Ray Tracing in Antartica

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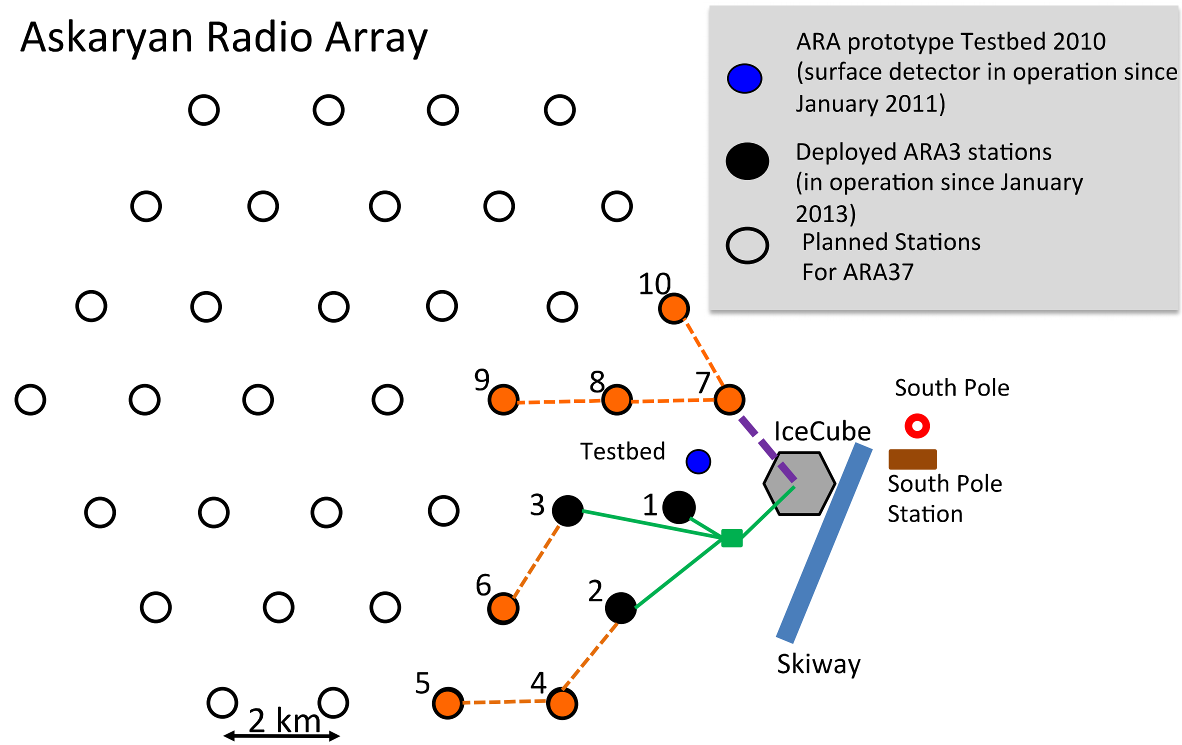
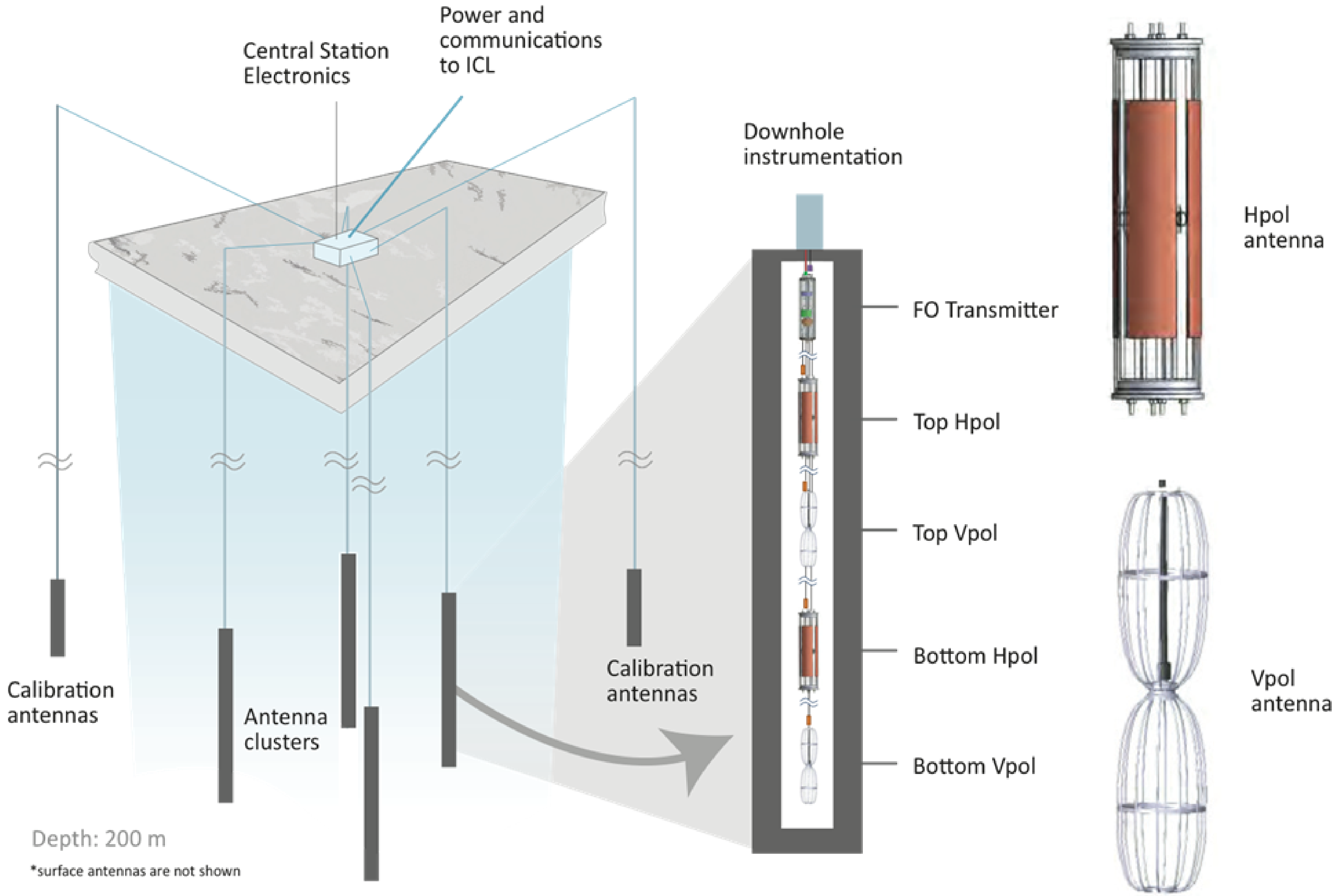
ARA

- Askaryan Radio Array
- Planned Hexagonal Array of 37 stations in the ice sheet 200 m deep.
- 3 of 37 planned stations currently deployed plus ARA testbed.
- Stations powered from the IceCube station.
- Bandwidth: 150 to 850 MHz
- Effective Volume:O(100 km^3).
- Expectation:
  - 0.2 GZK neutrinos in 10 months
  - 1000 impulsive RF events – non-thermal
  - Rest are thermal noise triggers (~150mil)
- Have to wait 1 year for data because ARA only gets 1 GB/day of satellite transfer (1% of IceCube) so tapes, now disks, have to be physically transported to Madison and placed online.

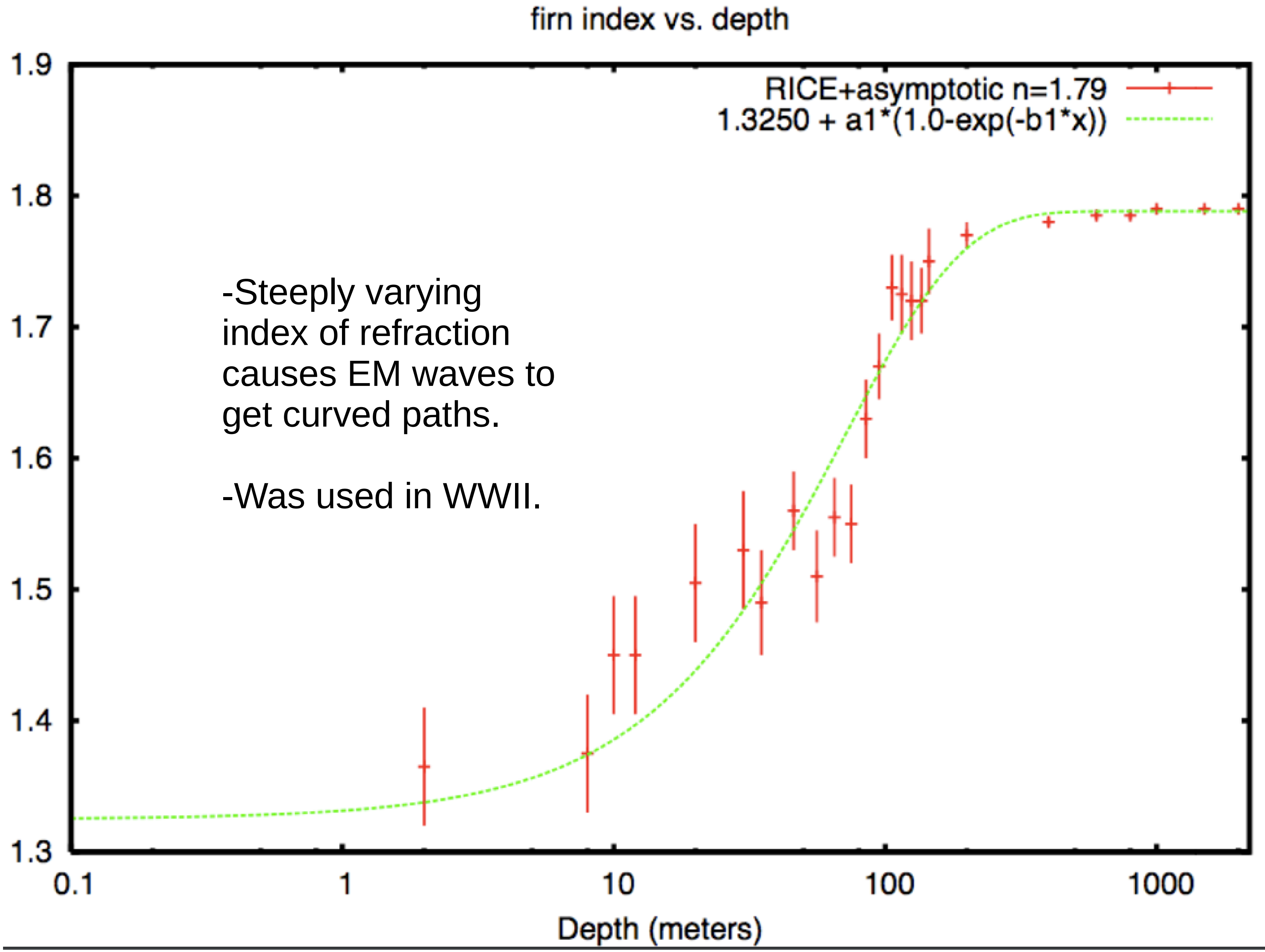


Basic Station Geometry



Important for ARA and for my thesis.

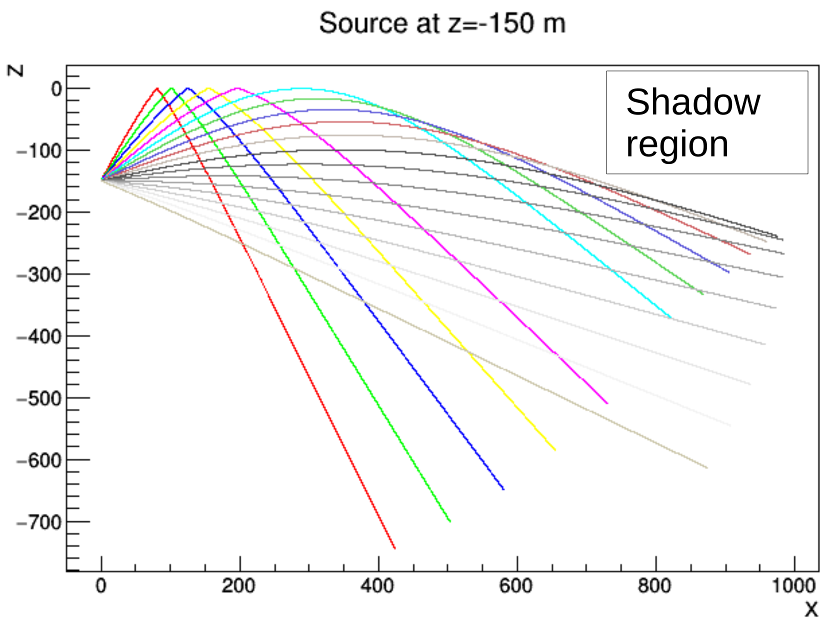
Raytracing

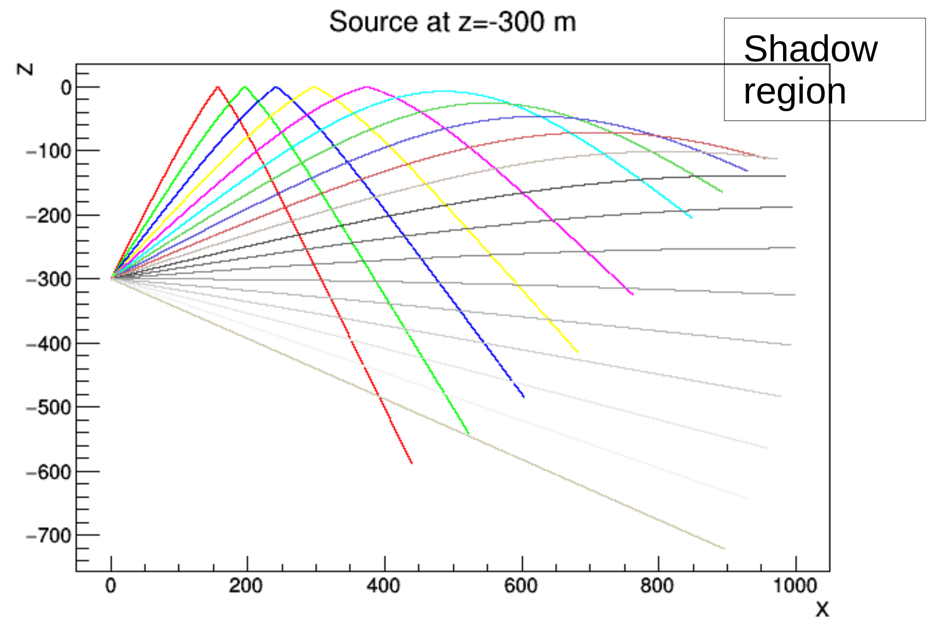


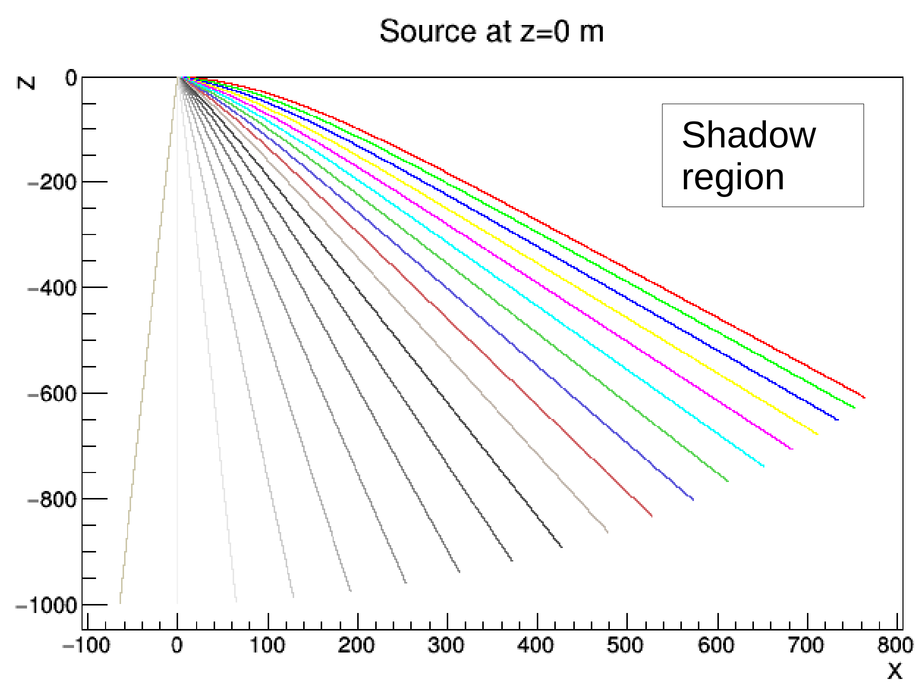
-Steeply varying index of refraction causes EM waves to get curved paths.

-Was used in WWII.

How Rays behave





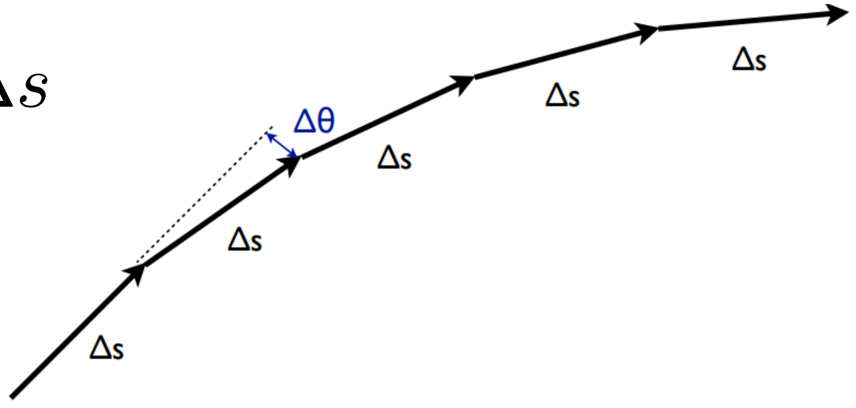


How to Simulate Raytracing

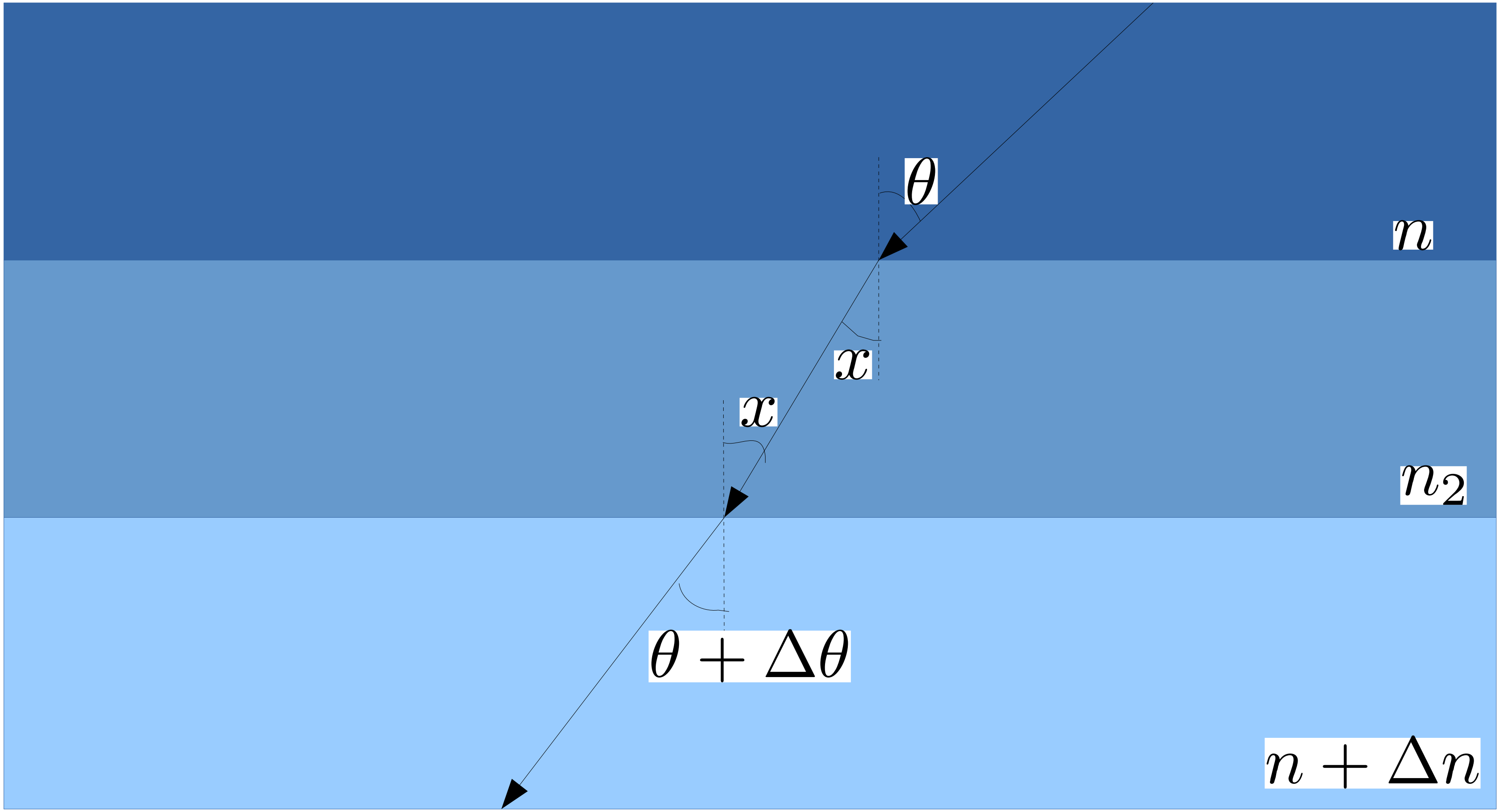
- I will be summarizing the work done by Christopher Weaver in Ohio state on this topic in 2010.

- **Most Simple method**: We can extrapolate curved paths by applying Euler's method to Snell's Law.
  - Simple but horribly slow

$$\Delta\theta = -\frac{1}{n(z)}\sin(\theta)\frac{dn}{dz}\Delta s$$



How-I?



$$n \sin(\theta) = n_2 \sin(x)$$

$$n_2 \sin(x) = (n + \Delta n) \sin(\theta + \Delta\theta)$$

$$\Rightarrow \frac{\sin(\theta + \Delta\theta)}{\sin(\theta)} = \frac{n}{n + \Delta n}$$

Still on How-I?

$$\text{if } \Delta\theta \ll 1 \text{ and } \Delta n \ll 1$$

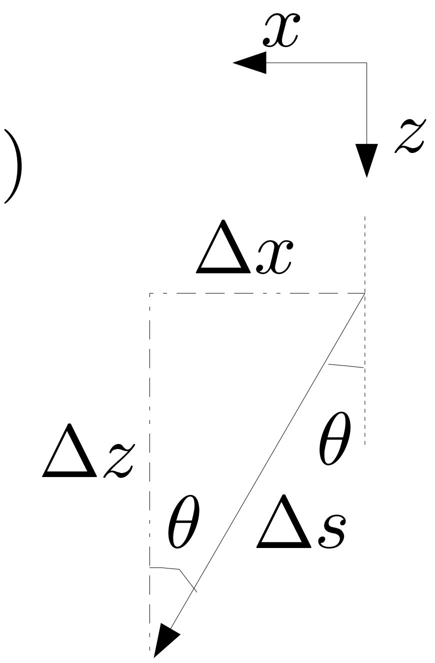
$$\Rightarrow \frac{\sin(\theta + \Delta\theta)}{\sin(\theta)} = \frac{n}{n + \Delta n}$$

$$\Rightarrow \frac{\sin(\theta)\cos(\Delta\theta)}{\sin(\theta)} + \frac{\cos(\theta)\sin(\Delta\theta)}{\sin(\theta)} = \left(1 + \frac{\Delta n}{n}\right)^{-1}$$

$$\Rightarrow 1 + \cot(\theta)\Delta\theta = 1 - \frac{\Delta n}{n}$$

$$\therefore \frac{\Delta n}{n} = -\cot(\theta)\Delta\theta$$

$$\therefore\ \frac{\Delta z}{\Delta s} = \cos(\theta)$$



$$\Rightarrow\ \frac{\Delta n}{n}\frac{\sin(\theta)}{\cos(\theta)} = -\Delta\theta$$

$$\Rightarrow \Delta\theta = -\frac{\Delta n}{n}\frac{\sin(\theta)\Delta s}{\Delta z}$$

$$\therefore\ \frac{d\theta}{ds} = -\frac{\sin(\theta)}{n(z)}\frac{dn}{dz}\bigg|_z$$

Semi-Analytical approach

- Use the following equations with Runge-Kutta (RK4) method:

$$\frac{dx}{ds} = \sin(\theta) \qquad \frac{d\theta}{ds} = -\frac{\sin(\theta)}{n(z)}\frac{dn}{dz}\bigg|_z$$

$$\frac{dz}{ds} = \cos(\theta) \qquad \frac{dt}{ds} = \frac{n(z)}{c} \qquad \frac{dA_0}{ds} = \frac{A_0}{L(z,\omega)}$$

Need initial conditions . Can provide with x_0, z_0 and x, z

What about the launch angle? θ_0

Can use an analytical expression to give us the value of θ_0.

Journey to find the launch angle

$$n(z) = A + Be^{Cz}$$

$$\frac{dn}{dz} = BCe^{Cz}$$

(x_0, z_0) to (x, z)

$$\Rightarrow dn = BCe^{Cz}dz$$

$$\Rightarrow \frac{dn}{n} = \frac{BCe^{Cz}}{A + Be^{Cz}}dz$$

Using $\frac{dn}{n} = -\cot\theta d\theta$

$$\Rightarrow \frac{BCe^{Cz}}{A + Be^{Cz}}dz = -\cot(\theta)d\theta$$

$$\Rightarrow \frac{BCe^{Cz}}{A+Be^{Cz}}dz = -\cot(\theta)d\theta$$

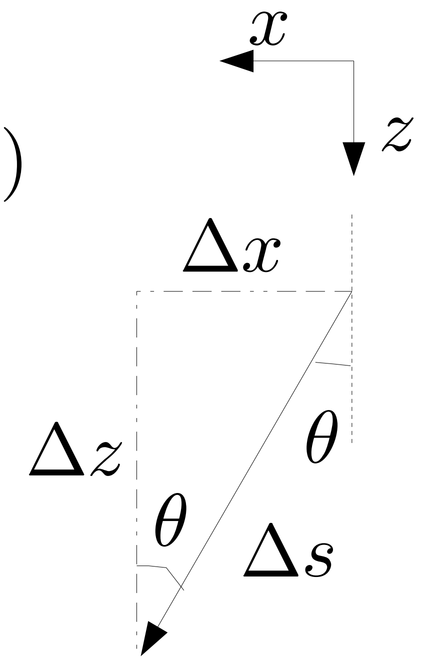
$$\Rightarrow \ln(A+Be^{Cz'})\Big|_{z_0}^{z} = -\ln(\sin(\theta'))|_{\theta_0}^{\theta}$$

$$\Rightarrow \frac{A+BCe^{Cz}}{A+Be^{Cz_0}} = \frac{\sin(\theta_0)}{\sin(\theta)}$$

$$\Rightarrow \theta = \arcsin\left(\sin(\theta_0)\frac{A+BCe^{Cz_0}}{A+Be^{Cz}}\right)$$

Step 2:

$$\therefore \frac{\Delta x}{\Delta z} = \tan(\theta)$$



$$\Rightarrow \frac{dx}{dz} = \tan(\theta)$$

$$\Rightarrow dx = \tan\left(\arcsin\left(\sin(\theta_0)\frac{A+BCe^{Cz_0}}{A+Be^{Cz}}\right)\right)dz$$

$$\Rightarrow x - x_0 = \int_{z_0}^{z} \tan\left(\arcsin\left(\sin(\theta_0)\frac{A+BCe^{Cz_0}}{A+Be^{Cz'}}\right)\right)dz'$$

$$\Rightarrow x - x_0 = \sin(\theta_0)\int_{z_0}^{z} \frac{A+BCe^{Cz_0}}{A+Be^{Cz'}\sqrt{1-\left(\frac{\sin(\theta_0)(A+Be^{Cz_0})}{A+Be^{Cz'}}\right)^2}}dz'$$

How-II?

$$\tan\left(\arcsin\left(\frac{a}{b}\right)\right) = \frac{a}{b\sqrt{1-\frac{a^2}{b^2}}}$$

$$\Rightarrow x - x_0 = \int_{z_0}^{z} \tan\left(\arcsin\left(\sin(\theta_0)\frac{A + BCe^{Cz_0}}{A + Be^{Cz'}}\right)\right) dz'$$

$$\Rightarrow x - x_0 = \sin(\theta_0)\int_{z_0}^{z} \frac{A + BCe^{Cz_0}}{A + Be^{Cz'}\sqrt{1-\left(\frac{\sin(\theta_0)(A+Be^{Cz_0})}{A+Be^{Cz'}}\right)^2}} dz'$$

Back to the problem!

Step 3:

Using substitution:

$$\sigma_0 = \sin(\theta_0)$$

$$u = A + Be^{Cz} \;\; \Rightarrow du = BCe^{Cz}dz \;\; \Rightarrow dz = \frac{du}{C(u-A)}$$

We can write

$$x - x_0 = \frac{\sigma_0 u_0}{C} \int_{u_0}^{u} \frac{du'}{u'(u'-A)\sqrt{1-\left(\frac{\sigma_0 u_0}{u'}\right)^2}}$$

$$\Rightarrow \frac{C}{\sigma_0 u_0}(x - x_0) = \int_{u_0}^{u} \frac{du'}{u'(u'-A)\sqrt{1-\left(\frac{\sigma_0 u_0}{u'}\right)^2}}$$

Using more substitution:

$$v = u - A \quad \Rightarrow \quad dv = du$$

$$\frac{C}{\sigma_0 u_0}(x - x_0) = \int_{u_0}^{u} \frac{du'}{u'(u' - A)\sqrt{1 - \left(\frac{\sigma_0 u_0}{u'}\right)^2}}$$

$$\Rightarrow \frac{C}{\sigma_0 u_0}(x - x_0) = \int_{v_0}^{v} \frac{dv'}{v'\sqrt{v'^2 + 2Av' + A^2 - (\sigma_0 u_0)^2}}$$

This has the form $\int \frac{dx}{x\sqrt{X}}$ where $X = a + bx + cx^2$

where in our case: $a = A^2 - (\sigma_0 u_0)^2$

$$b = 2A, c = 1$$

This has the form $\int \frac{dx}{x\sqrt{X}}$ where $X = a + bx + cx^2$

where in our case: $a = A^2 - (\sigma_0 u_0)^2 \quad b = 2A, c = 1$

- The solution to integrals of this form depends on the sign of a.
- For the Antarctic ice, A = 1.78 (asymptotic value of n) in the deep ice (below 200 m).
- σ0 ∈ [0, 1] and u0 is the index of refraction at the starting point of the ray, so u0 ≤ A.
- This means that the case of interest is a > 0 . In this case, the solution to the basic integral form is

$$\int \frac{dx}{x\sqrt{X}} = \frac{-1}{\sqrt{a}} \ln\left(\frac{\sqrt{X}+\sqrt{a}}{x} + \frac{b}{2\sqrt{a}}\right)$$

Finally! Wohoo!

$$u = A + Be^{Cz} \qquad \sigma_0 = \sin(\theta_0)$$

$$\frac{C\sqrt{A^2-(\sigma_0 u_0)^2}}{\sigma_0 u_0}(x_0 - x) = \ln\left(\frac{\sqrt{u^2-(\sigma_0 u_0)^2}+\sqrt{A^2-(\sigma_0 u_0)^2}}{u-A} + \frac{A}{\sqrt{A^2+({}_0 u_0)^2}}\right)$$

$$-\ln\left(\frac{\sqrt{u_0^2-(\sigma_0 u_0)^2}+\sqrt{A^2-(\sigma_0 u_0)^2}}{u_0-A} + \frac{A}{\sqrt{A^2+({}_0 u_0)^2}}\right)$$

- Cannot be solved for u
  - So cannot be solved for z as function of x.
- Cannot be solved for launch angle (σ0) to find the ray between and start and stop points.
- Can be rearranged to solved numerically for launch angle

$$0 = -\frac{C\sqrt{A^2-(\sigma_0 u_0)^2}}{\sigma_0 u_0}(x_0 - x) + \ln\left(\frac{\sqrt{u^2-(\sigma_0 u_0)^2}+\sqrt{A^2-(\sigma_0 u_0)^2}}{u-A} + \frac{A}{\sqrt{A^2+({}_0 u_0)^2}}\right)$$

$$-\ln\left(\frac{\sqrt{u_0^2-(\sigma_0 u_0)^2}+\sqrt{A^2-(\sigma_0 u_0)^2}}{u_0-A} + \frac{A}{\sqrt{A^2+({}_0 u_0)^2}}\right)$$