FORWARD KINEMATICS: DENAVIT-HARTENBERG CONVENTION ON A 6R ROBOTIC ARM EXAMPLE

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March 15, 2020

1 Introduction

In this paper we develop the forward or configuration kinematic equations for a rigid robot of 6R (6 Revolute Joints) using the Denavit-Hartenberg convention. The forward kinematics problem is concerned with the relationship between the individual joints of a robot manipulator and the position and orientation of the tool or end-effector. Formally, the forward kinematics problem is to determine the position and orientation of the end-effector, given joint variable values of the robot. The joint variables are the angles between the links in the case of revolute or rotational joints, and the link extension displacement in the case of prismatic or sliding joints. The forward kinematics problem is to be contrasted with the inverse kinematics problem, which will be explored at another time, which is concerned with the determination of values for the joint variables that achieve a desired position and orientation for the end-effector of the robot.

2 Kinematics Fundamentals

Intuitively, to be able to determine both the position and orientation of a robot end effector, one must know the parameters such as **link length**, **rotational link offset**, and the two possible joint variables: **link offset**, and **joint angle** represented by symbols d, α, a, and θ respectively. The usage of either joint variable at each link depends on the joint type at each link. This paper deals with a 6R arm, and therefore the joint angle will be variable, and the link distance will be static (there exist no P joints). To perform forward kinematics, it is beneficial to establish orthogonal *x,y,z* coordinate frames at each of the various robot links to keep track of the different parameters as the system configuration changes.

3 Denavit-Hartenberg Convention

It is important to keep in mind that the choices of the various coordinate frames at each link are not unique. Thus it is possible for different engineers to derive differing frame assignments for the links on the robot by using different conventions.

While it is possible to carry out analysis using arbitrary frame assignments, the Denavit-Hartenberg convention makes frame selection systematic and consistent. By this convention, resultant calculations are also simplified as the DH convention follows two assumptions:

(DH1) The axis x_i is $\perp$ to the axis z_{i-1}

(DH2) The axis x_i is coincident with the axis z_{i-1}

By following these assumptions, the need to reconcile a y-axis frame along every step is removed, and the forward kinematics problem becomes much easier to solve. As a part of the DH convention, the z axis of all frames are collinear with the joint axis, being the rotational axis in the case of a (R) revolute joint, and the linear actuation axis in the case of a (P) prismatic joint.

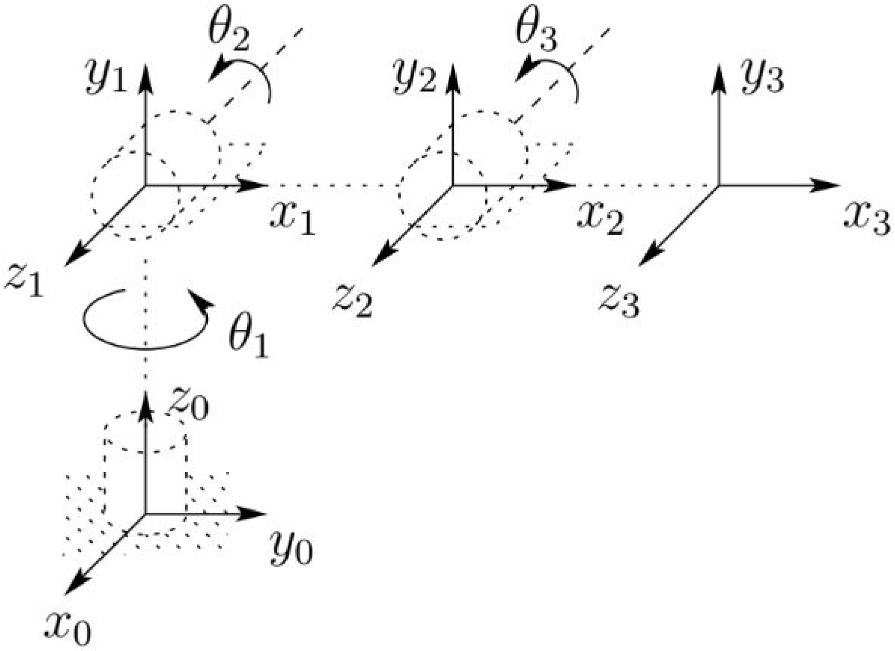


Figure 3.1: Coordinate frames attached to a 3R manipulator

A robot manipulator with *n* joints will have *n*+1 links (*Fig 3.1*) since each joint connects two links. Joints are numbered 1 to *n*, and links are numbered 0 to *n,* starting from the base. By this convention, joint *i* connects link *i*-1 to link *i*. *When joint i is actuated, link i moves.*

With these frames, it is possible to derive a transformation matrix T, that expresses the position and orientation of frame $o_j x_j y_j z_j$ with respect to a base frame $o_i x_i y_i z_i$. T_j^i is represented as a product of the separate transformation matrices that are based off of each link.

$$\begin{aligned} T_j^i &= A_{i+1}A_{i+2}\ldots A_{j-1}A_j \text{ if } i < j \\ T_j^i &= I \text{ if } i = j \\ T_j^i &= (T_i^j)^{-1} \text{ if } j > i. \end{aligned} \tag{3.2}$$

By this definition, it is possible to derive the position and orientation of not just the end effector, but any of the frame origins established along one of the joints.

Each homogeneous transformation A_i is of the form

$$
\begin{aligned}
A_i &= R_{z,\theta_i}\text{Trans}_{z,d_i}\text{Trans}_{x,a_i}R_{x,\alpha_i} \\
&= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
\tag{3.3}
$$

Thus the position and orientation of the end effector with respect to the base frame is given by homogeneous transformation matrix

$$H = T_n^0 = \mathrm{A}_1 \cdots \mathrm{A}_\mathrm{n} \tag{3.4}$$

4 Summary

We may summarize the above procedure based on the D-H convention in the following algorithm for deriving the forward kinematics for any manipulator

Step 4.l: Locate and label the joint axes z_0, ... ,z_{n-1}.

Step 4.2: Establish the base frame. Set the origin anywhere on the z_0-axis. The x_0 and y_0 axes are chosen conveniently to form a right-hand frame. For i = 1, ... , n − 1, perform Steps 3 to 5.

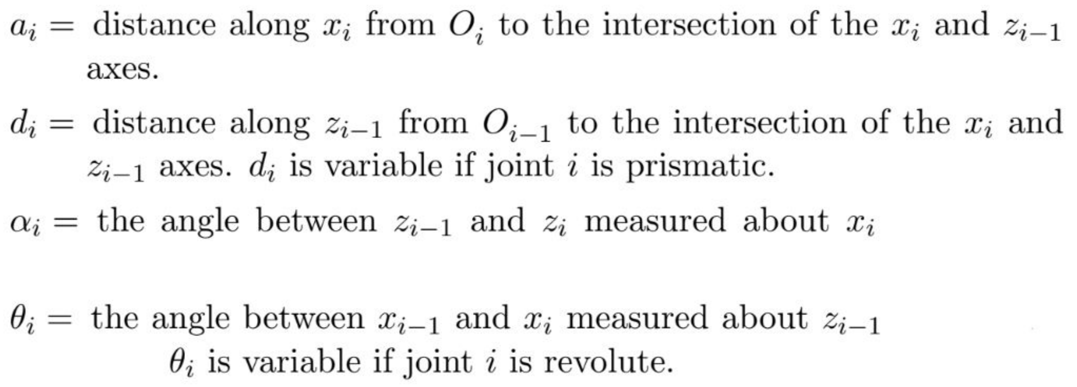
Step 4.3: Locate the origin O_i where the common normal to z_i and z_{i-1} intersects z_i. If z_i intersects z_{i-1} locate O_i at this intersection. If z_i and z_{i-1} are parallel, locate O_i in any convenient position along z_i.

Step 4.4: Establish x^i along the common normal between z_{i-1} and z_i through O_i, or in the direction normal to the $z_{i-1} - z_i$ plane if z_{i-1} and z_i intersect.

Step 4.5: Establish y_i to complete a right-hand frame.

Step 4.6: Establish the end-effector frame $o_n x_n y_n z_n$. Assuming the n-th joint is revolute, set z_n = a along the direction z_{n-1}. Establish the origin O_n conveniently along zn, preferably at the center of the gripper or at the tip of any tool that the manipulator may be carrying. Set y_n = s in the direction of the gripper closure and set x_n = n as s × a. If the tool is not a simple gripper set x_n and y_n conveniently to form a right-hand frame.

Step 4.7: Create a table of link parameters a_i, d_i, α_i, θ_i



$a_i =$ distance along x_i from O_i to the intersection of the x_i and z_{i-1} axes.

$d_i =$ distance along z_{i-1} from O_{i-1} to the intersection of the x_i and z_{i-1} axes. d_i is variable if joint i is prismatic.

$\alpha_i =$ the angle between z_{i-1} and z_i measured about x_i

$\theta_i =$ the angle between x_{i-1} and x_i measured about z_{i-1}
θ_i is variable if joint i is revolute.

Step 4.8: Form the homogeneous transformation matrices Ai by substituting the above parameters into (3.4).

Step 4.9: Form $H^0_n = T^0_n = A_1 \cdots A_n$. This then gives the position and orientation of the tool frame expressed in base coordinates.

5 Specific Embodiment of DH Convention on a 6R Example

By going through the steps outlined in section 4, one may arrive at the resultant H^0_n matrix for a rigid robot with any number of joints and any combination of joint types. In this section we attempt the outlined steps 4.1 through 4.9 on a specific example of a designed 6R robotic arm.

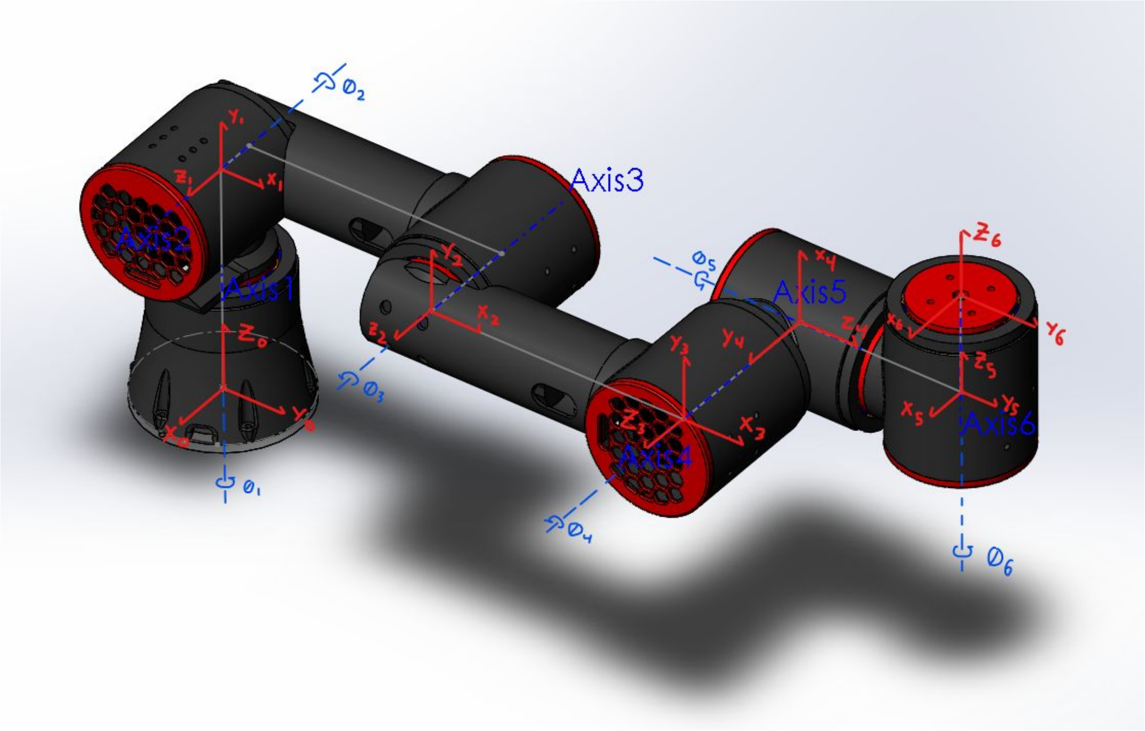


Figure 5.1: Specific embodiment of a 6R robot arm with frame assignment via DH convention

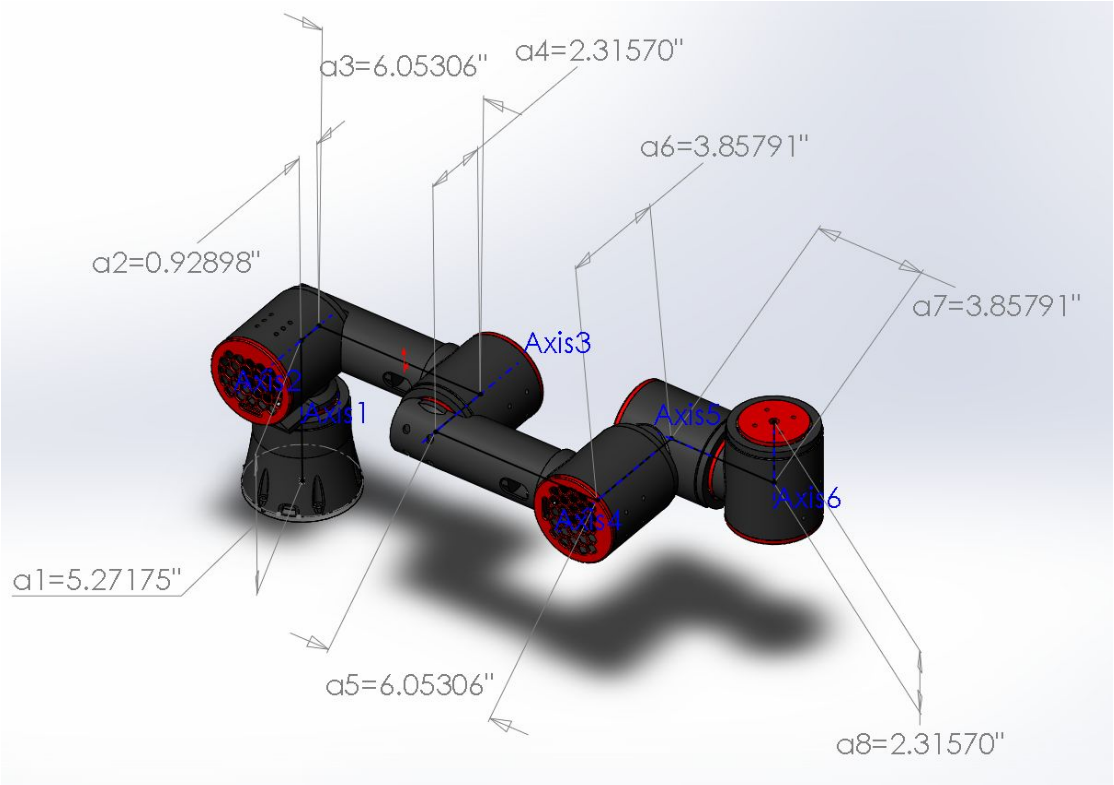


Figure 5.2: Link offsets and link lengths defined by a1...a8

In (Fig 5.1), it is evident that steps 4.1 and 4.2 are completed, and that 4.3-4.5 are satisfied for each link. Step 4.1 is simply the convention that the z axis at any link *i* should be collinear with the joint axis. Step 4.2 is the establishment of the base frame, which is grounded in our case and defined as the origin at $x_0y_0z_0$. Steps 4.3-4.5 are simply the assumptions of the Denavit-Hartenberg convention stated by **DH1** and **DH2** in section 3. Step 4.6 is also satisfied and is illustrated in (Fig 5.1) and the end effector frame origin is assigned as the point coplanar to and centered on the actuator surface. There exists no explicit joint *at* frame $x_6y_6z_6$, however, it is actuated by Axis6.

i **(Frame)**	$\boldsymbol{\theta_i}$	$\boldsymbol{\alpha_i}$	**a**** $_i$	$\boldsymbol{d_i}$
1	θ_1	90	0	a1
2	θ_2	0	a3	a4-a2
3	θ_3	0	a5	0
4	θ_4	90	0	-a6
5	θ_5	90	0	a7
6	θ_6	0	0	a8

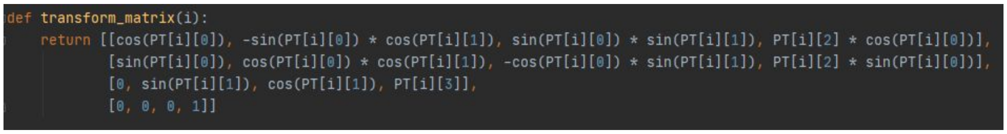
Table 5.3: Denavit-Hartenberg table for the specific embodiment of a 6R arm

By completing *Table 5.3* we have completed step 4.7. Notice that frame 0 of $x_0y_0z_0$ is not present, as all of the parameters within the table are based on the relation between frame *i,* and frame *i-1*. Thus, with frame 0 having no preceding frame to reference, it is not present within the table, but instead we look to establish parameters of frame 1 in relation to frame 0 in the first data row of the table.

Since the robot is of 6R, we must include variable parameters that account for the freely rotating joints. These variables are present in the θ_i column, since per the D-H convention, θ_i is a variable if joint *i* is revolute. To verify that the table was done correctly, all joint variables (θ_1...θ_6) as well as all link lengths (d_i) and link offsets (a_i) (a1-a8) should be accounted for in the table.

Next, we must establish each transformation matrix $A_{i,}$ which describes the transformational relation between frame i, and frame i-1, for each frame. This becomes difficult to compute by hand so we may use computational tools such as python to aid us.

$$\begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_ic_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_is_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



```
def transform_matrix(i):
    return [[cos(PT[i][0]), -sin(PT[i][0]) * cos(PT[i][1]), sin(PT[i][0]) * sin(PT[i][1]), PT[i][2] * cos(PT[i][0])],
            [sin(PT[i][0]), cos(PT[i][0]) * cos(PT[i][1]), -cos(PT[i][0]) * sin(PT[i][1]), PT[i][2] * sin(PT[i][0])],
            [0, sin(PT[i][1]), cos(PT[i][1]), PT[i][3]],
            [0, 0, 0, 1]]
```

Figure 5.4 & 5.5: Translation of transformation matrix A_i into a python function

Using the determined matrices for each frame, we may then dot them all together in the order of $A_1 \cdot \cdots \cdot A_6$ to determine transform matrix H_6^0, which describes the end effector position and orientation in relation to the base frame.

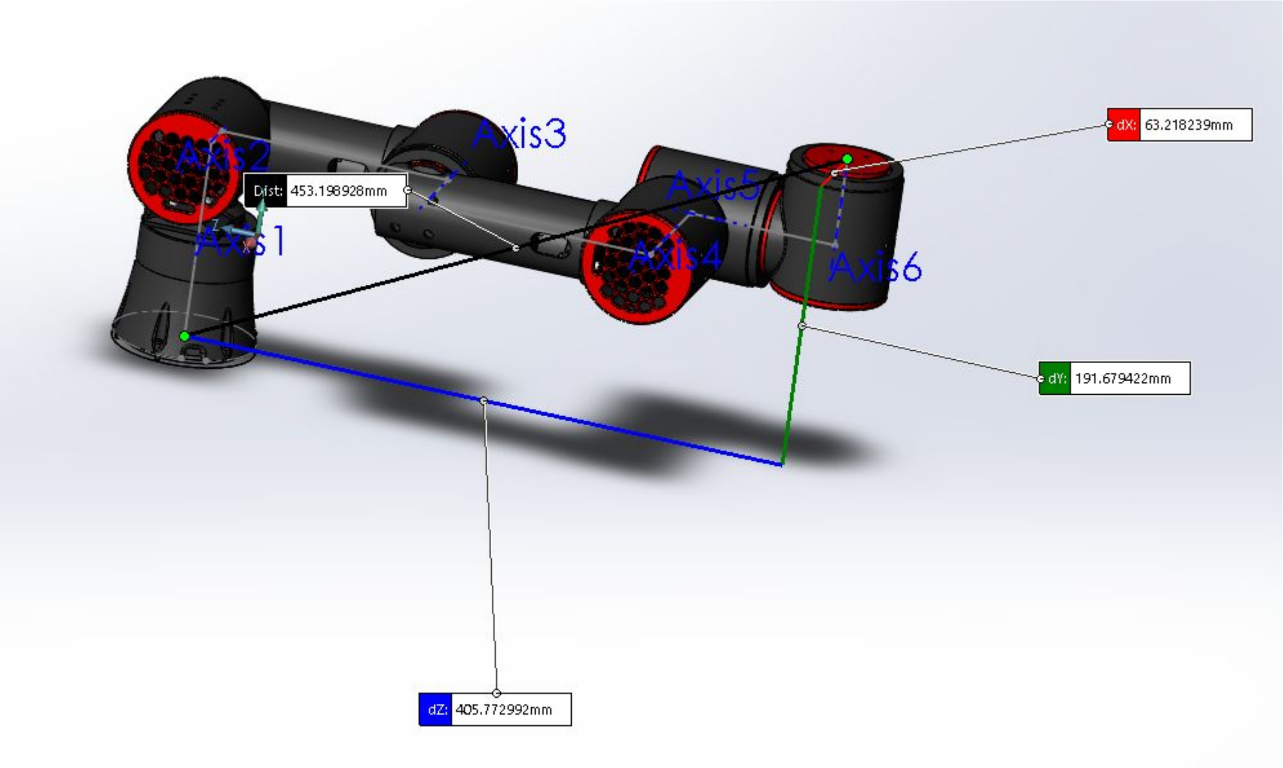
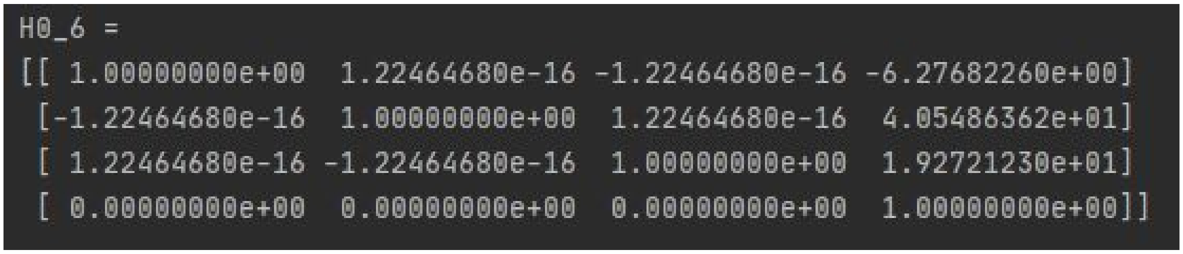


Figure 5.6: Solidworks model with measurement of end effector origin relative to base frame origin. Rounded measurements, X,Y,Z: (6.3, 40.5, 19.2) in cm



```
H0_6 =
[[ 1.00000000e+00  1.22464680e-16 -1.22464680e-16 -6.27682260e+00]
 [-1.22464680e-16  1.00000000e+00  1.22464680e-16  4.05486362e+01]
 [ 1.22464680e-16 -1.22464680e-16  1.00000000e+00  1.92721230e+01]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  1.00000000e+00]]
```

Figure 5.7: Python console output with resultant H_6^0 matrix mentioned by equation (3.4) Notice the first 3 entries of the last column of matrix, that are the X,Y,Z components of frame $x_6y_6z_6$; that is, X,Y,Z: (6.3, 40.5, 19.3) in cm

The numbers match closely! (Variance is due to hand-placed and orientated arm links in CAD model). The -6.3cm value in the python console is correctly negative. Base frame (grounded frame) utilizes an x_0 axis frame pointing in the opposite direction (Fig 5.6). As detailed in (Fig

5.7), end effector position is detailed by the first 3 entries of the final column of the matrix. End effector orientation is reconciled by the first 3x3 rows and columns of the matrix.

Sources Cited:

https://users.cs.duke.edu/~brd/Teaching/Bio/asmb/current/Papers/chap3-forward-kinematics.pdf