Polynomial Functors

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Contents

Chapter 1

Representable functors from the category of sets

1.1 Representable functors and the Yoneda Lemma

Definition 1.1.1 (Representable functors)**.** *For a set S, we denote by y^S : $\mathbf{Set} \to \mathbf{Set}$ the functor that sends each set X to the set $X^S := \mathbf{Set}(S, X)$ and each function $h : X \to Y$ to the function $h^S : X^S \to Y^S$, that sends $g : S \to X$ to $g \mathbin{⨾} h : S \to Y$.*

We call a functor (isomorphic to one) of this form a ***representable functor****, or a* ***representable****. In particular, we call y^S the functor represented by S, and we call S the representing set of y^S. As y^S denotes raising a variable to the power of S, we will also refer to representables as* ***pure powers****.*

Proposition 1.1.1. *For any function $f : R \to S$, there is an induced natural transformation $y^f : y^S \to y^R$; on any set X its X-components $X^f : X^S \to X^R$ is given by sending $g : S \to X$ to $f \mathbin{⨾} g : R \to X$.*

Proof. To prove that given any function $f : R \to S$ the construction $y^f : y^S \to y^R$ is a natural transformation, we must verify that, for any function $h : X \to Y$, the following commutative diagram commutes:

$$
\begin{tikzcd}
X^S \arrow[r, "h^S"] \arrow[d, "X^f"'] & Y^S \arrow[d, "Y^f"] \\
X^R \arrow[r, "h^R"] & Y^R
\end{tikzcd}
.
$$

By definition 1.1.1 we have that $h^S := - \mathbin{⨾} h$ and $C^f := f \mathbin{⨾} -$, for X, Y. Let

$s : S \to X$ and consider the naturality square

$$\begin{aligned} Y^f(h^S(s)) &= Y^f(s \mathbin{;} h) \\ &= f \mathbin{;} (s \mathbin{;} h) \\ &= (f \mathbin{;} s) \mathbin{;} h \\ &= h^R(f \mathbin{;} s) \\ &= h^R(X^f(s)) \end{aligned}$$

We see that by associativity of composition the diagram commutes. $\square$

Theorem 1.1.1 (Yoneda Lemma)**.** *Given a functor $F : \textbf{\textit{Set}} \to \textbf{\textit{Set}}$ and a set S, there is an isomorphism*

$$F(S) \simeq \textbf{\textit{Set}}^{\textbf{\textit{Set}}}(y^S, F) \tag{1.1}$$

where the right-hand side is the set of natural transformations $y^S \to F$. Moreover, (1.1) is natural in both S and F.

Proof. We will break down the proof into individual steps for clarity.

1. (First map) Given a natural transformation $m : y^S \to F$, consider applying it to the simplest object we have, namely the identity, id_S, which yields an element $m_S(id_S) \in F(S)$. For clarity:

$$m : y^S := \mathbf{Set}(S, -) \to F, \tag{1.2}$$

$$m_S : S^S \to F(S) \tag{1.3}$$

$$\tag{1.4}$$

2. (Second Map) Conversely, consider a, an element of $F(S)$. We want to construct a natural transformation relative to this data. Let's go backward and assume we have a natural transformation indexed by the element, $m^a : y^S \to F$ and then try to deduce how to define such a natural transformation.

   So this natural transformation acts on objects X as $m^a(X) : X^S \to F(X)$. That is, the components should send morphisms $g : S \to X$ to elements of F(X). Do we have a way to associate a morphism $g \in X^S$ with an element of $F(X)$? Well, we have $F(g) : F(S) \to F(X)$ and $a \in F(S)$ so we can define the behaviors of our component $m^a(X)$ as $g \mapsto F(g)(a)$. That it, we have defined our morphism indexed by a as

$$m^a : y^S \to F \tag{1.5}$$

$$m^a(X) : X^S \to F(X) \tag{1.6}$$

$$m^a(X)(g) = F(g(a)) \tag{1.7}$$

$$\tag{1.8}$$

3. (Naturality of second map) Now we must verify that this morphism is indeed a natural transformation by showing that the construction is natural in X (since X is the parameter it takes). So lets consider the following diagram

$$\begin{tikzcd} X^S \arrow[r, "h^S"] \arrow[d, "F(-)(a)_X"'] & Y^S \arrow[d, "F(-)(a)_Y"] \\ F(X) \arrow[r, "F(h)"] & F(Y) \end{tikzcd}.$$

   where $m^a(X) := F(-)(a)_X$ and likewise for Y.

   So for $g : S \to X$ we have

$$\begin{aligned} g \fatsemi h^s \fatsemi F(-)(a)_Y &= (g \fatsemi h) \fatsemi F(-)(a)_Y && (1.9) \\ &= F(g \fatsemi h)(a)_Y && (1.10) \\ &= F(g)(a)_X \fatsemi F(h) && (1.11) \\ &= g \fatsemi F(-)(a)_X \fatsemi F(h) && (1.12) \end{aligned}$$

   So we see that the diagram commutes.

4. (First inverse) Next we show that the maps $m \mapsto m_S(id_S)$ and $a \mapsto m^a$ mutually inverse. First, we have

$$\begin{aligned} m^{m_S(id_S)}(X)(g) &= F(g)(m_S(id_S)) && (1.13) \\ &= m_X(g^S)(id_S) && (1.14) \\ &= m_X(g) && (1.15) \end{aligned}$$

   Where the second line comes from the naturality of m:

$$\begin{tikzcd} S^S \arrow[r, "m_S"] \arrow[d, "g^S"'] & F(S) \arrow[d, "F(g)"] \\ X^S \arrow[r, "m_X"] & F(X) \end{tikzcd}.$$

   So we have $m^{m_S(id_S)} = m$.

5. (Second Inverse) Next we have to show that $m^a_S(id_S) = a$. By construction $m^a_S(id_S) = F(id_S)(a) = id_F S(a) = a$.

6. (Natural in functor parameter) Next we show that the diagram (1.1) is natural in F. It suffices to show that given two functors $F, G : \mathbf{Set} \to \mathbf{Set}$ and a natural transformation between then $\alpha : F \to G$ the naturality square

$$\begin{tikzcd} \mathbf{Set}^{\mathbf{Set}}(y^S, F) \arrow[r, "- \fatsemi \alpha"] \arrow[d, "\sim"'] & \mathbf{Set}^{\mathbf{Set}}(y^S, G) \arrow[d, "\sim"] \\ F(S) \arrow[r, "\alpha_S"] & G(S) \end{tikzcd}.$$

commutes. The commutativity of this square is trivial:

$$\begin{aligned}
&m \mapsto m_S(id_S) \mapsto \alpha_S(m_S(id_S)) \\
&m \mapsto m \fatsemi \alpha \mapsto (m \fatsemi \alpha)_S(id_S) = \alpha_S(m_S(id_S)).
\end{aligned}$$

7. (Natural in object parameter) It suffices to consider given a morphism $g : S \to X$, consider the diagram

$$\begin{array}{ccc}
\mathbf{Set}^{\mathbf{Set}}(y^S, F) & \xrightarrow{y^g \fatsemi -} & \mathbf{Set}^{\mathbf{Set}}(y^X, F) \\
\sim\downarrow & & \downarrow\sim \\
F(S) & \xrightarrow{\quad F(g) \quad} & F(X)
\end{array}.$$

Let $m : y^S \to F$, then

$$\begin{aligned}
m \mapsto y^g \fatsemi m \mapsto (y^g \fatsemi m)_X(id_X) &= m_X(X^g(id_X)) && (1.16)\\
&= m_X(g \fatsemi id_X) && (1.17)\\
&= m_X(g). && (1.18)
\end{aligned}$$

$$m \mapsto m_S(id_S) \mapsto F(g)(m_S(id_S)) = m_X(g). \quad (1.19)$$

where the last line is by (1.13 - 15), so the square commutes.

□

Corollary 1.1.1.1 (Yoneda Embedding)**.** *There is a full and faithful functor $y^- : \boldsymbol{Set}^{op} \to \boldsymbol{Set}^{\boldsymbol{Set}}$. In particular there is an isomorphism $S^T \simeq \boldsymbol{Set}^{\boldsymbol{Set}}(y^S, y^T)$. For this reason we call y^- the **Yoneda embedding***

Proof. By the Yoneda lemma we have

$$F(S) \simeq \mathbf{Set}^{\mathbf{Set}}(y^S, F)$$

We just have to set F $= (-)^T$, then we have the desired isomorphism

$$S^T \simeq \mathbf{Set}^{\mathbf{Set}}(y^S, y^T).$$

□

1.2 Sums and products of sets