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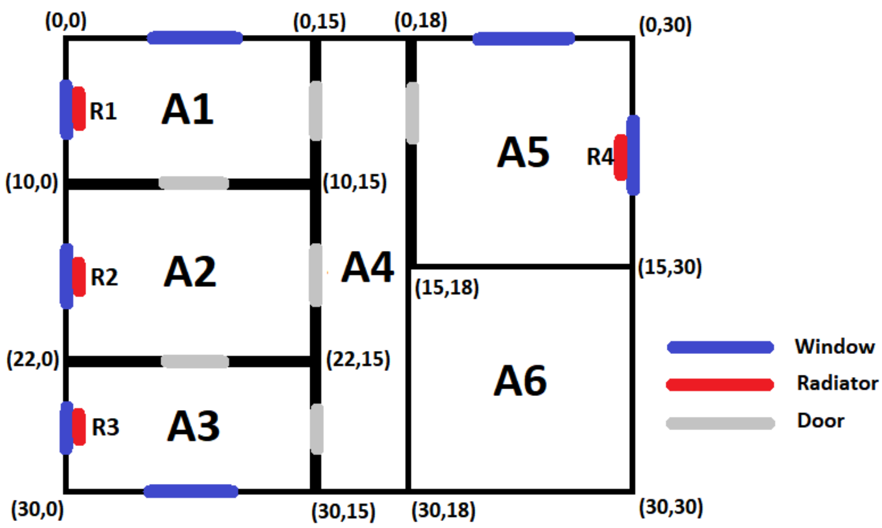
Apartment heating

1. Mathematical description of the model

Consider the following floor plan of a house consisting of five rooms (A1, A2, A3, A4, A5), as well as an outdoor area designated as A6, representing any open space outside the house.



Let $\Omega = \bigcup_{k=1}^{5} A_k$. Let D denote the coordinates of doors, W denote the coordinates of windows, R denote the coordinates of radiators, and S define the temperature ranges in which a particular radiator heats. The equation describing the temperature distribution in this house is as follows:

$$\begin{cases} \underset{\left[\frac{K}{s}\right]}{u_t(t,x,y)} = \underset{\left[\frac{m^2}{s}\right]}{\mathcal{D}}\underset{\left[\frac{K}{m^2}\right]}{\Delta u(t,x,y)} + \underset{\left[\frac{K}{s}\right]}{f(x,y,u(t,x,y))}, & (x,y) \in \{A_k : k = 1,\dots,5\} \cup \{D\} \\ \underset{\left[\frac{K}{s}\right]}{u_x(t,x,y)} = \underset{\left[\frac{K}{s}\right]}{g(t)} = 0, & (x,y) \in \partial\Omega \\ \underset{\left[\frac{K}{s}\right]}{u(t,x,y)} = \underset{\left[\frac{K}{s}\right]}{o(t)}, & (x,y) \in \{W\} \end{cases}$$

where

- $f(x,y,u(t,x,y))$ is a function that generates heat δ_{temp} from radiator R_i if the average temperature in room A_k falls within the temperature range S_i.

$$f(x,y,u(t,x,y)) = \sum_{i \in R} \delta_{temp} \cdot \mathbb{1}\{(x,y) : (x,y) \in R_i\} \cdot \mathbb{1}\left\{u : \int_{A_k} u(t,x,y)\ dx \ \in S_i\right\}$$

- $g(t)$ determines how the temperature changes at the boundary of the area A_k with respect to time t. If $g(t) = 0$, we say that no flux passes through this boundary.
- $o(t)$ is a function that defines the outside air temperature.
- $\mathcal{D}$ is the diffusion coefficient.

At the outset, let's solve the equation:

$$u_t(t,x,y) = \mathcal{D}\Delta u(t,x,y) + f(x,y,u(t,x,y)), \qquad (x,y) \in \{A_k : k = 1,\dots,5\} \cup \{D\}$$

We integrate the first equation with respect to time step h_t and get

$$\int_t^{t+h_t} u_t(s,x,y)\, ds = \int_t^{t+h_t} \mathcal{D}\Delta u(s,x,y)\, ds + \int_t^{t+h_t} f(x,u(s,x,y))\, ds$$

The left hand side is obvious. The right hand side can be approximated with *leftpoint rectangle* method of numerical integration. This results in the following equation

$$u(t+h_t,x) = u(t,x) + h_t\mathcal{D}\Delta u(x,t) + h_t f(x,u(t,x))$$

We are almost there. The second term on the right hand side of an equation can be discretized using the central difference method. We compute with space step h_x, h_y in both directions

$$\begin{aligned}
\Delta u(t,x,y) &\approx \frac{\partial^2 u(t,x,y)}{\partial x^2} + \frac{\partial^2 u(t,x,y)}{\partial y^2} \\
&\approx \frac{u(t,x+h_x,y) - 2u(t,x,y) + u(t,x-h_x,y)}{h_x^2} + \frac{u(t,x,y+h_y) - 2u(t,x,y) + u(t,x,y-h_y)}{h_y^2}
\end{aligned}$$

But we will usually take $h_x = h_y$ so the above equation simplifies to the following form

$$\Delta u(t,x,y) \approx \frac{1}{h_x^2}\Big(u(t,x+h_x,y) + u(t,x-h_x,y) + u(t,x,y-h_x) + u(t,x,y+h_x) - 4u(t,x,y)\Big)$$

Putting this all together yields the following formula:

$$\begin{aligned}
u(t+h_t,x,y) =& u(t,x,y) + \\
& \frac{\mathcal{D}h_t}{h_x^2}\Big(u(t,x+h_x,y) + u(t,x-h_x,y) + u(t,x,y-h_x) + u(t,x,y+h_x) - 4u(t,x,y)\Big) \\
& + h_t f(x,y,u(t,x,y))
\end{aligned}$$

Obviously this is super unreadable, hence we introduce the following notation.

$$A_k = [0,L_k] \times [0,M_k], k = 1, ...5, \text{ which is discretized into partitions as follows}$$

$$\Omega = \Big([x_0,x_1] \times [y_0,y_1]\Big) \cup \Big([x_1,x_2] \times [y_0,y_1]\Big) \cup \dots \cup \Big([x_{L_k-1},x_{L_k}] \times [y_0,y_1]\Big)$$

$$\cup$$

$$\dots$$

$$\cup$$

$$\Big([x_0,x_1] \times [y_{M_k-1},y_M]\Big) \cup \Big([x_1,x_2] \times [y_{M_k-1},y_{M_k}]\Big) \cup \dots \cup \Big([x_{L_k-1},x_{L_k}] \times [y_{M_k-1},y_{M_k}]\Big)$$

$$[0,T], \text{ which is discretized into partitions as follows}$$

$$[0,T] = [t_0,t_1] \cup [t_1,t_2] \cup \dots \cup [t_{T-1},t_T]$$

In this case it is convienient to think of a solution u of the above equation as 3-dimensional matrix. The first dimension represents time information, while the next two dimensions encapsulate spatial information. This is very convienient to imagine the solution u in a matrix form like this

$t = t_0 = 0$

$$\begin{pmatrix} u_{0,0,0} & u_{0,1,0} & \cdots & u_{0,L_k-1,0} & u_{0,L_k,0} \\ u_{0,0,1} & u_{0,1,1} & \cdots & u_{0,L_k-1,1} & u_{0,L_k,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ u_{0,0,M_k-1} & u_{0,1,M_k-1} & \cdots & u_{0,L_k-1,M_k-1} & u_{0,L_k,M_k-1} \\ u_{0,0,M_k} & u_{0,1,M_k} & \cdots & u_{0,L_k-1,M_k} & u_{0,L,M_k} \end{pmatrix}$$

$t = t_1$

$$\begin{pmatrix} u_{1,0,0} & u_{1,1,0} & \cdots & u_{1,L_k-1,0} & u_{1,L_k,0} \\ u_{1,0,1} & u_{1,1,1} & \cdots & u_{1,L_k-1,1} & u_{1,L_k,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ u_{1,0,M_k-1} & u_{1,1,M_k-1} & \cdots & u_{1,L_k-1,M_k-1} & u_{1,L_k,M_k-1} \\ u_{1,0,M_k} & u_{1,1,M_k} & \cdots & u_{1,L_k-1,M_k} & u_{1,L_k,M_k} \end{pmatrix}$$

$t = t_k$

$$\cdots$$

$t = t_{T-1}$

$$\begin{pmatrix} u_{T-1,0,0} & u_{T-1,1,0} & \cdots & u_{T-1,L_k-1,0} & u_{T-1,L_k,0} \\ u_{T-1,0,1} & u_{T-1,1,1} & \cdots & u_{T-1,L_k-1,1} & u_{T-1,L_k,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ u_{T-1,0,M_k-1} & u_{T-1,1,M_k-1} & \cdots & u_{T-1,L_k-1,M_k-1} & u_{T-1,L_k,M_k-1} \\ u_{T-1,0,M_k} & u_{T-1,1,M_k} & \cdots & u_{T-1,L_k-1,M_k} & u_{T-1,L_k,M_k} \end{pmatrix}$$

$t = t_T = T$

$$\begin{pmatrix} u_{T,0,0} & u_{T,1,0} & \cdots & u_{T,L_k-1,0} & u_{T,L_k,0} \\ u_{T,0,1} & u_{T,1,1} & \cdots & u_{T,L_k-1,1} & u_{T,L_k,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ u_{T,0,M_k-1} & u_{T,1,M_k-1} & \cdots & u_{T,L_k-1,M_k-1} & u_{T,L_k,M_k-1} \\ u_{T,0,M_k} & u_{T,1,M_k} & \cdots & u_{T,L_k-1,M_k} & u_{T,L_k,M_k} \end{pmatrix}$$

Using this notation we can simplify the formula as follows

$$u_{s+1,i,j,} = u_{s,i,j} + \frac{Dh_t}{h_x^2}\Big(u_{s,i+1,j} + u_{s,i-1,j} + u_{s,i,j+1} + u_{s,i,j-1} - 4u_{s,i,j}\Big) + h_t f(x_i, y_i, u_{s,i,j})$$

Next we need to calculate the Neumann boundary conditions

$$u_x(t,x,y)\Big|_{x\in\partial\Omega} = g(t)$$

Our goal is to use the following condition:

$$\frac{\partial u}{\partial n}(t,x,y) = \nabla u(t,x,y) n(x,y) = g(t)$$

First we easily discretize the gradient respectively with *leftpoint*, *rightpoint* methods and their combinations

$$\nabla u(t,x,y) = \Big(u_x(t,x,y), u_y(t,x,y)\Big) \approx \Big(\frac{u(t,x+h_x,y)-u(t,x,y)}{h_x}, \frac{u(t,x,y+h_x)-u(t,x,y)}{h_x}\Big),$$
$$\nabla u(t,x,y) = \Big(u_x(t,x,y), u_y(t,x,y)\Big) \approx \Big(\frac{u(t,x,y)-u(t,x-h_x,y)}{h_x}, \frac{u(t,x,y+h_x)-u(t,x,y)}{h_x}\Big),$$
$$\nabla u(t,x,y) = \Big(u_x(t,x,y), u_y(t,x,y)\Big) \approx \Big(\frac{u(t,x+h_x,y)-u(t,x,y)}{h_x}, \frac{u(t,x,y)-u(t,x,y-h_y)}{h_x}\Big),$$
$$\nabla u(t,x,y) = \Big(u_x(t,x,y), u_y(t,x,y)\Big) \approx \Big(\frac{u(t,x,y)-u(t,x-h_x,y)}{h_x}, \frac{u(t,x,y)-u(t,x,y-h_x)}{h_x}\Big),$$

which can be rewritten with our standard notation

$$\nabla u(t,x,y) \approx \Big(\frac{u_{s,i+1,j}-u_{s,i,j}}{h_x}, \frac{u_{s,i,j+1}-u_{s,i,j}}{h_x}\Big), \tag{1}$$
$$\nabla u(t,x,y) \approx \Big(\frac{u_{s,i,j}-u_{s,i-1,j}}{h_x}, \frac{u_{s,i,j+1}-u_{s,i,j}}{h_x}\Big), \tag{2}$$
$$\nabla u(t,x,y) \approx \Big(\frac{u_{s,i+1,j}-u_{s,i,j}}{h_x}, \frac{u_{s,i,j}-u_{s,i,j-1}}{h_x}\Big), \tag{3}$$
$$\nabla u(t,x,y) \approx \Big(\frac{u_{s,i,j}-u_{s,i-1,j}}{h_x}, \frac{u_{s,i,j}-u_{s,i,j-1}}{h_x}\Big). \tag{4}$$

We need to distinguish 4 cases

1. $x = 0, y \in (0, M_k)$
   In this case the normal vector is $n(x,y) = (-1,0)$
   Now we use the discretization (1) and get

$$\Big(\frac{u_{s,1,j}-u_{s,0,j}}{h_x}, \frac{u_{s,0,j+1}-u_{s,0,j}}{h_x}\Big) \cdot (-1,0) = \frac{u_{s,0,j}-u_{s,1,j}}{h_x} = g(t_s)$$

   Which translates into (with $s := s+1$)

$$u_{s+1,0,j} = u_{s+1,1,j} + h_x g(t_{s+1})$$

2. $x = L_k, y \in (0, M_k)$
   In this case the normal vector is $n(x,y) = (1,0)$
   Now we use the discretization (2) and get

$$\Big(\frac{u_{s,L,j}-u_{s,L-1,j}}{h_x}, \frac{u_{s,L,j+1}-u_{s,L,j}}{h_x}\Big) \cdot (1,0) = \frac{u_{s,L,j}-u_{s,L-1,j}}{h_x} = g(t_s)$$

   Which translates into

$$u_{s+1,L,j} = u_{s+1,L-1,j} + h_x g(t_{s+1})$$

3. $x \in (0, L_k), y = 0$
   In this case the normal vector is $n(x,y) = (0,-1)$
   Now we use the discretization (1) and get

$$\Big(\frac{u_{s,i,1}-u_{s,i,0}}{h_x}, \frac{u_{s,i,1}-u_{s,i,0}}{h_x}\Big) \cdot (0,-1) = \frac{u_{s,i,0}-u_{s,i,1}}{h_x} = g(t_s)$$

   Which translates into

$$u_{s+1,i,0} = u_{s+1,i,1} + h_x g(t_{s+1})$$

4. $x \in (0, L_k), y = M_k$
In this case the normal vector is $n(x, y) = (0, 1)$
Now we use the discretization (4) and get

$$\left(\frac{u_{s,i,M} - u_{s,i,M-1}}{h_x}, \frac{u_{s,i,M} - u_{s,i,M-1}}{h_x}\right) \cdot (0, 1) = \frac{u_{s,i,M} - u_{s,i,M-1}}{h_x} = g(t_s)$$

Which translates into

$$u_{s+1,i,M} = u_{s+1,i,M-1} + h_x g(t_{s+1})$$

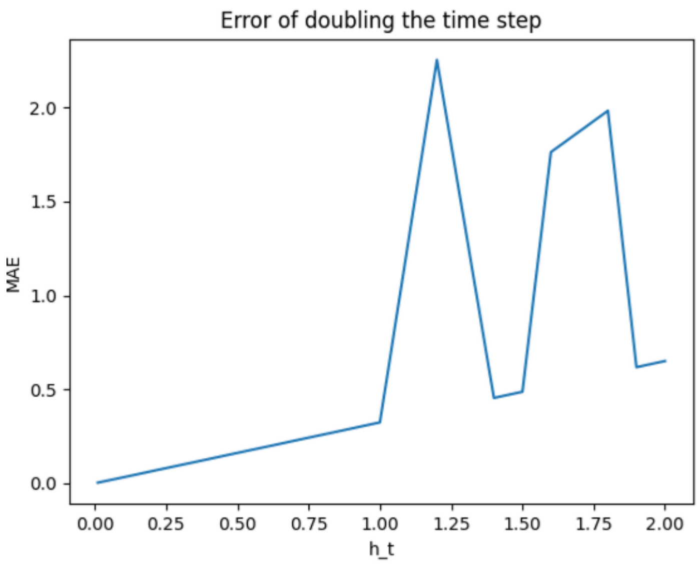
The final numerical scheme for $u_{s+1,i,j}$ is given by

$$\begin{cases} u_{s,i,j} + \frac{h_{step}}{h_{step}^2}\left(u_{s,i+1,j} + u_{s,i-1,j} + u_{s,i,j+1} + u_{s,i,j-1} - 4u_{s,i,j}\right) +, & (x_i, y_j) \notin \partial A_k \ \vee \ (x_i, y_j) \in \{D\}, \\ \qquad +f(x_i, y_j, u_{s,i,j}) & \\ u_{s+1,L_k-1,j}, & x_i = L, 0 \leqslant y_j \leqslant M \wedge (x_i, y_j) \notin \{D\}, \\ u_{s+1,1,j}, & x_i = 0, 0 \leqslant y_j \leqslant M_k \wedge (x_i, y_j) \notin \{D\}, \\ u_{s+1,i,M_k-1}, & 0 \leqslant x_i \leqslant L_k, y_j = M_k \wedge (x_i, y_j) \notin \{D\}, \\ u_{s+1,i,1}, & 0 \leqslant x_i \leqslant L_k, y_j = 0 \wedge (x_i, y_j) \notin \{D\}, \\ o(t_{s+1}), & (x_i, y_j) \in \{W\} \end{cases}$$

2. Model analysis and simulations

2.1. Error of doubling the time step

At the outset, we will examine the influence of the time step, denoted as h_t, on the accuracy of the simulation. In numerical analysis, especially when solving differential equations, the choice of the time step is crucial as it affects both the precision and efficiency of computations.



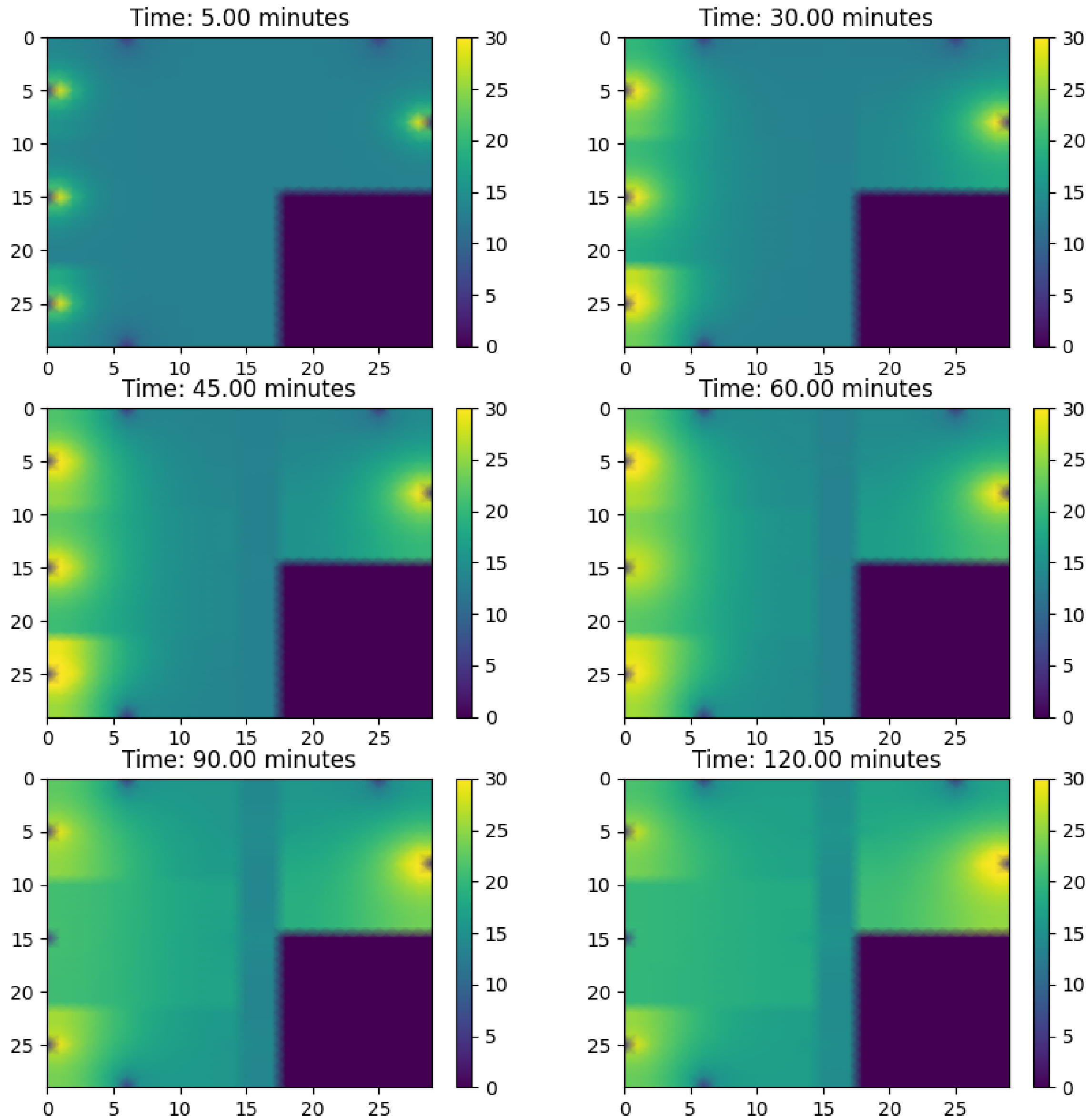
From the above plot, we can infer that the error associated with doubling the time step is generally small. It is evident that the Mean Absolute Error (MAE) decreases as the value of h_t decreases.

2.2. Radiator location and settings

In this section, we will delve into key aspects of optimizing the heating system by gaining a thorough understanding of radiator placement and precisely adjusting their temperature settings. We will explore the factors influencing heating efficiency based on the placement of radiators in individual rooms and learn how to tailor settings to achieve the perfect balance between comfort and energy savings.

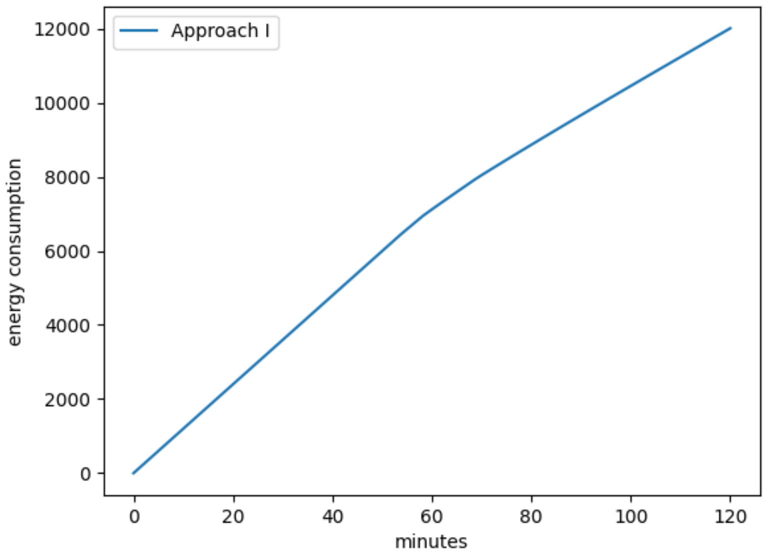
Let's consider the following radiator configurations:

Radiator	R1	R2	R3	R4
Maximum temperature [°C]	19	19	19	22



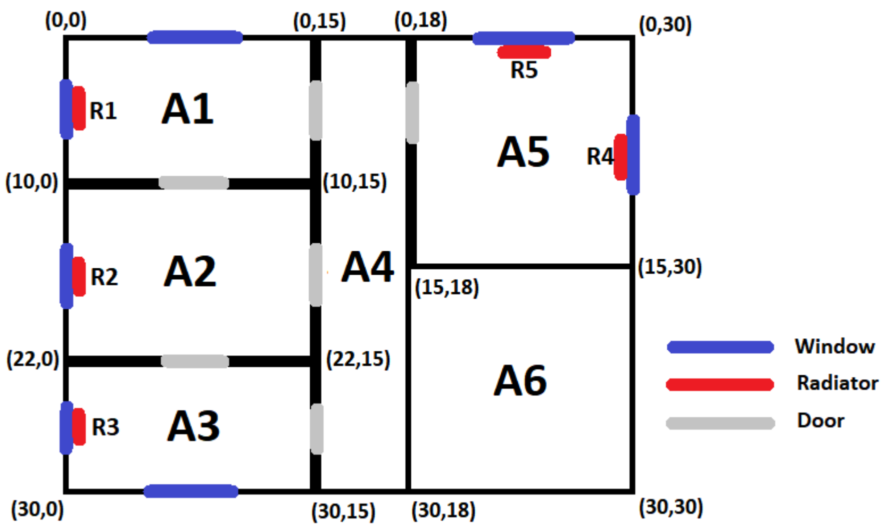
From the above diagram, we can observe that room A2 heats up the fastest. This could be attributed to its specific location - A2 has only one window and is situated between rooms A1 and A3, from which heat is transferred to A2 through the door. Additionally, we can note that the lowest temperature is maintained in room A4 (corridor), which is not directly heated but receives warmth from other rooms through its doors. Room A5 poses the greatest challenge in achieving optimal temperature, as it is large, has two windows, and lacks direct adjacency to heated spaces.

Let's now examine the total energy consumption in our home over time.

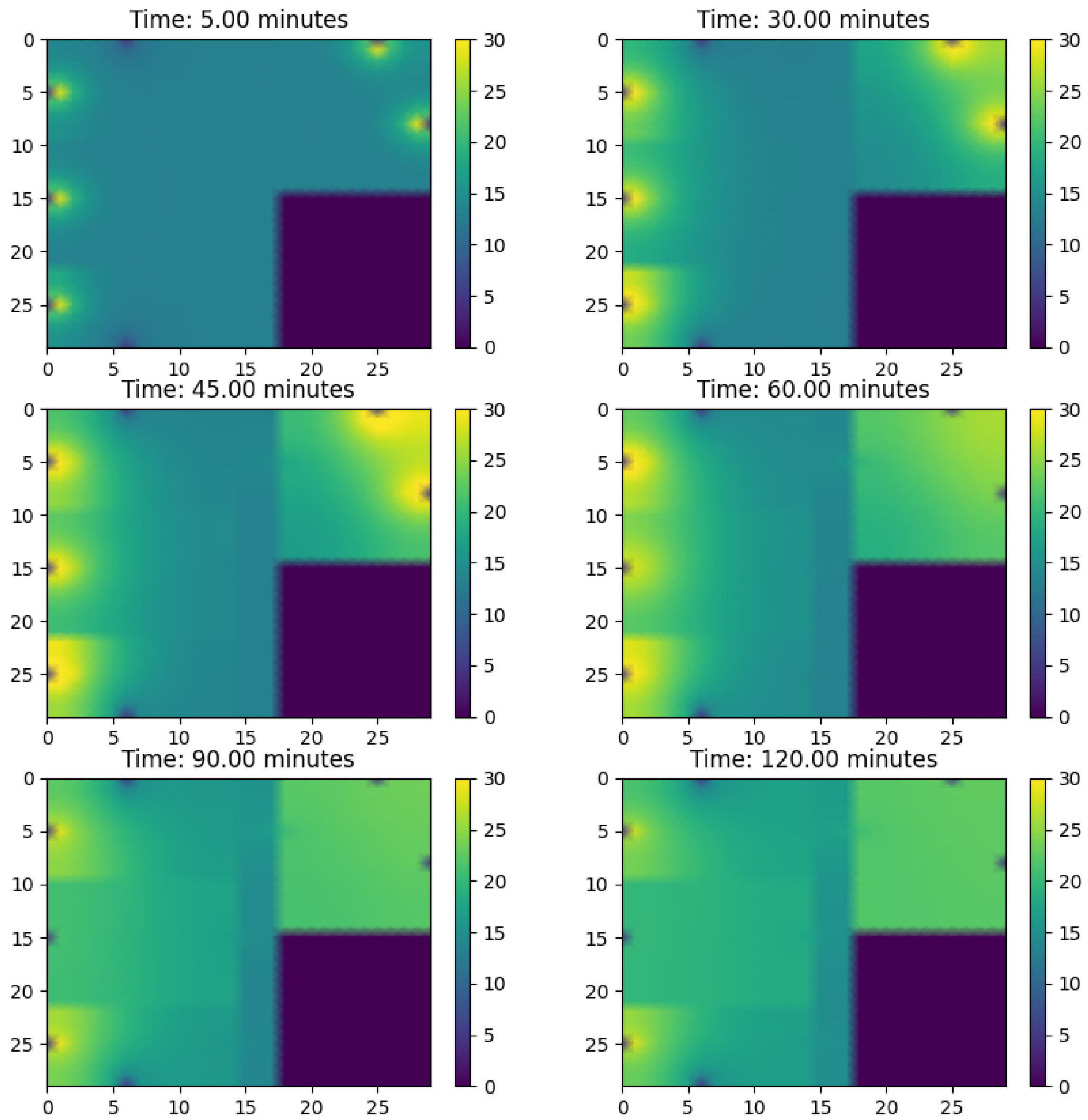


We observe a clear decrease in total energy consumption around the 60-minute mark. This is likely a result of fully heating room A2, leading to the shutdown of the radiator in that area

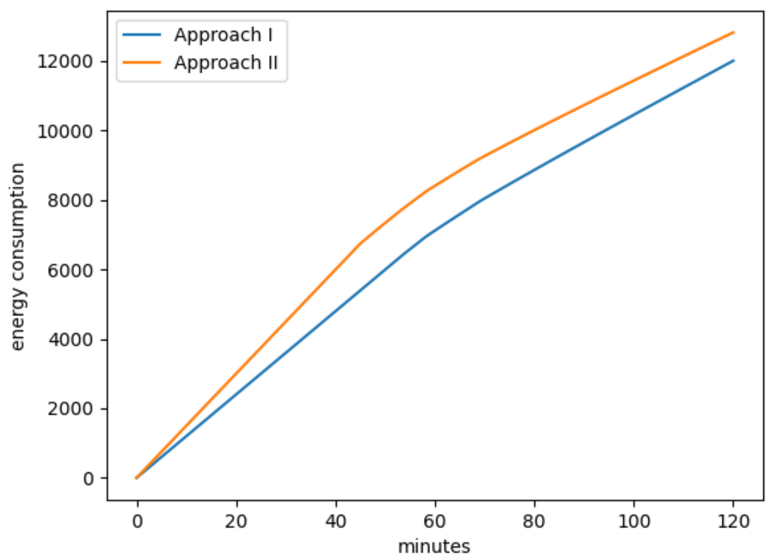
Let's see how the situation in our apartment will change if we add a radiator to room A5, placed by the window, which will heat up until reaching a temperature of 22 degrees Celsius. Now, our home plan looks as follows:



Radiator	R1	R2	R3	R4	R5
Maximum temperature [°C]	19	19	19	22	22

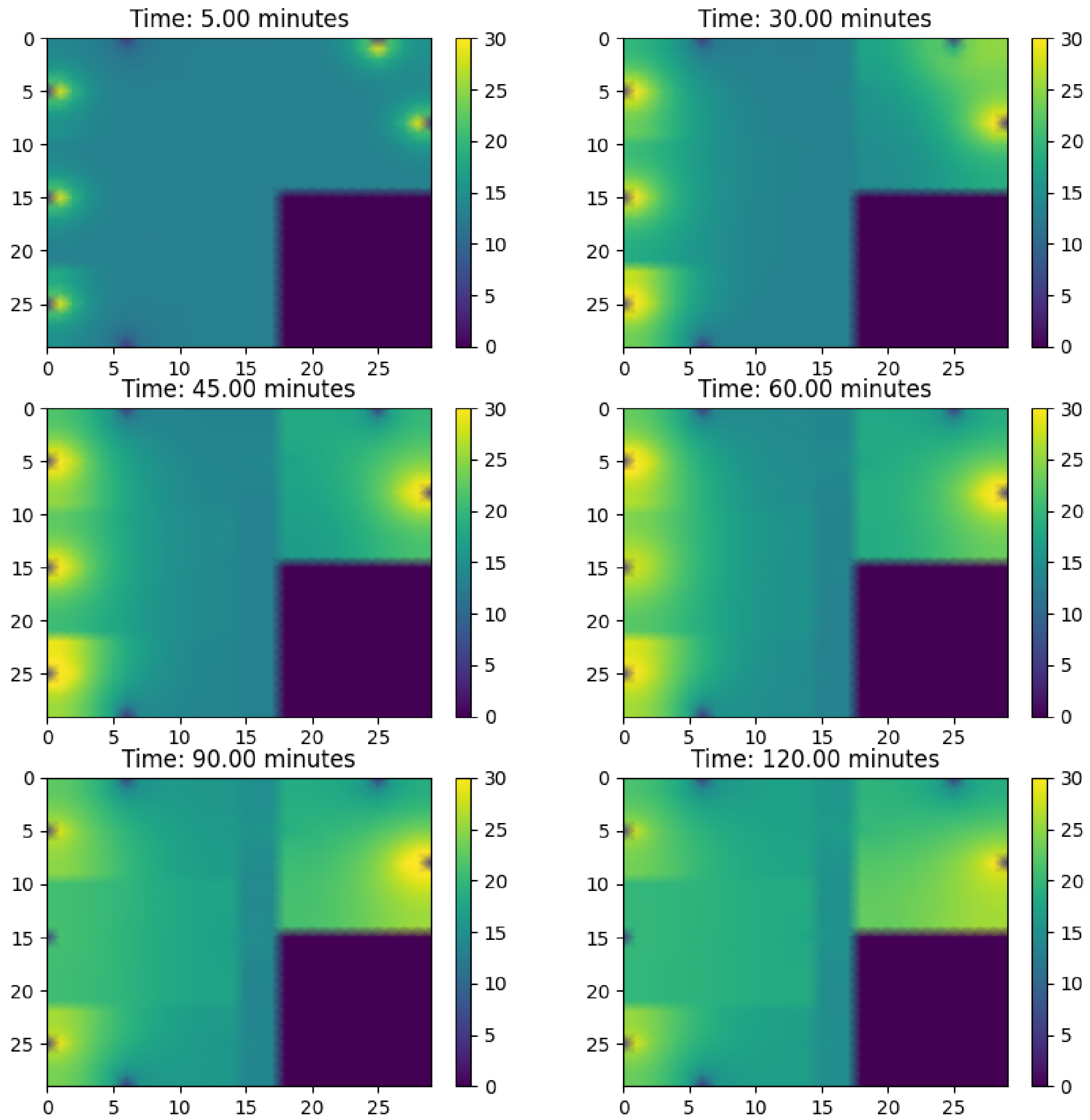


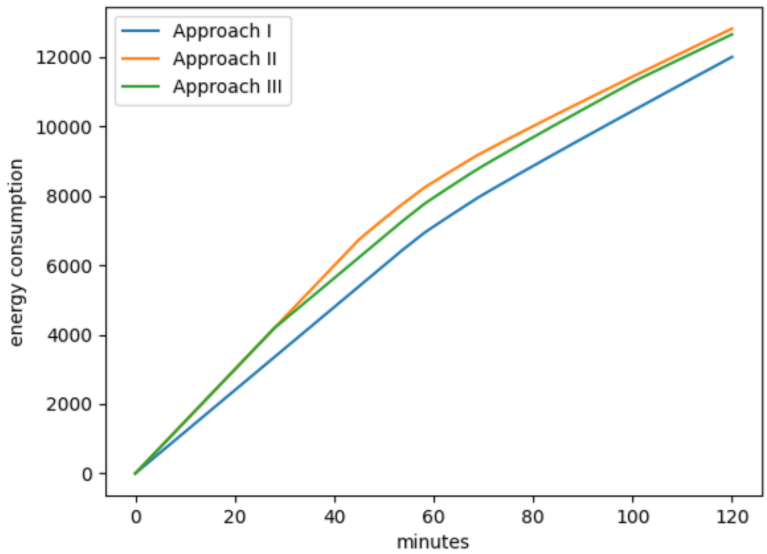
It is evident that the introduction of a second radiator in room A5 has significantly accelerated the heating of that space. Let's now examine how the total energy consumption in the house looks like.



There is a slight increase in total energy consumption noticeable. However, it should be noted that in room A5, both radiators are set to a temperature of 22 degrees Celsius. Let's examine how the situation will change when the maximum temperature on radiator R5 is set to 19 degrees Celsius.

Radiator	R1	R2	R3	R4	R5
Maximum temperature [°C]	19	19	19	22	19

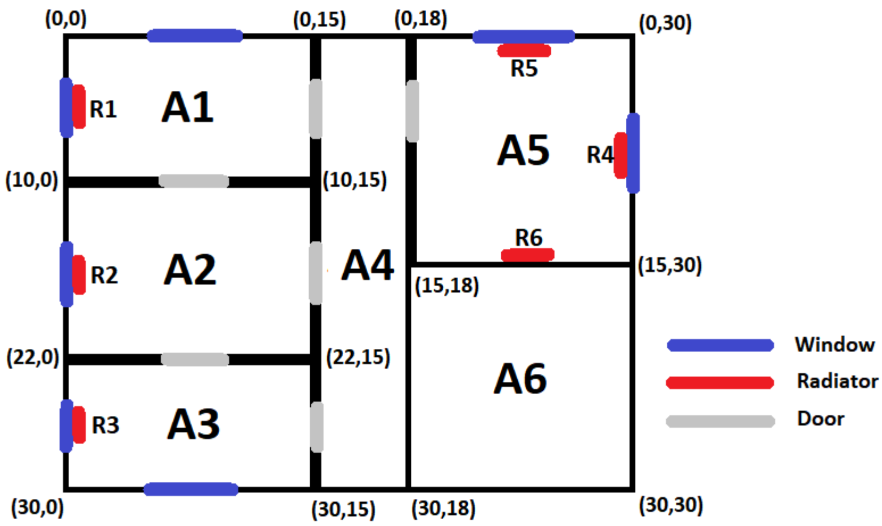


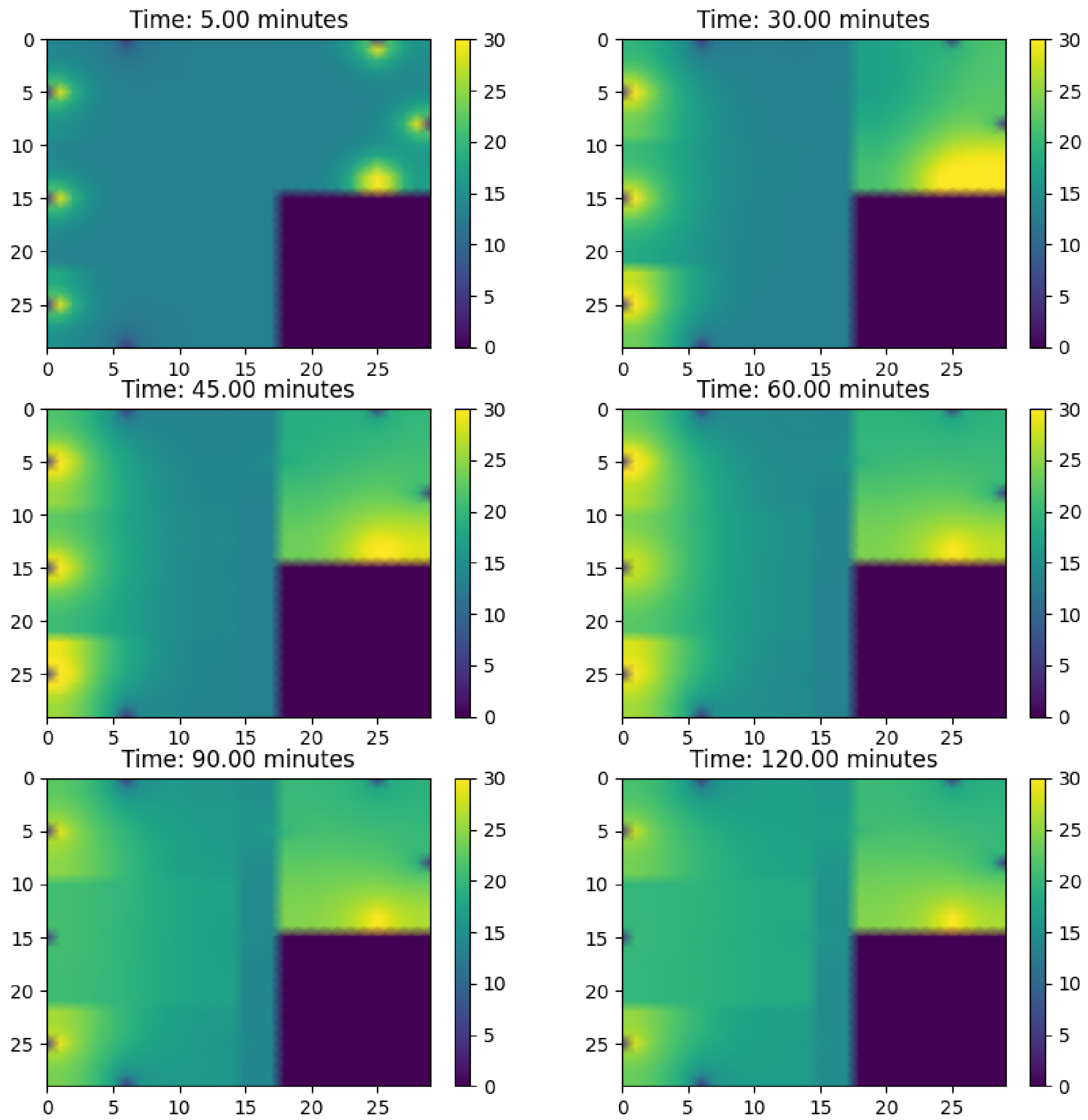


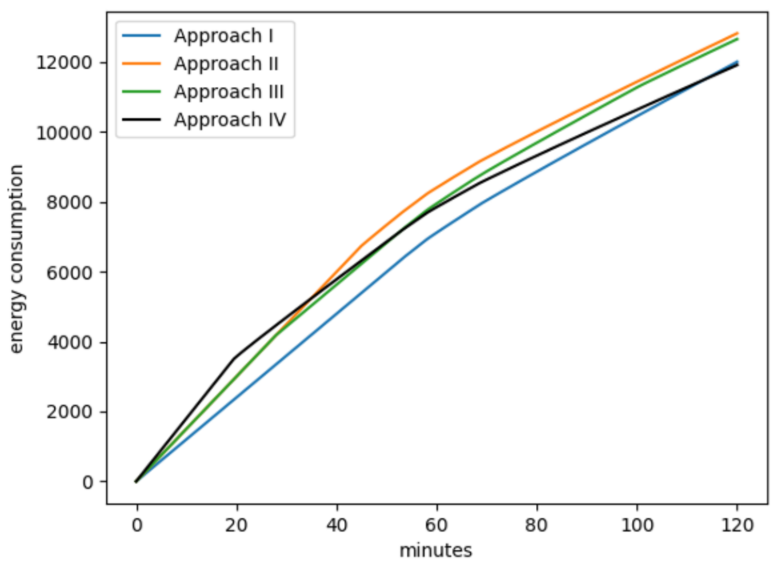
Analysis of the above charts indicates that the total energy consumption at the radiator R5 set to both 19 degrees Celsius and 22 degrees Celsius is nearly identical. However, it is noticeable that when the temperature of radiator R5 is set to 22 degrees Celsius, the room heats up slightly faster. This is particularly evident when one of the radiators in room A5 operates at the maximum temperature of 22 degrees Celsius, and the other at 19 degrees Celsius. In such a scenario, the radiator set to 22 degrees Celsius eventually maintains the room temperature independently. Therefore, a decision has been made to set the temperatures on radiators R4 and R5 to 22 degrees Celsius, which appears to be more effective for the heating process.

Let's see how adding a third radiator in room A5 will change the situation. Set this radiator to a temperature of 22 degrees Celsius as well.

Radiator	R1	R2	R3	R4	R5	R6
Maximum temperature [°C]	19	19	19	22	22	22



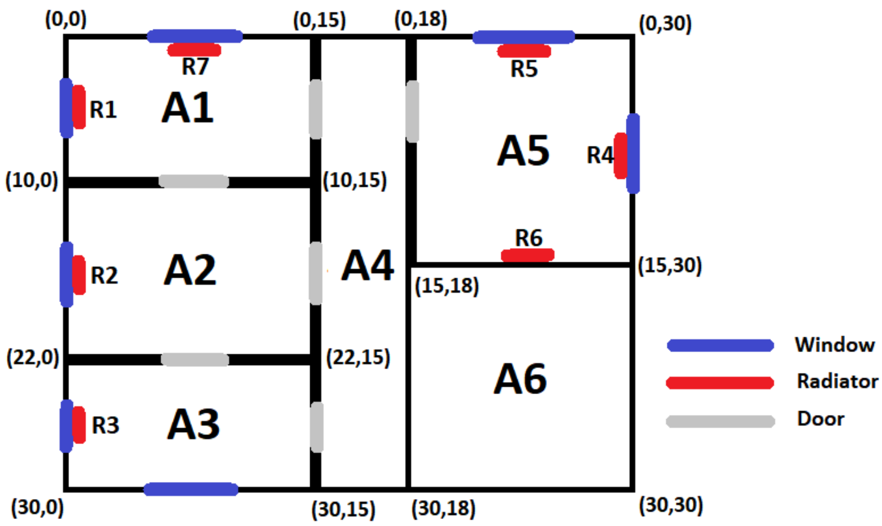


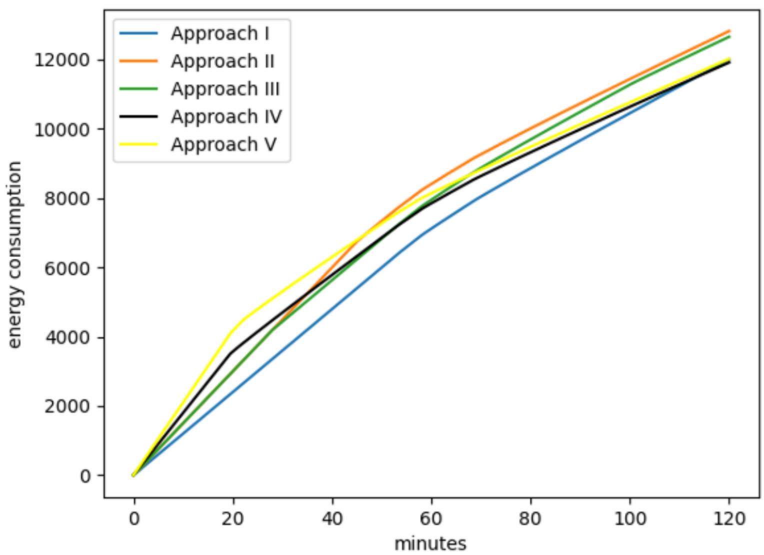


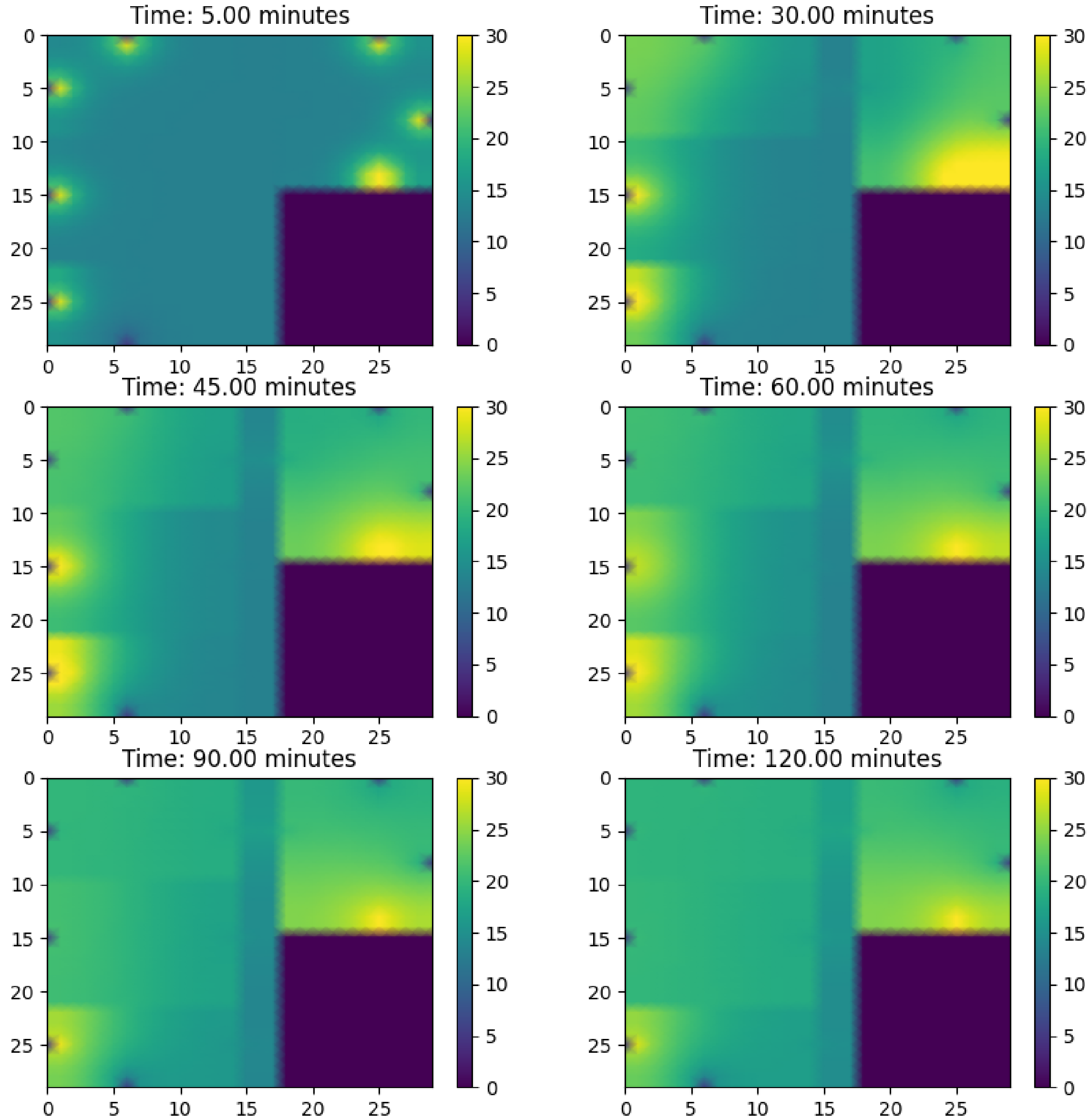
The analysis of the above charts reveals that the addition of a third radiator turned out to be a game-changer. Our results, as shown in the total energy consumption chart, are significantly better than in the two previous cases. Although initially it might have seemed that adding a third radiator would significantly increase energy consumption, the effect turns out to be quite the opposite

As for rooms A1 and A3, the situation in them is not as critical. It's worth considering three different scenarios: heating only room A1, only A3, or both simultaneously. The choice depends on our financial capabilities and the expected comfort in our apartment. Let's assume that one of the rooms, for example, A3, is a place where we don't spend much time during the day. Therefore, we decide to provide additional heating only to room A1. Set the radiator by the window in room A1 to a maximum temperature of 19 degrees Celsius and see how the situation changes.

Radiator	R1	R2	R3	R4	R5	R6	R7
Maximum temperature [°C]	19	19	19	22	22	22	19







We can observe that introducing an additional radiator in room A1 was a very good idea, as this room heats up much faster. Additionally, we can see that the temperature in our corridor (A4) is also higher than in the previous cases. It is worth noting that adding a radiator in room A1 did not lead to as significant an increase in energy consumption as was the case after adding a second radiator in room A5. This may be attributed to the difference in the set temperatures on the radiators – 22 degrees Celsius in room A5 and 19 degrees Celsius in room A1.

Additionally: It is worthwhile to view the animation (file named "animation.gif"), as it allows for a detailed observation of how heat propagates in each minute!