# Convex Functions: Weekly Exercise 2



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## Exercise 1

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Assume $J$ is l.s.c. To show that the epigraph is closed, we need to show that all sequences $\{(u_n, \alpha_n)\}_{n\in\mathbb{N}} \subset \text{epi}(J)$ attain their limit $(u_n,\alpha_n) \to (u^*, \alpha^*)$ as $n \to \infty$ in epi(J), i.e. $(u^*, \alpha^*) \in \text{epi}(J)$.

$$\because J(u^*) \leq \liminf_{n\to\infty} J(u_n) \quad \text{[Definition of l.s.c.]}$$

$$\Rightarrow J(u^*) \leq \liminf_{n\to\infty} \alpha_n \quad \left[\begin{array}{l}\because (u_n,\alpha_n) \in \text{epi}(J) \\ \Rightarrow J(u_n) \leq \alpha_n\end{array}\right]$$

$$\therefore J(u^*) \leq \alpha^* \quad \left[\because \liminf_{n\to\infty} \alpha_n = \alpha^*\right]$$

$$\therefore (u^*, \alpha^*) \in \text{epi}(J) \quad \text{[Definition of epi(J)]}$$

Hence, epi(J) is closed.

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Conversely, if $J$ is <u>NOT</u> l.s.c., then $\{(u_n, d_n)\}_{n\in\mathbb{N}} \subset epi(J)$ should ~~not~~ have its limit point outside $epi(J)$, i.e. $(u^*, d^*) \notin epi(J)$, which is what we need to show.

As $J$ is not l.s.c., hence for some $\{u_n\}_{n\in\mathbb{N}} \subset dom(J)$ such that $\lim_{n\to\infty} u_n = u^* \in dom(J)$, the l.s.c. property doesn't hold, i.e:

$$J(u^*) > \liminf_{n\to\infty} J(u_n)$$

$\Rightarrow \exists n^*$, $J(u^*) > J(u_n)$ for $n \geq n^*$, or

$\underbrace{J(u^*) - \varepsilon \geq J(u_n)}_{(i)}$ for some $\varepsilon > 0$.

From (i) we can note that $(u_n, J(u^*) - \varepsilon) \in epi(J)$ but the limit point $(u^*, J(u^*) - \varepsilon) \notin epi(J)$ since $J(u^*) - \varepsilon \not\geq J(u^*)$ for $\varepsilon > 0$ (definition of $epi(J)$).

Hence, if $J$ is NOT l.s.c., then $epi(J)$ is not closed.

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Exercise 2

Assume $J: E \to R$ is convex with $\text{dom}(J) = \mathbb{R}^n$, and bounded above on $\mathbb{R}^n$, but $J$ is <u>NOT</u> a constant function.

Then, $\exists u, v \in \text{dom}(J)$, $J(u) > J(v)$.

Since $J$ is convex, we have:

$$\Rightarrow J(u) \leq \theta J\left(\frac{u-(1-\theta)v}{\theta}\right) + (1-\theta)J(v) \quad , \theta \in (0,1)$$

The above follows from the definition of convexity $J(\theta u' + (1-\theta)v') \leq \theta J(u') + (1-\theta)J(v')$, with $u = \theta u' + (1-\theta)v'$ and $v = v'$.

$$\Rightarrow \frac{J(u)-(1-\theta)J(v)}{\theta} \leq J\left(\frac{u-(1-\theta)v}{\theta}\right)$$

$$\Rightarrow \frac{J(u)-J(v)}{\theta} + J(v) \leq J\left(\frac{u-(1-\theta)v}{\theta}\right)$$

$$\Rightarrow \underset{\theta \to 0^+}{\text{limit}} \left\{ \frac{J(u)-J(v)}{\theta} + J(v) \right\} \leq \underset{\theta \to 0^+}{\text{limit}} \left\{ J\left(\frac{u-(1-\theta)v}{\theta}\right) \right\}$$

$$\Rightarrow \underset{\theta \to 0^+}{\text{limit}} \left( \frac{J(u)-J(v)}{\theta} \right) + J(v) \leq J(\omega) \quad , \omega \in \text{dom}(J)$$

such that $J(\omega) = \underset{\theta \to 0^+}{\text{limit}} \left\{ J\left(\frac{u-(1-\theta)v}{\theta}\right) \right\} < C$, since $J$ is bounded.

Now, since $J(u) > J(v) \Rightarrow J(v) - J(v) > 0$

$$\Rightarrow \lim_{\theta \to 0^+} \left\{ \frac{J(u) - J(v)}{\theta} \right\} = \infty$$

$\therefore J(w) \geq \infty$, which is a contradiction since $J$ is bounded and therefore $J(w) < C < \infty$.

Hence proved that $J$ is a constant function.

## Exercise 3

**Part A** Let $u, v \in \mathbb{E}$, then for $\theta \in [0, 1]$

$$\Rightarrow J(\theta u + (1-\theta)v) := \|\theta u + (1-\theta)v\| \leq \|\theta u\| + \|(1-\theta)v\| \quad \text{[Triangle Inequality]}$$

$$= |\theta| \|u\| + |1-\theta| \|v\| \quad \text{[Homogenity of norms of degree 1.]}$$

$$= \theta \|u\| + (1-\theta)\|v\| = \theta J(u) + (1-\theta) J(v) \quad [\text{since } \theta \in [0,1]]$$

Hence $J(u) := \|u\|$ is convex since $\mathrm{dom}(J) := \mathbb{E}$ (i.e. the normed vector space) is also convex following from the closure property of vector spaces w.r.t. addition and scalar multiplication.

**Part B** Let $u, v \in \mathrm{dom}(J)$, then for $\theta \in [0,1]$

$$\Rightarrow J(\theta u + (1-\theta)v) := F(K(\theta u + (1-\theta)v)) = F(\theta Ku + (1-\theta)Kv)$$

$$\leq \theta F(Ku) + (1-\theta) F(Kv) \quad [\text{since } F \text{ is convex}]$$

$$= \theta J(u) + (1-\theta) J(v)$$

Hence, $J(u) := F(Ku)$ is convex over $\mathrm{dom}(J)$.

convex

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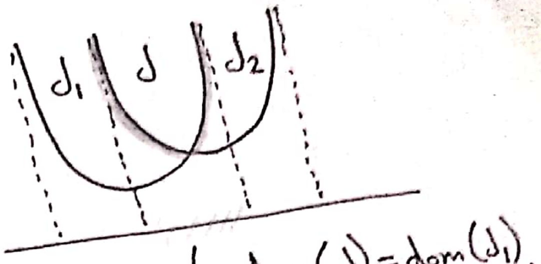
# Part C




↳ $\text{dom}(J) = \text{dom}(J_1) \cap \text{dom}(J_2)$

Note that for

$J(u) := \max\{J_1(u), J_2(u)\}$,

$\text{dom}(J) = \text{dom}(J_1) \cap \text{dom}(J_2)$ as depicted in the adjoining figure. Since intersection preserves convexity, hence $\text{dom}(J)$ is convex for convex $\text{dom}(J_1)$ and $\text{dom}(J_2)$.

Now, for $u, v \in \text{dom}(J)$,

$$\Rightarrow J(\theta u + (1-\theta)v) = \max\{J_1(\theta u + (1-\theta)v), J_2(\theta u + (1-\theta)v)\}$$, then:

Assume $J_1(\theta u + (1-\theta)v) > J_2(\theta u + (1-\theta)v)$

$$\Rightarrow J(\theta u + (1-\theta)v) = J_1(\theta u + (1-\theta)v)$$

$$\leq \theta J_1(u) + (1-\theta) J_1(v) \quad [\because J_1 \text{ convex}]$$

$$\leq \theta \max(J_1(u), J_2(u)) + (1-\theta)\max(J_1(v), J_2(v))$$

~~[∵ max() maintains~~ $[\because \max(x, y) \geq x]$

$$\therefore J(\theta u + (1-\theta)v) \leq \theta J(u) + (1-\theta) J(v)$$

The same can be shown for $J_2(\theta u + (1-\theta)v) > J(\theta u + (1-\theta)v)$

Hence, $J(u) := \max\{J_1(u), J_2(u)\}$ is convex for $J_1$ and $J_2$ convex.

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Assume $J$ is convex. Then $\forall u, u+td \in U, t>0$ (possible since $U$ is open), the Taylor expansion yields:

$$\because J(u+td) = J(u) + t d^T \nabla J(u) + \frac{t^2}{2} d^T \nabla^2 J(u) d + o(t^2)$$

$$\Rightarrow J(u+td) - J(u) - t d^T \nabla J(u) = \frac{t^2}{2} d^T \nabla^2 J(u) d + o(t^2)$$

$$\because J(u+td) - J(u) - t d^T \nabla J(u) \geq 0 \quad [\because J \text{ convex and } u+td, u \in U]$$

$$\Rightarrow 0 \leq \frac{t^2}{2} d^T \nabla^2 J(u) d + o(t^2)$$

$$\Rightarrow 0 \leq d^T \nabla^2 J(u) d + \frac{2 o(t^2)}{t^2} \quad [\text{Multiplying by } 2/t^2]$$

$$\therefore 0 \leq d^T \nabla^2 J(u) d \quad [\text{for } t \to 0]$$

Hence $\nabla^2 J(u)$ is positive semidefinite for convex $J$.

Now, assume $\nabla^2 J(u)$ positive semidefinite with no convex assumption of $J$. For ~~...~~ $\forall u, u+d \in U$

$$\Rightarrow J(u+d) = J(u) + d^T \nabla J(u) + \frac{1}{2} d^T \nabla^2 J(u+td) d$$

$$\Rightarrow J(u+d) - J(u) - d^T \nabla J(u) = \frac{1}{2} d^T \nabla^2 J(u+td) d \geq 0$$

(since $\nabla^2 J(u)$ is positive semi definite for $u \in U$ and $u+td \in U$ for $u, u+d \in U$ as the former is a convex combination of the latter.)

$$\Rightarrow J(u+d) - J(u) - d^T \nabla J(u) \geq 0$$

$$\therefore J(u+d) - J(u) \geq d^T \nabla J(u)$$

Hence $J$ is convex as $u, u+d \in U$ by assumption and $d = (u+d) - (u)$.