Convex Optimization for Machine Learning and Computer Vision

Lecture: Dr. Tao Wu
Exercises: Zhenzhang Ye
Winter Semester 2019/20

Computer Vision Group
Institut für Informatik
Technische Universität München

Weekly Exercises 5

Room: 02.09.023

Wednesday, 27.11.2019, 12:15-14:00

Submission deadline: Monday, 25.11.2019, 16:15, Room 02.09.023

Convex conjugate (8+6 Points)

Exercise 1 (4 points)**.** Given a $X \in \mathbb{R}^{m\times n}$, compute the subdifferential of the $1,2$-norm, i.e.

$$\partial||X||_{1,2} = \partial(\sum_{i=1}^{m}(\sum_{j=1}^{n} X_{i,j}^2)^{1/2}).$$

Exercise 2 (4 Points)**.** Consider following problems of convex conjugate:

- Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex. Show that the convex conjugate of the perspective function $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \cup \{\infty\}$

  $$g(x,t) = \begin{cases} tf(x/t), & \text{if } t > 0 \\ \infty, & \text{otherwise} \end{cases}$$

  is given by

  $$g^*(y,s) = \begin{cases} 0, & \text{if } f^*(y) \leq -s \\ \infty, & \text{otherwise} \end{cases}$$

- Show that the biconjugate of the persepective function g is given by

  $$g^{**}(x,t) = \begin{cases} tf(x/t), & \text{if } t > 0 \\ \sigma_{\text{dom}(f^*)}(x), & \text{if } t = 0 \\ \infty, & \text{if } t < 0 \end{cases}$$

  where $\sigma_{\text{dom}(f^*)}(x) = \sup_{y\in\text{dom}(f^*)}\langle x, y\rangle$ is the *support function* of $\text{dom}(f^*)$.

Exercise 3 (6 Points)**.** Let $A \in \mathbb{R}^{m\times n}$ be a linear operator and $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ a convex function. Then $Af : \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ defined as

$$(Af)(u) := \begin{cases} \inf\limits_{v\in\mathbb{R}^n,\, Av=u} f(v) & \text{if } \exists\, v \in \mathbb{R}^n \text{ s.t. } Av = u \\ \infty & \text{otherwise.} \end{cases}$$

is called the image of f under A.

1. Show that the convex conjugate $(Af)^*$ of Af is given as $f^* \circ A^\top$ where $(f^* \circ A^\top)(v) := f^*(A^\top v)$.

2. Name the properties that we require for $A^\top f^* = (f \circ A)^*$ to hold. What theorem from the lecture applies here?

3. Give an example of a closed, convex and non-empty set C and a linear operator A s.t. $AC := \{Ax : x \in C\}$ is not closed.

4. Let f be closed, (convex) and proper. Argue that Af does not need to be closed.