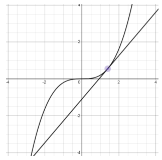
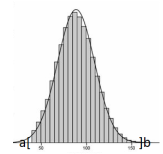
Fundamental Theorem of Calculus - A Proof by Definition



Derivative

$\frac{\Delta f(x)}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$

$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{df(x)}{dx}$



$\Delta x = \frac{b-a}{n}$

Integral

$\sum_{i=a}^{b} f(x_i)\Delta x$

$\lim_{\Delta x \to 0} \sum_{i=a}^{b} f(x_i)\Delta x = \int_a^b f(x)dx$

The Fundamental Theorem of Calculus

If a function f is continuous on the closed interval [a, b] and F is an anti-derivative of f on the continuous interval [a, b], then $\int_a^b f(x)dx = F(b) - F(a)$

Approach - take the integral of a derivative of a function *f(x)*

Given that integration and differentiation are inverse operations taking the integral of the derivative should return the function itself f(x) in other words: $\int \frac{df(x)}{dx}dx = f(x)$

Intuition: $¿\sum \frac{\Delta f(x)}{\Delta x}\Delta x = \sum \frac{\Delta f(x)}{\cancel{\Delta x}}\cancel{\Delta x} = \sum \Delta f(x) = f(x)$?

$\therefore \quad ¿\int \frac{df(x)}{\cancel{dx}}\cancel{dx} = \int df(x) = df(x_1) + df(x_2) + df(x_3) + ...df(x_n) = f(x) + c$?

The fundamental theorem of calculus can be rewritten using the approach

$\int_a^b \frac{dF(x)}{dx}dx = F(b) - F(a)$ | This can be proven as follows:

$\int_a^b \frac{dF(x)}{\cancel{dx}}\cancel{dx} = \int_a^b dF(x)$

Given $\Delta f(x) = f(x+\Delta x) - f(x) \quad \therefore \quad df(x) = f(x+dx) - f(x)$ | The Integral can be written as an alternating series

$\int_a^b dF(x) = F(a+dx) - F(a) + F((a+dx)+dx) - F(a+dx) + F(((a+dx)+dx)+dx) - F((a+dx)+dx) + ... + F(a+ndx) - f(a+(n-1)dx)$ | Every positive term can be canceled by subsequent negative term except for the negative term in the first sum and the positive term in the last sum

$\int_a^b dF(x) = \cancel{F(a+dx)} - F(a) + \cancel{F((a+dx)+dx)} - \cancel{F(a+dx)} + \cancel{F(((a+dx)+dx)+dx)} - \cancel{F((a+dx)+dx)} + ... + F(a+ndx) - \cancel{f(a+(n-1)dx)}$

$\int_a^b dF(x) = F(a+ndx) - F(a)$ | $b = a + n\Delta x \quad \therefore \quad a + ndx \to b$

$\int_a^b dF(x) = F(b) - F(a) \quad \square$

Indefinite Integral

$\int \frac{dF(x)}{dx}dx = \int \frac{dF(x)}{\cancel{dx}}\cancel{dx} = \int dF(x) = F(x) + C$ | This can be proven as follows:

$\int dF(x) = \int_{x_0}^{x} dF(t)$ | The indefinite integral can be rewritten using the change of variable technique as the definite integral $\int_{x_0}^{x} dF$ from an initial point x_0 to the variable x

Given $\Delta f(x) = f(x+\Delta x) - f(x) \quad \therefore \quad df(x) = f(x+dx) - f(x)$ | The Integral can be written as an alternating series: $F(x_0+dt) - F(x_0) + F((x_0+dt)+dt) - F(x_0+dt) + F(((x_0+dt)+dt)+dt) - F((x_0+dt)+dt) + ... + F(a+ndt) - f(a+(n-1)dt)$ | The alternating terms cancel out except two terms

$\int_{x_0}^{x} dF(t) = F(x_0+ndx) - F(x_0)$ | $x_0 + ndx \to x$

$\int_{x_0}^{x} dF(t) = F(x) - F(x_0)$ | x_0 is a point $F(x_0) \to C$

$\int dF(x) = F(x) + C \quad \square$