(*) No branch error :

* Lognormal rates : $\mu_i \sim \text{Lognormal}(\theta, \sigma^2)$

$$r_i = \frac{b_i}{\mu T_i} = \frac{\mu_i T_i}{\mu T_i} = \frac{\mu_i}{\mu} \ ; \ r_i \sim \text{Lognormal}(\theta - \ln\mu, \sigma^2)$$

$$\Rightarrow \begin{cases} \log r_i \sim \mathcal{N}(\theta - \ln\mu, \sigma^2) \Rightarrow \text{skewness} = 0 \\ 1/r_i \sim \text{Lognormal}(-\theta + \ln\mu, \sigma^2) \Rightarrow \text{skew} \end{cases}$$

+ Let $\mu = \text{Median}(\mu_i)$, $r_i = \frac{b_i}{\mu T_i} = \frac{\mu_i t_i}{\mu T_i} = \frac{\mu_i}{\mu}$

$\Rightarrow \mu = e^{\theta} \Rightarrow \theta - \ln(\mu) = 0$

Then $r_i \sim \text{Lognormal}(0, \sigma^2)$

$\Rightarrow \log(r_i) \sim \mathcal{N}(0, \sigma^2) \Rightarrow$ Least square estimator

+ Let $\mu = \text{Mode}(\mu_i)$, $r_i = \frac{b_i}{\mu T_i} = \frac{\mu_i}{\mu}$

$\Rightarrow \mu = \exp(\theta - \sigma^2)$

$r_i \sim \text{Lognormal}(\theta - \ln(\mu), \sigma^2)$

$$\text{Mode}(r_i) = \exp(\theta - \underbrace{\ln(\mu)}_{\theta - \sigma^2} - \sigma^2) = 1$$

Proved : $\sum_i \log^2 r_i$ maximizes joint prob.

* Gamma rates : $\mu_i \sim \Gamma(\overset{\text{shape}}{\alpha}, \overset{\text{rate}}{\frac{\alpha}{\mu}})$

$$r_i = \frac{\mu_i}{\mu} \sim \Gamma(\alpha, \alpha)$$

$$\Rightarrow \begin{cases} 1/r_i \sim \Gamma^{-1}(\overset{\text{shape}}{\alpha}, \overset{\text{scale}}{\alpha}) \Rightarrow \text{skewness} = \frac{4\sqrt{\alpha - 2}}{\alpha - 3}, \ \alpha > 3 \\ \log r_i \sim \text{Log-gamma}(\alpha, \alpha) \Rightarrow \text{skewness} = \frac{\text{trigamma}(\alpha)}{(\text{digamma}(\alpha))^{3/2}} \end{cases}$$

⊛ With branch error

* gamma - Poisson mixture

Assume $\begin{cases} \mu_i \sim \Gamma\left(\alpha, \frac{\alpha}{\mu}\right) \Rightarrow s\mu_i\tau_i \sim \Gamma\left(\alpha, \frac{\alpha}{s\mu\tau_i}\right) \\ s\hat{b}_i \sim \text{Poisson}(s\mu_i\tau_i) \end{cases}$

$\Rightarrow s\hat{b}_i \sim NB\left(\frac{s\mu\tau_i}{\alpha}, \frac{\alpha}{\alpha+1}\right)$