Ch 7.2-7.4: Step Functions, Basis Functions, Start Splines

Lecture 23 - CMSE 381

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Announcements

Last time:

- 7.1 Polynomial regression
- 7.2 Step functions

This lecture:

- 7.2 Step functions
- 7.3 Basis functions
- 7.4 Regression Splines (Finish next lecture)

Announcements:

- Hw #6 Due Monday

Section 1

Last time

Polynomial regression

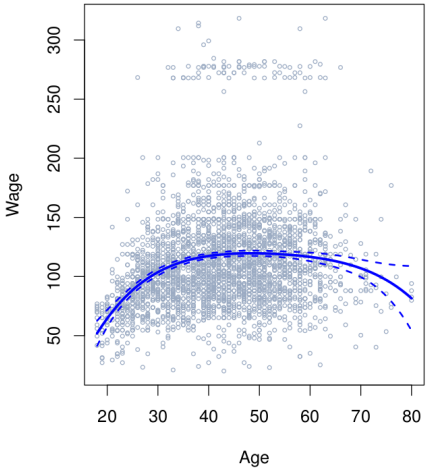
Replace linear model

$$y_i = \beta_0 + \beta_1 x_1 + \varepsilon_i$$

with

$$y_i = \beta_0 + \beta_1 x_1 + \beta_2 x_i^2 + \cdots + \beta_d x_i^d + \varepsilon_i$$

Example with wage data



$$-184.1542 + 21.24552 * age + -0.56386 * age^2 + 0.00681 * age^3 + -3e-05 * age^4$$

Step functions

- $\mathrm{I}(X < c)$
- $\mathrm{I}(c_1 \leq X < c_2)$
- $\mathrm{I}(c \leq X)$

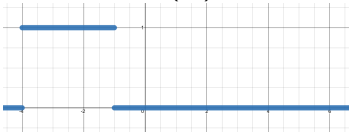
$$
\begin{aligned}
C_0(X) &= I(X < c_1), \\
C_1(X) &= I(c_1 \leq X < c_2), \\
C_2(X) &= I(c_2 \leq X < c_3), \\
&\vdots \\
C_{K-1}(X) &= I(c_{K-1} \leq X < c_K), \\
C_K(X) &= I(c_K \leq X),
\end{aligned}
$$

Learned model:

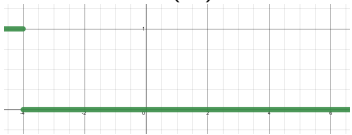
$$y_i = \beta_0 + \beta_1 C_1(x_i) + \beta_2 C_2(x_i) + \cdots + \beta_K C_K(x_i) + \varepsilon_i$$

Example: Knots at -4, -1, 3, 6

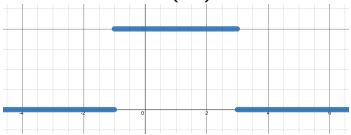
$C_1(X)$



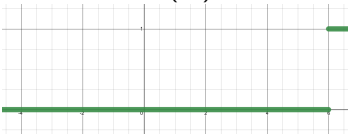
$C_0(X)$



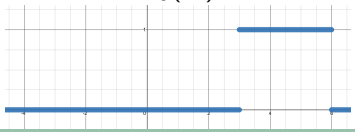
$C_2(X)$



$C_4(X)$

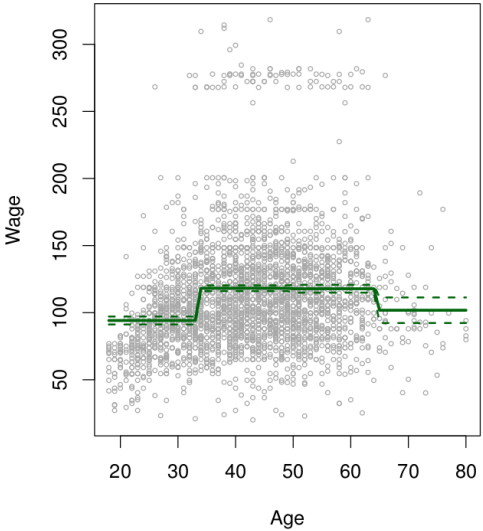


$C_3(X)$



Coding with step functions

Step function example

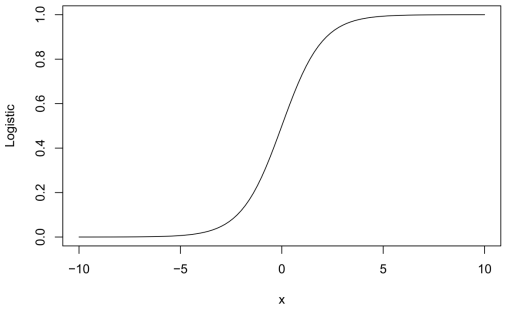


Section 2

Classification versions

Remember logisitic regression?

$$y = \frac{e^x}{1 + e^x}$$



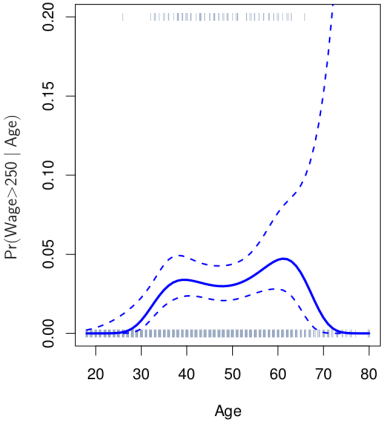
$$p(X) = \frac{e^{\beta_0+\beta_1 X}}{1 + e^{\beta_0+\beta_1 X}}$$

Multiple features:

$$p(X) = \frac{e^{\beta_0+\beta_1 X_1+\cdots+\beta_p X_p}}{1 + e^{\beta_0+\beta_1 X_1+\cdots+\beta_p X_p}}$$

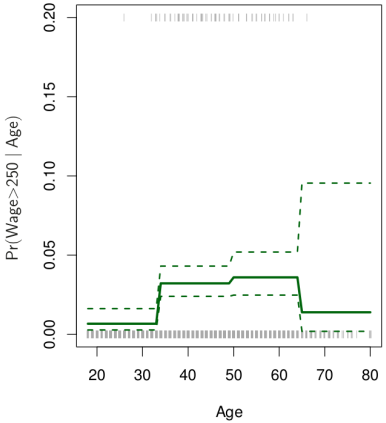
Classification version: Polynomial regression

$$\Pr(y_i > 250 \mid x_i) = \frac{\exp(\beta_0 + \beta_1 x_i + \cdots + \beta_d x_i^d)}{1 + \exp(\beta_0 + \beta_1 x_i + \cdots + \beta_d x_i^d)}$$

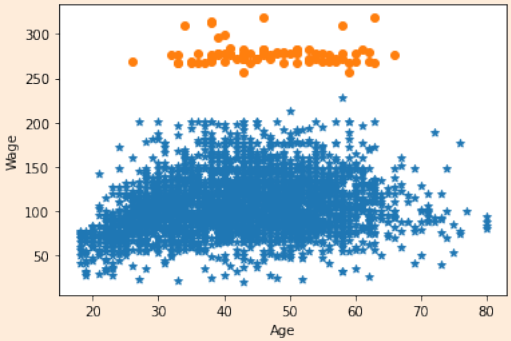


Classification version: Step functions

$$\Pr(y_i > 250 \mid x_i) = \frac{\exp(\beta_0 + \beta_1 C_1(x_i) + \beta_2 C_2(x_i) + \cdots + \beta_K C_K(x_i))}{1 + \exp(\beta_0 + \beta_1 C_1(x_i) + \beta_2 C_2(x_i) + \cdots + \beta_K C_K(x_i))}$$



Coding bit: classification version



A few more comments on step functions

Section 3

Basis functions

Basis Functions Setup

Polynomial and piecewise-constant regression models are special cases of a *basis function* approach.

$$y_i = \beta_0 + \beta_1 b_1(x_i) + \beta_2 b_2(x_i) + \cdots + \beta_K b_K(x_i) + \varepsilon_i$$

Section 4

Regression Splines

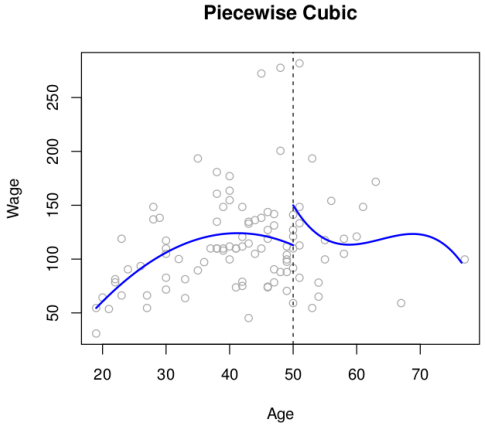
Piecewise polynomials

- Fit a polynomial regression

$$y_i = \beta_0 + \beta_1 x_1 + \beta_2 x_i^2 + \cdots + \beta_d x_i^d + \varepsilon_i$$

- Let the β_i's be different at different locations of the range.

Example of piecewise polynomial

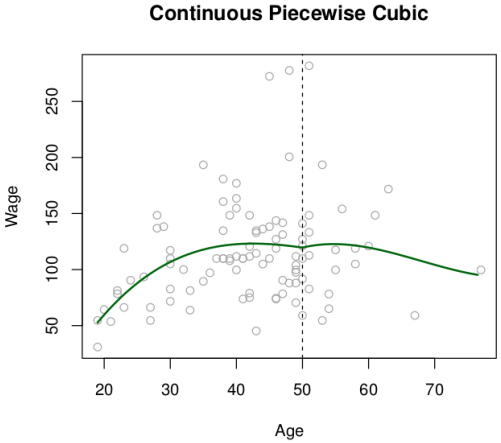


Example:

$$y_i = \begin{cases} \beta_{01} + \beta_{11} x_i + \beta_{21} x_i^2 + \beta_{31} x_i^3 + \epsilon_i & \text{if } x_i < c \\ \beta_{02} + \beta_{12} x_i + \beta_{22} x_i^2 + \beta_{32} x_i^3 + \epsilon_i & \text{if } x_i \geq c. \end{cases}$$

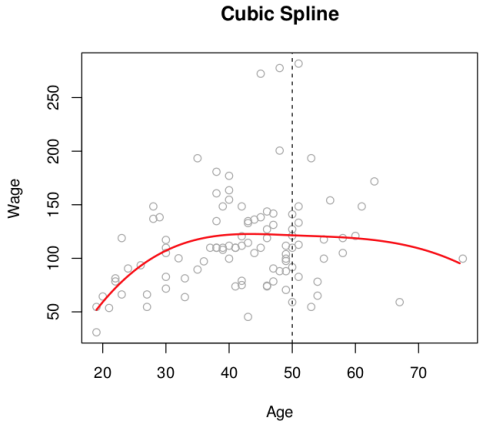
The fix

- Fit piecewise polynomial
- Require continuity at knots



The better fix: Cubic splines

- Fit piecewise polynomial
- Require continuity at knots
- Require the first and second derivatives to be continuous at knots

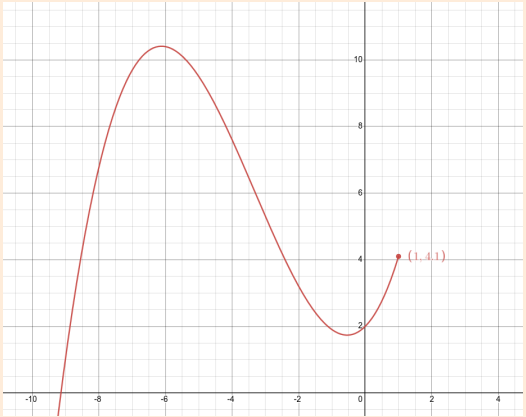


Example

We have the following piecewise cubic polynomial:

$$f(x) = \begin{cases} 2 + x + x^2 + 0.1x^3 & x \leq 1 \\ b_0 + b_1 x + b_2 x^2 - x^3 & x > 1 \end{cases}$$

What are b_1, b_1, and b_2 to make this a cubic spline?



Check your answers: desmos.com/calculator/ns4tr7mw0n

More space for work

$$f(x) = \begin{cases} 2 + x + x^2 + 0.1x^3 & x \leq 1 \\ b_0 + b_1 x + b_2 x^2 - x^3 & x > 1 \end{cases}$$

Next time

Status	Lec #	Date			Reading	Homeworks
Done	20	Fri	Oct 27	Dimension Reduction	6.3	
Done	21	Mon	Oct 30	More dimension reduction; High dimensions	6.4	
Done	22	Wed	Nov 1	Polynomial & Step Functions	7.1,7.2	
	23	Fri	Nov 3	Step Functions; Basis functions; Start Splines	7.2 - 7.4	
	24	Mon	Nov 6	Regression Splines	7.4	HW #6 Due
	25	Wed	Nov 8	Decision Trees	8.1	
	26	Fri	Nov 10	Random Forests	8.2.1, 8.2.2	
	27	Mon	Nov 13	Maximal Margin Classifier	9.1	
	28	Wed	Nov 15	SVC	9.2	
	29	Fri	Nov 17	SVM	9.3, 9.4	
	30	Mon	Nov 20	Single layer NN	10.1	
	31	Wed	Nov 22	Virtual: Project office hours		
		Fri	Nov 24	No class - Thanksgiving		
		Mon	Nov 27	**Review**		
		Wed	Nov 29	**Midterm #3**		
	32	Fri	Dec 1	Multi Layer NN	10.2	
	33	Mon	Dec 4	CNN	10.3	
	34	Wed	Dec 6	Unsupervised Learning & Clustering	12.1, 12.4	
	35	Fri	Dec 8	Virtual: Project office hours		Project due