# Raytracing in South Pole Ice

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## 1 RayTracing in South Pole Ice

Once we have propagated the E-fields from air shower particles to the ice surface then we have to understand the propagation mechanism that takes them from the ice surface to the ARA antennas inside the ice sheet. For this we need to perform raytracing, i.e., tracing the path of the radiowaves as they travel through the ice sheet. In so far as ARA is a detector of neutrino-induced showers in the ice sheet, raytracing forms an extremely important part of the neutrino vertex reconstruction required to perform energy reconstruction of the neutrino. Rays are refracted in the ice sheet owing to the depth-dependent density, and therefore index of refraction. The density profile of the South Pole ice is expected to follow an exponential form which corresponds to a refractive index profile:

$$n(z) = A + Be^{Cz} \tag{1}$$

here if $z$ is negative then $C$ has to be positive and if $C$ is positive then $z$ has to be negative, with $z$ defined as depth inside the ice. For the model currently used by ARA these parameters have the values $A = 1.78$, $B = -0.43$ and $C = 0.0132$ m$^{-1}$ [?].

This causes the bending of rays and creates 'shadow zones' which are areas from which we should technically not see any radio signals. This is important as it helps quantify the effective area or volume over which the ARA stations can detect neutrino-induced showers.

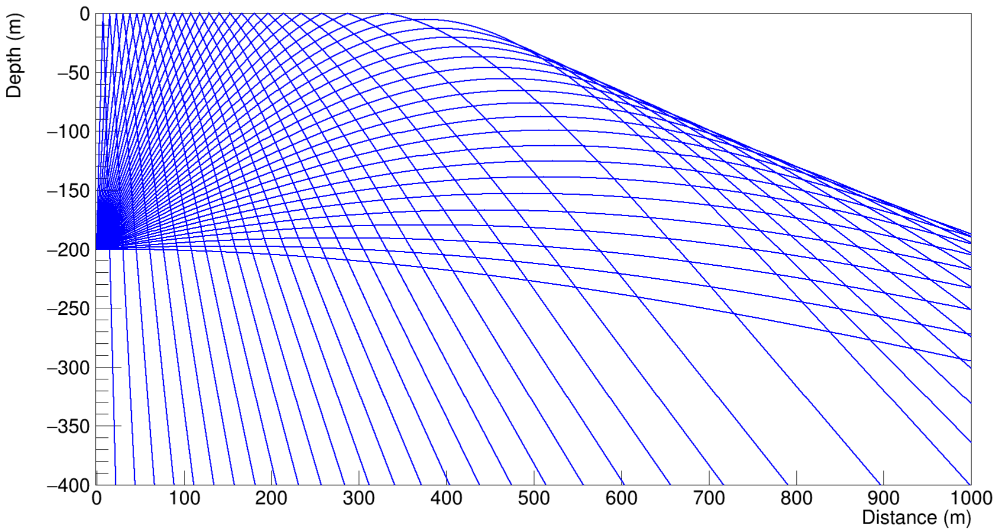


Figure 1: Raytracing simulation for a source at $z = -200$ m depth, emitting radiowaves in launch angle steps of 2°. Rays are reflected first from the ice surface at $z = 0$ and then begin to refract downwards as the source migrates laterally from the receiver. The area where the rays do not penetrate is called the 'shadow zone'.

As is visible in Figure 1, there are in general two possible indirect ray trajectories from each source. – those that reflect off the surface and those which refract downwards because of the refractive index profile. The refracted ray that takes the shortest possible time to reach the target is called the direct ray.

### 1.1 Numerical Raytracing using the Runge-Kutta Method

There are two ways in which ray trajectories can be calculated: numerically and analytically. The set of following equations can be solved using the fourth order Runge Kutta Method to trace out the ray path:

$$\frac{dx}{ds} = \sin(\beta)\ , \quad \frac{dz}{ds} = \cos(\beta)\ , \quad \frac{dt}{ds} = \frac{n(z)}{c} \tag{2}$$

$$\frac{d\beta}{ds} = -\frac{\sin(\beta)}{n(z)}\frac{dn(z)}{dz}\bigg|_z\ , \quad \frac{dA_0}{ds} = \frac{A_0}{L(z,f)} \tag{3}$$

here $\beta$ in this case is the ray angle w.r.t the vertical, from below, $t$ is the time of the ray, $A_0$ is the initial signal amplitude and $L(z, f)$ is the attenuation length, which itself is dependent on frequency and the depth of the ray. A schematic of these conventions is shown in the right panel of Figure 2.

To solve these we need initial conditions. The coordinates of the source $(x_0, z_0)$ and also the coordinates of the target $(x_1, z_1)$ are given, but we do not know, *a priori*, the initial launch angle $\theta_0$ required to reach the receiver. The unknown launch angle can be found in several different ways. We can either use trial-and-error, and keep launching rays until we hit the target, or we can try to derive an analytical expression that gives us the launch angle for the reflected and the direct rays, as discussed in the next section.

The advantage of the brute force trial-and-error method is that it will work with any refractive index model. However, to predict the ray times accurately requires very small step sizes, of order 1 mm. If the rays are being traced over lengths on the order of kilometers (the usual case), then the computational time is large.

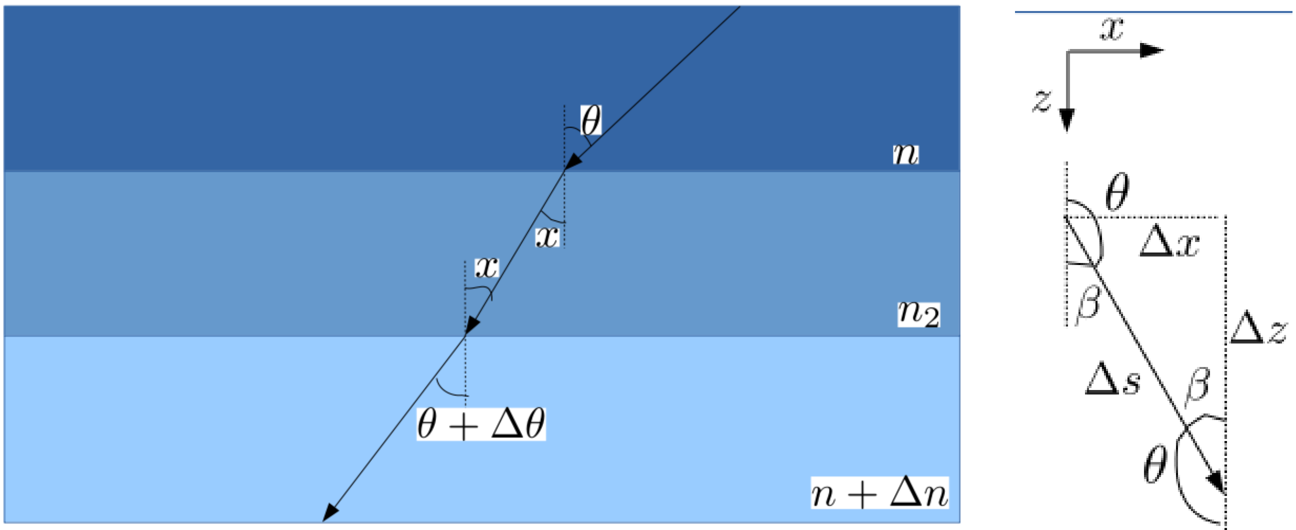


Figure 2: (**Left**) A schematic of the ice sheet, approximated as having multiple layers. (**Right**) A schematic of the ray path conventions being used in the equations.

### 1.2 Analytical Solution for Raytracing

To start the derivation we start from the ice layer schematic shown on the left of Figure 2. Here the ice sheet is being approximated as a combination of multiple thin layers with different values of refractive index. Using Snell's law, we can write:

$$n\sin(\theta) = n_2 \sin(x), \quad n_2 \sin(x) = (n + \Delta n)\sin(\theta + \Delta\theta). \tag{4}$$

Substituting one of the above equations into the other gives us:

$$\frac{\sin(\theta + \Delta\theta)}{\sin(\theta)} = \frac{n}{n + \Delta n}. \tag{5}$$

Using double angle identities and some algebraic manipulation we can write:

$$\frac{\sin(\theta)\cos(\Delta\theta)}{\sin(\theta)} + \frac{\cos(\theta)\sin(\Delta\theta)}{\sin(\theta)} = \left(1 + \frac{\Delta n}{n}\right)^{-1}; \tag{6}$$

taking the limits $\delta\theta \ll 1$ and $\Delta n \ll 1$ we can rewrite the above equation as:

$$1 + \cot(\theta)\Delta\theta = 1 - \frac{\Delta n}{n} \quad \Rightarrow \quad \frac{\Delta n}{n} = -\cot(\theta)\Delta\theta, \tag{7}$$

reducing to:

$$\boxed{\frac{d\theta}{dn} = -\frac{\tan(\theta)}{n(z)}} \tag{8}$$

From here, using Equation 8 and the chain rule, we obtain:

$$\frac{d\theta}{dz} = \frac{d\theta}{dn}.\frac{dn}{dz} \quad \Rightarrow \quad -\frac{\tan(\theta)}{n(z)}\, n'(z), \text{ here we have } n'(z) = \frac{dn}{dz} \tag{9}$$

If we integrate both sides of the above equation we obtain

$$L = n(z)\sin(\theta), \tag{10}$$

where $\theta$ is as defined in the schematic on the right of Figure 2, and $L$ is the integration constant. In this case, $L$ corresponds to the initial condition of the ray and its value is obtained from the source depth ($z_0$) and the launch angle ($\theta_0$):

$$\boxed{L = n(z_0)\sin(\theta_0)} \tag{11}$$

Using Equation 10 and the value of $L$ we can now write:

$$\theta = \arcsin\left(\frac{L}{n(z)}\right), \tag{12}$$

Now using the above result, we can be write for any ray path that:

$$\frac{dx}{dz} = \tan(\theta) = \tan\left(\arcsin\left(\frac{L}{A + Be^{Cz}}\right)\right), \tag{13}$$

Here the functional form of $n(z)$ has been explicitly stated. If we integrate this differential equation, we finally obtain the analytic form of the ray path:

$$\boxed{x(L,z) = \frac{L}{C}\frac{1}{\sqrt{A^2 - L^2}}\left(Cz - \log\left(A(A + Be^{Cz}) - L^2 + \sqrt{A^2 - L^2}\sqrt{(A + Be^{Cz})^2 - L^2}\right)\right)} \tag{14}$$

The analytic solution that we found in Equation (14) has several important properties:

1. When $A = L$ or $A = n(z_0)\sin(\theta_0)$, the solution becomes undefined. The condition $A < L$ therefore must always be fulfilled, putting a limit on our launch angle $\theta_0$ for a given source depth, and limiting launch angles to less than 90°. Since ray paths are reversible, the target and the source can be swapped. Thus this technique allows us to trace rays even if the source is shallower than the target.

2. This solution has two roots: one for $C > 0$ and $z < 0$ and the other for $C < 0$ and $z > 0$.

3. The solution also becomes undefined when $n(z) = L$ or $A + Be^{Cz} = L$. This gives us a limit on the minimum depth (peak point, or turning point) that a ray launched at a certain angle $\theta$ will attain. This point is called $z_{max}$, which can be written as:

$$\boxed{z_{max} = \frac{1}{C}\log\left(\frac{L - A}{B}\right)} \tag{15}$$

### 1.2.1 Finding the launch angle

Given $(x_0, z_0)$ and $(x_1, z_1)$, we still do not have the value of $\theta_0$ which is needed so that the value of $L$ can be calculated. Then the ray can be traced out between the source and the target.

To find $\theta_0$, the analytical solution needs to be minimized w.r.t the launch angle. We start by writing the solution in Equation (14) as $x = f(L, z)$. First we start the ray from zero distance, such that $x_0$ corresponds to $x = 0$. To fulfill this condition, it is required:

1. For Direct rays: $f_1'(L, z) = f_1(L, z) - f_1(L, z_0)$.
2. For Reflected and Refracted rays: $f_2'(L, z) = f_2(L, z) - f_2(L, z_0) - 2x_{max}$.

Here $x_{max}$ is the value of $x$ at the turning point $(x_{max}, z_{max})$. Then, if the target coordinates are to lie on this ray path, they must fulfill the condition:

1. For Direct rays: $f_1'(L, z_1) - x_1 = 0$.
2. For Reflected rays: $f_2'(L, z_1) - x_1 = 0$, for which $x_{max} = f_2(L, 0) - f_2(L, z_0)$
3. For Refracted rays: $f_2'(L, z_1) - x_1 = 0$, for which $x_{max} = f_2(L, z_{max}) - f_2(L, z_0)$.

Now the value of $L$ or the launch angle $\theta_0$ for which these conditions are fulfilled can be calculated. This is done in ROOT using a function $TF1 :: GetX()$ which uses Brent's method and returns the value of the launch angle for the zeroes of the above function [**?**].

### 1.2.2 Finding the travel time of the rays

For calculating the travel time, one can use Fermat's least time principle, which leads to the integral:

$$t = \int dz \sqrt{1 + \left(\frac{dx}{dz}\right)^2} \frac{n(z)}{c} \tag{16}$$

Here we can substitute Equation (13) for $dx/dz$ and integrate to get:

$$\begin{aligned} t(L, z) = &\left(\frac{1}{c\ C\sqrt{n(z)^2 - L^2}}\right) \\ &\left[n(z)^2 - L^2 + \left[Cz - \log\left(A\ n(z) - L^2 + \sqrt{A^2 - L^2}\sqrt{n(z)^2 - L^2}\right)\right] \frac{A^2\sqrt{n(z)^2 - L^2}}{\sqrt{A^2 - L^2}}\right. \\ &\left. + A\sqrt{n(z)^2 - L^2} \log\left[n(z) + \sqrt{n(z)^2 - L^2}\right]\right] \end{aligned} \tag{17}$$

Equation (17) gives us the time at a particular point; to find the time taken by the ray between two points we have:

1. For Direct rays: $\Delta t = t(L, z_0) - t(L, z1)$
2. For Reflected rays: $\Delta t = t(L, z_0) - t(L, 0) + t(L, z_1) - t(L, 0)$
3. For Refracted rays: $\Delta t = t(L, z_0) - t(L, z_{max}) + t(L, z_1) - t(L, z_{max})$