24x(-x^2+1)}{(x^2+1)^4}$

$f^{(5)}(x)=\frac{24(5x^4-10x^2+1)}{(x^2+1)^5}$

Maclaurin Series:

$P(x)=f(0)+f'(0)x+f''(0)\cdot\frac{1}{2}x^2+f'''(0)\frac{1}{3!}\cdot x^3$

$+\cdots+f^{(n)}(0)\frac{x^n}{n!}$

$\Rightarrow f(0)=0$, $f'(0)=1$, $f''(0)=0$, $f'''(0)=-2$

$f^{(4)}(0)=0$, $f^{(5)}(0)=24$

$\Rightarrow P(x)=0+x+0+\frac{x^3}{3}+0+\frac{x^5}{5}+\cdots$

~~$\therefore \tan^{-1}(x)=\sum_{n \text{ is odd}}^{\infty}\frac{x^n}{n}$~~

$\therefore \tan^{-1}(x)=\sum_{n/2}^{\infty}\frac{x^{n-1}}{n-1}$

2) $\because$ Centered at $x=0$, $\arctan x$, same as (a)

$\therefore P_n(x)=P_{n-1}(x)+\frac{x^{n-1}}{n-1}$

c)

$$\pi \approx T_n = 16\, P_n\left(\tfrac{1}{5}\right) - 4P_n\left(\tfrac{1}{239}\right)$$

From Question b):

$$P_n(x) = P_{n-1}(x) + \frac{x^{n-1}}{n-1}$$

$$\Rightarrow P_n\left(\tfrac{1}{5}\right) = P_{n-1}\left(\tfrac{1}{5}\right) - \frac{5}{4x^{4/5}}$$

$$P_n\left(\tfrac{1}{239}\right) = P_{n-1}\left(\tfrac{1}{239}\right) - \frac{239}{238x^{238/239}}$$

$$\Rightarrow T_n = 16\left(P_{n-1}\left(\tfrac{1}{5}\right) - \frac{5}{4x^{4/5}}\right) - 4\left(P_{n-1}\left(\tfrac{1}{239}\right) - \frac{239}{238x^{238/239}}\right)$$

Absolute Error:

$$T_1 = 16\left(P_0\left(\tfrac{1}{5}\right) - \frac{5}{4x^{4/5}}\right) - 4\left(P_0\left(\tfrac{1}{239}\right) - \frac{239}{238x^{238/239}}\right)$$

$$T_1 = 16\left(-5 - \frac{5}{4x^{4/5}}\right) - 4\left(-239 - \frac{239}{238x^{238/239}}\right)$$

$$|T_n - \pi|\Big|_{n=1} = \left|\left[876 - \frac{5}{4x^{4/5}} + \frac{239}{238x^{238/239}}\right] - \pi\right|$$