# Polynomial Functors

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# Chapter 1

# Representable functors from the category of sets

## 1.1 Representable functors and the Yoneda Lemma

**Definition 1.1.1** (Representable functors)**.** *For a set $S$, we denote by $y^S$ : $\mathbf{Set} \to \mathbf{Set}$ the functor that sends each set $X$ to the set $X^S := \mathbf{Set}(S, X)$ and each function $h : X \to Y$ to the function $h^S : X^S \to Y^S$, that sends $g : S \to X$ to $g \mathbin{⨟} h : S \to Y$.*

*We call a functor (isomorphic to one) of this form a **representable functor**, or a **representable**. In particular, we call $y^S$ the functor represented by $S$, and we call $S$ the representing set of $y^S$. As $y^S$ denotes raising a variable to the power of $S$, we will also refer to representables as **pure powers**.*

**Proposition 1.1.1.** *For any function $f : R \to S$, there is an induced natural transformation $y^f : y^S \to y^R$; on any set $X$ its $X$-components $X^f : X^S \to X^R$ is given by sending $g : S \to X$ to $f \mathbin{⨟} g : R \to X$.*

*Proof.* To prove that given any function $f : R \to S$ the construction $y^f : y^S \to y^R$ is a natural transformation, we must verify that, for any function $h : X \to Y$, the following commutative diagram commutes:

$$\begin{array}{ccc} X^S & \xrightarrow{h^S} & Y^S \\ {\scriptstyle X^f}\downarrow & & \downarrow{\scriptstyle Y^f} \\ X^R & \xrightarrow{h^R} & Y^R \end{array} \quad .$$

By definition 1.1.1 we have that $h^S := - \mathbin{⨟} h$ and $C^f := f \mathbin{⨟} -$, for $X, Y$. Let

$s : S \to X$ and consider the naturality square

$$\begin{aligned} Y^f(h^S(s)) &= Y^f(s \mathbin{⨾} h) \\ &= f \mathbin{⨾} (s \mathbin{⨾} h) \\ &= (f \mathbin{⨾} s) \mathbin{⨾} h \\ &= h^R(f \mathbin{⨾} s) \\ &= h^R(X^f(s)) \end{aligned}$$

We see that by associativity of composition the diagram commutes. $\square$

**Theorem 1.1.1** (Yoneda Lemma)**.** *Given a functor $F$ : $\textbf{\textit{Set}} \to \textbf{\textit{Set}}$*