Categorical Logic

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Introduction

The book is divided into two sections, the first being on Topos theory, and the second being on Type theory. The first few chapters of the Topos theory section are taken from [1].

Acknowledgements

I want to thank...

Contents

Part I

Topos Theory

Chapter 1

The Topos-Logos Duality

In this chapter we describe how a topos can be presented as a spacial concept, dual to the algebraic notion of a logos.

1.1 First Definition

Definition 1.1.1. *A* ***reflective localization*** *is a functor $L : \mathcal{E} \to \mathcal{F}$ admitting a fully faithful right adjoint. In particular, since it is a left adjoint it is a co-continuous functor. A* ***left-exact*** *(or* ***lex****) localization is a refective localization L that preserves finite limits.*

Definition 1.1.2 (Logos)**.** *A* ***logos*** *is a category $\mathcal{E}$ that can be presented as a left-exact localization of a presheaf category $Pr(\mathcal{C})$, on a small category $\mathcal{C}$. A morphism of logoi $f^* : \mathcal{E} \to \mathcal{F}$ is a functor preserving (small) colimits and finite limits.*

The category of Logoi will be denoted ***Logos****. It is a 2-category if we take into account the natural transformations $f^* \to g^*$ between morphisms.*

Definition 1.1.3. *The category of morphisms of logoi is the full subcategory $\{\mathcal{E}, \mathcal{F}\}^{lex}_{cc} \subset \{\mathcal{E}, \mathcal{F}\}$ spanned by functors preserving small colimits and finite limits.*

Definition 1.1.4 (Topos)**.** *A* ***topos*** *is defined to be an object of $Logos^{op}$. The 2-category of Topoi is defined as*

$$\boldsymbol{Topos} := Logos^{op}. \tag{1.1}$$

We shall refer to a morphism between topoi as a ***geometric morphism****, or* ***topos morphism****.*

If $\mathcal{X}$ is a topos, we shall denote by $\mathrm{Sh}(\mathcal{X})$ the corresponding logos. The objects of $\mathrm{Sh}(\mathcal{X})$ are called the **sheaves** on $\mathcal{X}$.

For $u : \mathcal{Y} \to \mathcal{X}$ a topos morphism, we denote by $u^* : Sh(\mathcal{X}) \to Sh(\mathcal{Y})$ the corresponding logos morphism.

Given F in Sh($\mathcal{X}$), the object u^*F in Sh(Y) is called the **pullback**, or **base-change** of F along u.

A logos $\mathcal{E}$ always has a terminal object 1 (since it has finite limits); a map $1 \to F$ in $\mathcal{E}$ shall be called a **global section** of F.

The definition motivating a logos is the category of sheaves of sets on a space. Let X be a topological space. The category of sheaves on X is a reflective localization of the presheaf category $Pr(\mathcal{O}(X)) := [\mathcal{O}(X)^{op}, Set]$. This is in fact a left exact localization and referred to as the **sheafification functor**, therefore Sh(X) is a logos. Since we will never focus on topological spaces we will refer to the topos dual to this logos as X.

Proposition 1.1.1. *The sheafification functor above is a left exact localization.*

Proof. Let X be a topological space and denote by $\mathcal{E}t(X)$ the full subcategory of $Top_{/X}$ spanned by local homeomorphisms (or etale maps) $Y \to X$. Any such map defines a presheaf of local sections, namely $\Gamma(X)_Y$. Now we will construct a functor $F : Pr(\mathcal{O}(X)) \to \mathcal{E}t(X)$ which we will show is a left exact localization, hence that $\mathcal{E}t(X)$ is a category of sheaves. For any point $x \in X$ let U(x) be the filter of neightborhoods containing X

$$U(x) = \{V \in \mathcal{O}(X) \mid x \in V\}$$

The **stalk** of F at x is the colimit F(x) := $colim_{V \in U(x)} F(V)$. Since U(x) is a filter and filtered colimits preserve finite limits the map $F \mapsto F(x)$ is left exact. Any point $x \in V$ defines a map $s : F(V) \to F(x)$, which sends a local section to its **germ** s(x) at x. Then, the underlying set of F(V) is $\coprod_{x \in X} F(x)$ and a basis for the topology is given by the set of germs $\{s(x) \mid x \in U\}$ for any s in F(U). This geometric contruction produces a functor $F : Pr(\mathcal{O}(X)) \to \mathcal{E}t(X)$ that is left exact because the construction of the stalk is

$$\begin{tikzcd}
Pr(\mathcal{O}(X)) \arrow[r, "sheafification"] & \mathcal{E}t(X) equivalence \arrow[dl, "equivalence"] \\
Sh(X) \arrow[u, hook] &
\end{tikzcd}$$

□

1.2 Presheaf Logoi and Alexandrov Topoi

Definition 1.2.1 (Alexandrov Logos)**.** *The **Alexandrov logos** of a small category $\mathcal{C}$ is defined to be the category of set-valued $\mathcal{C}$-diagrams $[\mathcal{C}, Set] = Pr(\mathcal{C}^{op})$. The **Alexandrov topos** is defined to be the dual topos, and we shall denote it by $\mathbb{B}\mathcal{C}$*

The above definition defines a contraviant 2-functor $[-, Set] : Cat^{op} \to$ *Logos* and a covariant 2-functor $\mathbb{B} : Cat \to Topos$, where Cat denotes the 2-category of small categories.

Chapter 2

Elements of Topos Geometry

2.1 Free Logoi and Affine Topoi

Definition 2.1.1. *Let $\mathcal{C}$ be a small category and $\mathcal{C}^{lex}$ the free completion of $\mathcal{C}$ for finite limits, i.e. $\mathcal{C}^{lex}$ is the initial object in the comma category of finite complete categories under $\mathcal{C}$: For any finite complete category $\mathcal{D}$*

$$Hom(\mathcal{C}, \mathcal{D}) \simeq Hom(\mathcal{C}^{lex}, \mathcal{D}),$$

$$\begin{tikzcd}
\mathcal{C} \arrow[d] \arrow[dr] & \\
\mathcal{C}^{lex} \arrow[r, dotted] & \mathcal{D}.
\end{tikzcd}$$

*Then the functor category $Set[\mathcal{C}] := Pr(\mathcal{C}^{lex}) = [(\mathcal{C}^{lex})^{op}, Set]$ is a logos called the **free logos** on $\mathcal{C}$. The free logos has the following universal property: Let $\mathcal{E}$ be a logos. For any cocontinuous and left exact functor $Set[\mathcal{C}] \to \mathcal{E}$ we have*

$$Hom(\mathcal{C}, \mathcal{E}) \simeq Hom(Set[\mathcal{C}], \mathcal{E}),$$

$$\begin{tikzcd}
\mathcal{C} \arrow[d] \arrow[dr] & \\
Set[\mathcal{C}] \arrow[r, dotted] & \mathcal{E}.
\end{tikzcd}$$

Inspired by algebraic geometry we called the topos corresponding to $Set[\mathcal{C}]$ an **affine topos** and denote it by $\mathbb{A}^{\mathcal{C}}$

2.2 The Category of Points

The category **Logos** has as initial object the category **Set**. The category of topoi has the corresponding terminal object $\mathbb{1}$.

Definition 2.2.1. *A* ***point*** *of a topos $\mathcal{X}$ is a morphism $x : \mathbb{1} \to \mathcal{X}$. Equivalently, a point is a morphism of logoi $x^* : Sh(\mathcal{X}) \to Set$. The* ***category of points*** *of $\mathbb{X}$ is*

$$Pt(\mathcal{X}) := Hom_{\boldsymbol{Topos}}(\mathbb{1}, \mathcal{X}) \simeq Hom_{\boldsymbol{Logos}}(Sh(\mathcal{X}), Set) = [Sh(\mathcal{X}), Set]^{lex}_{cc},$$

which is the full subcategory of $[Sh(\mathcal{X}), Set]$ spanned by the functors preserving colimits and finite limits.

Geometrically, a point x of $\mathcal{X}$ sends a sheaf F on $\mathcal{X}$ to its stalk F(x) := x^*F at x.

2.3 Quotient Logoi and Embeddings of Topoi

Definition 2.3.1. *A morphism of logoi $\mathcal{E} \to \mathcal{F}$ shall be called a* ***quotient*** *if it is a left-exact localization. The corresponding morphism of topoi shall be called an* ***embedding****. If $\mathcal{Y} \hookrightarrow \mathcal{X}$ is an embedding, we shall also say that $\mathcal{Y}$ is a* ***subtopos*** *of $\mathcal{X}$.*

At the level of points, the functor $Pt(\mathcal{Y}) \to Pt(\mathcal{X})$ induced by the embedding is fully faithful.

Classically, a quotient $\mathcal{E} \to \mathcal{F}$ is encoded by the data of a Lawvere-Tierney topology on $\mathcal{E}$. In the case where $\mathcal{E} = Pr(\mathcal{C})$ is a presheaf logos, this is also equivalent to the data of a Grothendeick topology on the category $\mathcal{C}$.

2.4 Products of Topoi

A reflective localization of a presheaf category is called a **presentable category**, it is basically a logos with forgotten finite limits. A functor of presentable categories is a cocontinuous functor.

The tensor product of logoi is defined at the level of their underlying presentable categories.

Given three presentable categories $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$, a functor $\mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is called **bilinear** if it is cocontinuous in each variable. A bilinear functor $\mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is determined by the same data as a morphism of presentable categories $\mathcal{A} \otimes \mathcal{B} \to \mathcal{C}$ for a certain presentable category $\mathcal{A} \otimes \mathcal{B}$. Thsi category can be described as $\mathcal{A} \otimes \mathcal{B} = \{\mathcal{A}^{op}, \mathcal{B}\}^c$ (c means continuous functors). This formula shows in particular that Set is the unit of this product.

2.5 Fiber Products of Topoi

The fact that topoi live in a 2-category requires the use of the so-called pseudo fiber product.

Definition 2.5.1. *Let $\mathcal{X}, \mathcal{Y}$, and $\mathcal{Z}$ be topoi. Their (pseudo) fiber product is the topos $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ defined as in the diagram*

$$\begin{tikzcd} \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \arrow[r] \arrow[d, "f^* g"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{Y} \arrow[d, "g"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Z} \end{tikzcd}$$

where the pullback is up to isomorphism rather than equality.

2.6 Étale Domains

Definition 2.6.1. *Let $\mathcal{E}$ be a logos and F an object of $\mathcal{E}$. The base change along the map $F \to 1$ provides a morphism of logoi $\epsilon_F^* : \mathcal{E} \to \mathcal{E}_{/F}$ called an **étale morphism**.*

*If $\mathcal{E} = Sh(\mathcal{X})$, the corresponding morphisms of topoi will be denoted $\epsilon_F : \mathcal{X}_F \to \mathcal{X}$ and called an **étale morphism** or **local homeomorphism**.*

The étale extension $\epsilon_F^* : \mathcal{E} \to \mathcal{E}_{/F}$ has an important universal property. The object $\epsilon_F^*(F)$ in $\mathcal{E}_{/F}$ corresponds to the map $p_1 : F^2 \to F$, which admits a canonical section given by the diagonal $\Delta : F \to F^2$. Then the pair (ϵ_F^*, Δ) is universal for creating a global section of F. More precisely, if $u^* : \mathcal{E} \to \mathcal{F}$ is a logos morphism and $\delta : 1 \to u^*F$ a global section of F in $\mathcal{F}$, there exists a unique factorization of u^* via $\mathcal{E}_{/F}$

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "u^*"] \arrow[dr, "\epsilon_F^*"'] & & \mathcal{F} \\ & \mathcal{E}_{/F} \arrow[ur, "v^*"'] & \end{tikzcd}$$

such that $v^*(\Delta) = \delta$.

2.7 Open Domains

Definition 2.7.1. *We define an **open embedding** of a topos $\mathcal{X}$ to be an étale morphism $\mathcal{Y} \to \mathcal{X}$ that is also an embedding. The corresponding morphisms of logoi will be called **open quotients.***

Proposition 2.7.1. *For an object U in a logos $\mathcal{X}$, the functor $\epsilon_U^* : Sh(\mathcal{X}) \to Sh(\mathcal{X})_{/U}$ is a **quotient** iff the canonical morphism $U \to 1$ is a monomorphism.*

2.8 Closed Embeddings

Part II

Categorical Logic

Chapter 3

HoTT Semantics

3.1 2-categorical preliminaries

Definition 3.1.1 (Cartesian morphism)**.** *We shall say that a morphism $f : X \to Y$ is* ***cartesian*** *with respect to a functor $p : \mathcal{C} \to \mathcal{D}$, or that f is* ***p-cartesian****, if for every object $C \in \mathcal{C}$, and every pair of morphisms*

$$u : C \to Y \quad \text{and} \quad v : p(C) \to p(X)$$

such that

$$\begin{tikzcd} p(C) \arrow[d, "v"'] \arrow[dr, "p(u)"] & \\ p(X) \arrow[r, "p(f)"] & p(Y), \end{tikzcd} \qquad p(u) = p(f) \circ v,$$

there exists a unique morphism $w : C \to X$ such that

$$\begin{tikzcd} C \arrow[d, dotted, "w"'] \arrow[dr, "u"] & \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad f \circ w = u,$$

and $p(w) = v$,

$$\begin{tikzcd} C \arrow[d] \arrow[dr, dotted, "w"] \arrow[drr, bend left, "u"] & & \\ p(C) \arrow[dr, "v=p(w)"'] \arrow[drr] & X \arrow[r, "f"] \arrow[d, "p(u)"] & Y \arrow[d] \\ & p(X) \arrow[r, "p(f)"'] & p(Y). \end{tikzcd}$$

Dually we have ***p-cocartesian*** *morphisms if the opposite morphisms is cartesian with respect to the opposite functor.*

Definition 3.1.2 (Grothendieck fibration)**.** *We shall say that a functor $p : \mathcal{C} \to \mathcal{D}$ is a* ***Grothendieck fibration*** *if for every object $Y \in \mathcal{C}$ and every morphism $g \in Mor(\mathcal{D})$ with codomain $p(Y)$ there exists a p-cartesian morphism $f \in Mor(\mathcal{X})$ with codomain Y such that $p(f) = g$. If this f is unique we refer to this as a* ***discrete Grothendieck fibration****. Dually if $p^{op} : \mathcal{C}^{op} \to \mathcal{D}^{op}$ is a Grothendieck fibration then p is called a* ***Grothendieck opfibration****. A functor that is both a Grothendieck fibration and Grothendieck opfibration is called a* ***Grothendieck bifibration***

Definition 3.1.3. *A morphism $f : X \to Y$ in a 2-category $\mathcal{K}$ is an* ***internal fibration*** *if each induced functor $Hom_{\mathcal{K}}(Z, X) \to Hom_{\mathcal{K}}(Z, Y)$ is a Grothendieck fibration. Explicitly, this means for any $g : Z \to X$ and 2-cell*

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[dr, "h"'] & X \arrow[d, "f"] \\ & Y, \arrow[from=1-1, to=2-2, phantom, "\Nearrow\alpha"] \end{tikzcd} \qquad \alpha : h \to f \circ g,$$

there is a cartesian lift

$$\begin{tikzcd} Z \arrow[r, bend left, "k"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\beta"] & X \end{tikzcd} \qquad \beta : k \to g$$

with

$$\begin{tikzcd} Z \arrow[r, "k"] \arrow[dr, "h"'] & X \arrow[d, "f"] \\ & Y, \end{tikzcd} \qquad h = f \circ k$$

and

$$\begin{tikzcd} Z \arrow[r, bend left, "k"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\beta"] & X \arrow[r, "f"] & Y \end{tikzcd} = \begin{tikzcd} Z \arrow[r, "g"] \arrow[dr, "h"'] & X \arrow[d, "f"] \\ & Y, \arrow[from=1-1, to=2-2, phantom, "\Nearrow\alpha"] \end{tikzcd} \qquad \alpha = f \circ \beta.$$

If β is unique then we call the internal fibration a ***discrete internal fibration****.*

Let $\mathcal{E}$ be a (large, locally small) category and write PSH($\mathcal{E}$) = [$\mathcal{E}^{op}$, Grpd] for the (very large) 2-category of contravariant pseudofunctors from $\mathcal{E}$ to large groupoids, and pseudonatural transformations between them.

Definition 3.1.4. *A* ***strict fibration*** *in PSH($\mathcal{E}$) is a strictly natural transformation $X \to Y$ such that each component $X(A) \to Y(A)$ is a fibration of groupoids. Similarly we define a* ***strictly discrete fibration****.*

Part III

Type Theory

Chapter 4

Categorical Semantics

4.1 Natural Models of Homotopy Type Theory

The reference for this section is [2].

Homotopy type theory is an interpretation of constructive Martin-Lof type theory into abstract homotopy theory. It shows that the objects in the constructive system can be homotopy types.

4.1.1 Natural Models

Definition 4.1.1. *Let $\mathbb{C}$ be a small category. A natural transformation*

$$f : Y \to X$$

of presheaves on $\mathcal{C}$ is called ***representable*** *if all of its fibers are representable objects, in the following sense: for every $C \in \mathcal{C}$ and $x \in X(C)$, there is a $D \in \mathcal{C}$, a $p : D \to C$, and a $u \in Y(D)$ such that the following square is a pullback,*

$$\begin{tikzcd}
yD \arrow[r, "y"] \arrow[d, "yp"'] \arrow[dr, phantom, "\ulcorner", very near start] & T \arrow[d, "f"] \\
yC \arrow[r, "x"'] & X
\end{tikzcd}$$

Note that we have obviously used the Yoneda Lemma to identify elements $x \in X(C)$ with natural maps $x : yC \to X$.

Denote by **Fam** the category of families of (small sets). An object of **Fam** is a pair $(I, (A_i)_{i \in I})$ consisting of a set I together with an I-indexed family of sets $(A_i)_{i \in I}$. A morphism from $(I, (A_i)_{i \in I})$ to $(J, (B_j)_{j \in J})$ is a pair $(f, (g_i)_{i \in I})$ consisting of a function $f : A \to B$ and an I-indexed family of functions $(g_i : A_i \to B_{f(i)})_{i \in I}$

Definition 4.1.2 (Category with families)**.** *A* ***category with families*** *is a category $\mathbb{C}$ with distinguished terminal object 1, together with the following data:*

- *A functor $T : \mathbb{C} \to$ **Fam** - we write $T(\Gamma) = (Ty(\Gamma), Tm(\Gamma, A)_{A \in Ty(\Gamma)})$ and denote $A[\sigma] \in Ty(\Delta)$ and $a[\sigma] \in Tm(\Delta, A[\sigma])$ the result of applying $T(\sigma : \Delta \to \Gamma)$ to an element $A \in Ty(\Gamma)$ and $a \in Tm(\Gamma, A)$, respectively;*

- *For each $\Gamma \in ob(\mathbb{C})$ and each $A \in Ty(\Gamma)$, an object $\Gamma.A$ of $\mathbb{C}$, a morphism $p_A : \Gamma.A \to \Gamma$ of $\mathbb{C}$ and an element $q_A \in Ty(\Gamma.A, A[p_A])$;*

such that, given any object Δ of $\mathbb{C}$, a morphism $\sigma : \Delta \to \Gamma$ and element $a \in Tm(\Delta, A[\sigma])$, there is a unique morphism $\langle \sigma, a\rangle : \Delta \to \Gamma.A$ such that $\sigma = p_A \circ \langle \sigma, a\rangle$ and $a = q_A[\langle \sigma, a\rangle]$

$$\begin{tikzcd} \Delta \arrow[r, dotted, "\langle\sigma,a\rangle"] \arrow[dr, "\sigma"'] & \Gamma.A \arrow[d, "p_A"] \\ & \Gamma \end{tikzcd}$$

As the notation suggests, in a category with families we view the elements of $Ty(\Gamma)$ as dependent types $\Gamma \vdash A$, and the elements of $Tm(\Gamma, A)$ as terms $\Gamma \vdash a : A$.

Proposition 4.1.1. *Let $p : \tilde{\mathcal{U}} \to \mathcal{U}$ be a natural transformation of presheaves on a small category $\mathbb{C}$ with terminal object 1. Then p is representable iff $(\mathbb{C}, p)$ is a category with families.*

Proof. We will prove ($\Longrightarrow$), it is almost trivial to construct ($\Longleftarrow$).

Let $\mathbb{C}$ be an arbitrary category, we will write its objects as $\Delta, \Gamma, ...$, etc, and arrows as $\sigma : \Delta \to \Gamma, ...$, etc. Intuitively we think of **C** as a category of contexts.

Let $p : \tilde{\mathcal{U}} \to \mathcal{U}$ be a representable map of presheaves. and write its elements as:

$$A \in \mathcal{U}(\Gamma) \iff \Gamma \vdash A : type \tag{4.1}$$
$$a \in \tilde{\mathcal{U}}(\Gamma) \iff \Gamma \vdash a : A, \tag{4.2}$$

where $A = p \circ a$ as indicated in:

$$\begin{tikzcd} & \tilde{\mathcal{U}} \arrow[d, "p"] \\ y\Gamma \arrow[ur, "a"] \arrow[r, "A"] & \mathcal{U}. \end{tikzcd}$$

Thus we regard $\mathcal{U}$ as the presheaf of types, with $\mathcal{U}(\Gamma)$ the set of all types in context Γ, and $\tilde{\mathcal{U}}$ the set of all terms in context Γ, while p is the typing of the terms in context Γ.

Observe that the naturality of $p : \tilde{\mathcal{U}} \to \mathcal{U}$ means that for any "substitution" $\sigma : \Delta \to \Gamma$, we have an action on types and terms:

$$\Gamma \vdash A : type \implies \Delta \vdash A\sigma : type \tag{4.3}$$
$$\Gamma \vdash a : A \implies \Delta \vdash a\sigma : A\sigma \tag{4.4}$$

$$\begin{tikzcd}
 & & \tilde{\mathcal{U}} \arrow[dd, "p"] \\
y\Delta \arrow[r, "\sigma"] \arrow[urr, bend left, "a\sigma"] \arrow[drr, bend right, "A\sigma"'] & y\Gamma \arrow[ur, "a"] \arrow[dr, "A"] & \\
 & & \mathcal{U}.
\end{tikzcd}$$

While by functoriality, given any further $\tau : \Delta' \to \Delta$, we have:

$$A(\sigma)\tau = A(\sigma \circ \tau) \tag{4.5}$$

$$a(\sigma)\tau = a(\sigma \circ \tau) \tag{4.6}$$

$$A(1) = A \tag{4.7}$$

$$a1 = a \tag{4.8}$$

where $1 = id_\Gamma : \Gamma \to \Gamma$.

Finally, the representability of $p : \tilde{\mathcal{U}} \to \mathcal{U}$ is exactly the operation of **context extension**: given any $\Gamma \vdash A : type$, by Yoneda we have the corresponding map $A : y\Gamma \to \mathcal{U}$, and we let $p_A : \Gamma . A \to \Gamma$ be the resulting fiber of p as in definition (4.1.1). We therefore have the pullback square:

$$\begin{tikzcd}
y(\Gamma . A) \arrow[r, "q_A"] \arrow[d, "y(p_A)"'] \arrow[dr, phantom, "\ulcorner", very near start] & \tilde{\mathcal{U}} \arrow[d, "p"] \\
y\Gamma \arrow[r, "A"] & \mathcal{U}
\end{tikzcd} \tag{4.9}$$

where the map $q_A : y(\Gamma . A) \to \tilde{\mathcal{U}}$ determines a term

$$\Gamma . A \vdash q_a : Ap_A.$$

The fact that (4.9) is a pullback means that given any $\sigma : \Delta \to \Gamma$ and $\Delta \vdash a : A\sigma$, there is a map

$$\langle \sigma, a \rangle : \Delta \to \Gamma . A$$

That satisfies the following equations:

$$p_A \circ \langle \sigma, a \rangle = \sigma \tag{4.10}$$

$$q_A(\sigma, a) = a, \tag{4.11}$$

as indicated in the following diagram

$$\begin{tikzcd}
y\Delta \arrow[dr, dotted, "y\langle \sigma, a \rangle"] \arrow[drr, bend left, "a"] \arrow[ddr, bend right, "y\sigma"'] & & \\
 & y(\Gamma . A) \arrow[r, "q_A"] \arrow[d, "y(p_A)"'] \arrow[dr, phantom, "\ulcorner", very near start] & \tilde{\mathcal{U}} \arrow[d, "p"] \\
 & y\Gamma \arrow[r, "A"] & \mathcal{U}.
\end{tikzcd} \tag{4.12}$$

Moreover, by the uniqueness of $\langle \sigma, a \rangle$, for any $\tau : \Delta' \to Delta$, we also have
:

$$\langle \sigma, a \rangle \circ \tau = \langle \sigma \circ \tau, a\tau \rangle \tag{4.13}$$
$$\langle p_A, q_A \rangle = 1. \tag{4.14}$$

□

The notion of a category with families is a "variable-free" way of presenting dependent type theory, including contexts and substitutions, types and terms in context, and context extension. Accordingly, we may think of a natural model as a "type theory over $\mathbb{C}$" - With $\mathbb{C}$ serving as the category of contexts and substitutions (the requirement that $\mathbb{C}$ has a terminal object, representing the empty context, purely conventional).

Definition 4.1.3. *By a* ***natural model of type theory*** *on a small category $\mathbb{C}$ we mean a representable map of presheaves,*

$$p : \tilde{\mathcal{U}} \to \mathcal{U}.$$

Corollary 4.1.0.1. *Natural models of type theory are closed under composition, coproducts, and pullbacks along arbitrary maps $\mathcal{U}' \to \mathcal{U}$.*

Proof. All structures are inherited by Hom sets respecting those structures. For instance suppose we have two representable transformations $fY \to X$ and $g : Z \to Y$, then since the identity is trivially representable we have the diagram

$$\begin{tikzcd}
yE \arrow[r, "z"] \arrow[d, "yq"'] \arrow[ddd, bend right, "y(p\circ q)"'] & Z \arrow[d, "g"] \arrow[ddd, bend left, "f\circ g"] \\
yD \arrow[r, "y"] \arrow[d, "="'] & Y \arrow[d, "="] \\
yD \arrow[r, "y"] \arrow[d, "yp"'] & Y \arrow[d, "f"] \\
yC \arrow[r, "x"] & X
\end{tikzcd} \tag{4.15}$$

That is, we construct the bottom diagram and then use the object D in the above diagram. □

4.2 Algebraic character

Chapter 5

Multimodal Dependent Type Theory

5.1 The Syntax of MTT

We fix a small 2-category $\mathcal{M}$, henceforth called a **mode theory**. Intuitively the objects of the mode theory index type theories and modalities between modes act as a sort of syntactic functor between the object indexed type theories.

5.1.1 The Type Theory at Each Mode

. Each mode is inhabited by a standard Martin-Lof Type Theory (MLTT), and accordingly includes the usual judgements.

We use a Coquand-style universe, where universe types come with an explicit isomorphism between types and terms of the universe type. The universe type must be a large type to avoid Girard's paradox. Both levels are closed under all the standard type formers. The judgement $\Gamma \vdash A\ \mathit{type}_0 \ @\ m$ states that A is a small-type, and $\Gamma \vdash A\ \mathit{type}_1 \ @\ m$ that A is a large type. We introduce an operator that lifts a type to a larger one:

$$\frac{\ell \leq \ell' \qquad \Gamma \vdash A\ \mathit{type}_\ell \ @\ m}{\Gamma \vdash \Uparrow A\ \mathit{type}_{\ell'} \ @\ m}$$

This operator is defined to commute definitionally with all type formers.

Only large types will be allowed in contexts and the judgment $\Gamma \vdash M : A @\ m$ will presuppose that A is large.

Following this stratification into small and large types we introduce operations that exhibit the isomorphism

$$\frac{\Gamma \vdash M : U \ @\ m}{\Gamma \vdash El(M)\ \mathit{type}_0 \ @\ m} \qquad \frac{\Gamma \vdash A\ \mathit{type}_0 \ @\ m}{\Gamma \vdash Code(A) : U \ @\ m} \tag{5.6}$$

along with the equations $Code(El(M)) = M$ and $El(Code(A)) = A$.

We will often suppress the $\Uparrow$ and universe isomorphism.

$$\Gamma \vdash A\,type_\ell\ @\ m \tag{5.1}$$

$$\frac{\Gamma\ ctx\ @\ m}{\Gamma \vdash U\ type_1\ @\ m} \qquad \frac{\Gamma\ ctx\ @\ m}{\Gamma \vdash \mathbb{B}\ type_\ell\ @\ m} \qquad \frac{\Gamma\ ctx\ @\ m \qquad \Gamma \vdash A\ type_\ell\ @\ m \qquad \ell \leq \ell'}{\Gamma \vdash \Uparrow A\ type_{\ell'}\ @\ m} \tag{5.2}$$

$$\frac{\Gamma\ ctx\ @\ m \qquad \Gamma \vdash A\ type_\ell\ @\ m \qquad \Gamma \vdash M, N\ :\Uparrow A\ @\ m}{\Gamma \vdash Id_A(M,N)\ type_\ell\ @\ m} \tag{5.3}$$

$$\tag{5.4}$$

$$\frac{\Gamma\ ctx\ @\ m \qquad \Gamma \vdash A\ type_\ell\ @\ m \qquad \Gamma, x :\Uparrow A \vdash B\ type_\ell\ @\ m}{\Gamma \vdash \Pi_{x:A}B(x)\ type_\ell\ @\ m \qquad \Gamma \vdash \Sigma_{x:A}B(x)\ type_\ell\ @\ m} \tag{5.5}$$

Figure 5.1: Selected mode-local rules

5.1.2 Introducting a Modality

Part IV

Cubical Homotopy Theory

Chapter 6

Cartesian Cubical Model Categories

6.1 Cartesian Cubical Sets

Definition 6.1.1. *The objects $[n]$ of the* ***Cartesian cube category*** $\square$*, called* ***n-cubes****, are finite sets of the form*

$$[n] = \{0, x_1, ..., x_n, 1\},$$

where the $x_1, ..., x_n$, are arbitrary but distinct, elements, and $0, 1$ are further distinct, distinguished elements. The arrows,

$$f : [m] \to [n],$$

are arbitrary bipointed maps $f' : [n] \to [m]$(note the reversal of direction). Thus $\mathbb{B} = \square^{op}$ is the category of finite, strictly bipointed sets.

Definition 6.1.2. *The category cSet of* ***Cartesian cubical sets*** *is the category of presheaves on the Cartesian cube category $\square$,*

$$cSet := Set^{\square^{op}}.$$

By the density lemma, it is generated by the representable presheaves $y[n]$, to be written

$$I^n = y[n]$$

and referred to as the ***geometric n-cubes****.*

Note that the representables I^n are closed under finite products: $I^m \times I^n = I^{m+n}$. We write $I = I^1$ and $1 = I^0$, which is terminal.

Proposition 6.1.1. *The n-cubes I^n are tiny, in the sense that the endofunctor $X \mapsto X^{I^n}$ is a left adjoint.*

Proof. For any cubical set X, the exponential X^{I^n} is a "shift by n dimensions":

$$X^{I^n}(m) \simeq Hom(I^m, X^{I^n}) \simeq Hom(I^{m+n}, X) \simeq X(m+n).$$

Since cubes are stable under products, it sufficed only to consider the case n=1 as mentioned in the paper.

Thus X^{I^n} is given by precomposition with the "plus n" functor $\square \to square$ with $[m] \mapsto [m+n]$. Since precomposition always has a right adjoint[1], which in this case we shall write as

$$(-)^I \dashv (-)_I$$

and call X_I the I^{th}**-root** of X. $\square$

The exponential X^I will be called the **path object** of X.It classifies "paths" in X; so the 0-cubes $p \in (X^I)_0$,

$$(X^I)_0 := X^I(0) \simeq Hom(I^0, X^I) := Hom(1, X^I) \simeq Hom(I, X) \simeq X(1) := X_1$$

correspond to 1-cubes $p \in X_1$, the "endpoints" of which $p_0, p_1 \in X_0$ are given by composing with the evaluation maps

$$\epsilon_0, \epsilon_1 : X^I \rightrightarrows X$$

at the points $\delta_0, \delta_1 : 1 \rightrightarrows I$. More generally, higher cubes $c \in X_{n+1}$ correspond to maps $c : I^{n+1} \to X$, which are thus paths between the n-cubes $c_0, c_1 : I^n \to X$, corresponding to $c_0, c_1 \in X_n$. Note that, as a left adjoint, the path objects functor $X \mapsto X^I$ preserves all colimits. (LAPC)

Lemma 6.1.1. *The pushforward functor along any map $f : X \to Y$ preserves pathobjects; for any object $A \to X$ over X, the pathobjects of the pushforward f_*A is (canonically isomorphic over Y to) the pushforward of the pathobject,*

$$(f_*A)^I \simeq f_*(A^I).$$

Proof. This is proved by using the Beck-Chavalley condition... Somehow? $\square$

Lemma 6.1.2. *The pulled-back interval $I * I = I \times I \to I$ in $cSet_{/I}$ is also tiny.*

6.2 The Cofibration Weak Factorization System

[1] actually it always has both left and right adjoint given by left and right Kan extensions, respectively.

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