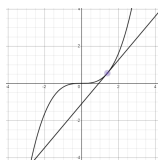


Fundamental Theorem of Calculus - A Proof by Definition



Derivative

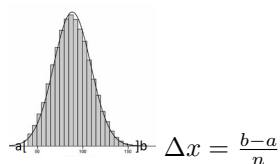
$$\frac{\Delta f(x)}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x}$$

$$\lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{df(x)}{dx}$$

Integral

$$\sum_{i=a}^b f(x_i) \Delta x$$

$$\lim_{\Delta x \rightarrow 0} \sum_{i=a}^b f(x_i) \Delta x = \int_a^b f(x) dx$$



The Fundamental Theorem of Calculus

If a function f is continuous on the closed interval $[a, b]$ and F is an anti-derivative of f on the continuous interval $[a, b]$, then $\int_a^b f(x) dx = F(b) - F(a)$

Approach - take the integral of a derivative of a function $f(x)$

Given that integration and differentiation are inverse operations taking the integral of the derivative should return the function itself $f(x)$ in other words: $\int \frac{df(x)}{dx} dx = f(x)$

Intuition: $\int \frac{df(x)}{dx} dx = \int df(x) = df(x)_1 + df(x)_2 + df(x)_3 + \dots df(x)_n = f(x) + c$?

The fundamental theorem of calculus can be rewritten using in terms of F as

$$\int_a^b \frac{dF(x)}{dx} dx = F(b) - F(a) \quad | \text{ This can be proven as follows:}$$

$$\int_a^b \frac{dF(x)}{dx} dx = \int_a^b dF(x)$$

Given $\Delta f(x) = f(x + \Delta x) - f(x) \quad \therefore \quad df(x) = f(x + dx) - f(x) \quad | \text{ The Integral can be written as an alternating series}$

$\int_a^b dF(x) = F(a + dx) - F(a) + F((a + dx) + dx) - F(a + dx) + F(((a + dx) + dx) + dx) - F((a + dx) + dx) + \dots + F(a + ndx) - F(a + (n-1)dx) \quad | \text{ Every positive term can be canceled by subsequent negative term except for the negative term in the first sum and the positive term in the last sum}$

$$\int_a^b dF(x) = \cancel{F(a + dx)} - F(a) + \cancel{F((a + dx) + dx)} - \cancel{F(a + dx)} + \cancel{F(((a + dx) + dx) + dx)} - \cancel{F((a + dx) + dx)} + \dots + F(a + ndx) - \cancel{f(a + (n-1)dx)}$$

$$\int_a^b dF(x) = F(a + ndx) - F(a) \quad | \quad b = a + n\Delta x \quad \therefore \quad a + ndx \rightarrow b$$

$$\int_a^b dF(x) = F(b) - F(a) \quad \square$$

Indefinite Integral

$$\int \frac{dF(x)}{dx} dx = \int \frac{dF(x)}{dx} dx = \int dF(x) = F(x) + C \quad | \text{ This can be proven as follows:}$$

$\int dF(x) = \int_{x_0}^x dF(t) \quad | \text{ The indefinite integral can be rewritten using the change of variable technique as the definite integral } \int_{x_0}^x dF \text{ from an initial point } x_0 \text{ to the variable } x$

Given $\Delta f(x) = f(x + \Delta x) - f(x) \quad \therefore \quad df(x) = f(x + dx) - f(x) \quad | \text{ The Integral can be written as an alternating series: } F(x_0 + dt) - F(x_0) + F((x_0 + dt) + dt) - F(x_0 + dt) + F(((x_0 + dt) + dt) + dt) - F((x_0 + dt) + dt) + \dots + F(a + ndt) - f(a + (n-1)dt) \quad | \text{ The alternating terms cancel out except two terms}$

$$\int_{x_0}^x dF(t) = \cancel{F(x_0 + dt)} - F(x_0) + \cancel{F((x_0 + dt) + dt)} - \cancel{F(x_0 + dt)} + \cancel{F(((x_0 + dt) + dt) + dt)} - \cancel{F((x_0 + dt) + dt)} + \dots + F(a + ndt) - \cancel{f(a + (n-1)dt)}$$

$$\int_{x_0}^x dF(t) = F(x_0 + ndx) - F(x_0) \quad | \quad x_0 + ndx \rightarrow x$$

$$\int_{x_0}^x dF(t) = F(x) - F(x_0) \quad | \quad x_0 \text{ is a point } F(x_0) \rightarrow C$$

$$\int dF(x) = F(x) + C \quad \square$$