E3081 Circuit Analysis Laboratory **Fall 2025**

Experiment 4: Second Order Systems

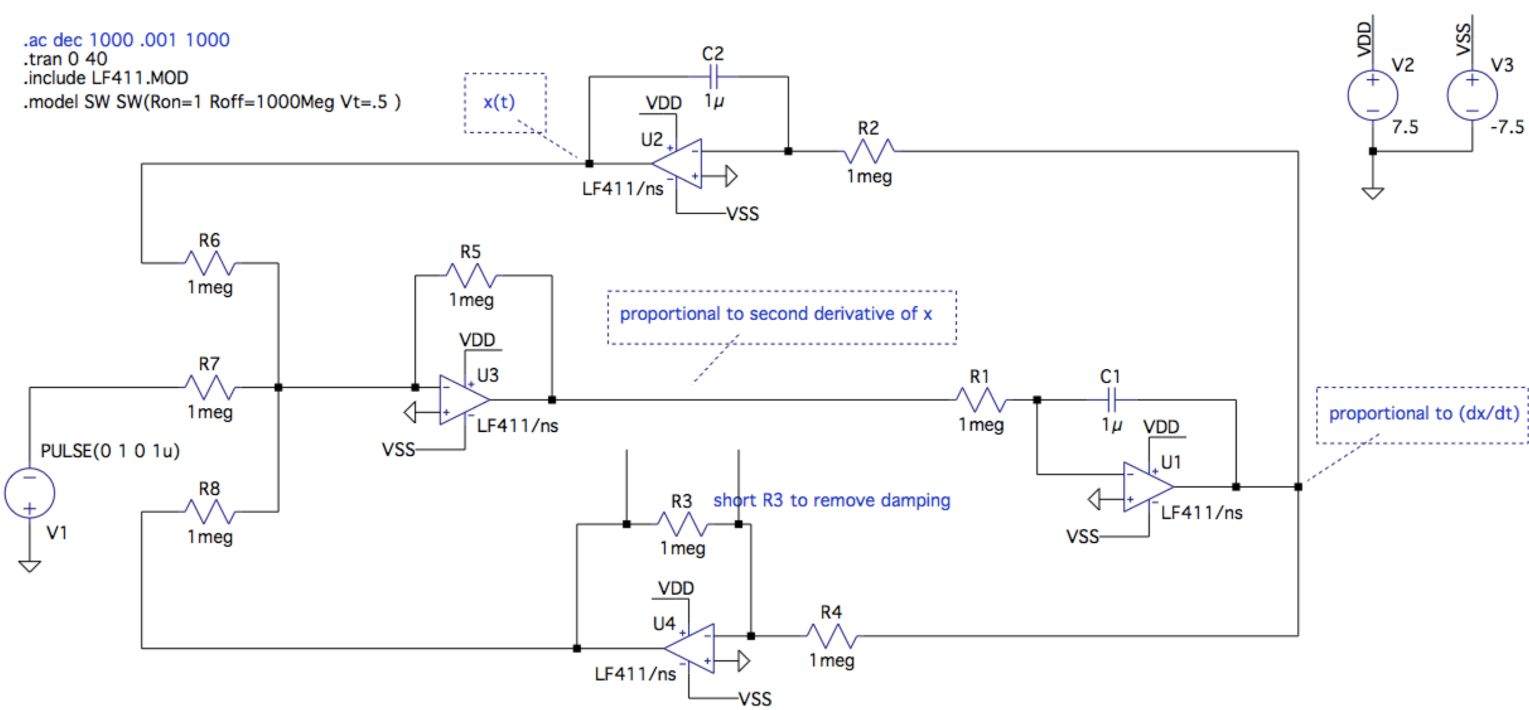
If the output of the second integrator of the analog computer in Experiment 3 is fed back to the input of its first integrator, a constant-coefficient second order differential equation can be simulated. Further, viewed in the frequency domain, it becomes a second order filter. In this experiment you'll investigate this circuit both ways.

We'll drop the correspondence between a real physical system (like the rocket) and this analog computer, focusing instead only on the characteristics of second order systems in the time and frequency domain. We could have used an RLC circuit instead of taking the opamp-based approach, but the analog computer is more versatile and inductors often are plagued by nonidealities.

Prelab: Read experiment, do calculations, and start simulations.

1. A second order analog computer (LTspice)

Let the output of opamp U2 be designated as x(t). Since U2 is part of an integrator, the input to this stage, which comes from U1, must be proportional to the time derivative of x(t). Applying similar reasoning, the output of U3 must be proportional to the second derivative of x(t). The circuit, as shown below, can then be analyzed as shown on the following page.



$$-x\,(R_5/R_6) + v_1\,(R_5/R_7) - R_2C_2\,x'\,(R_3/R_4)\,(R_5/R_8) = R_2C_2R_1C_1\,x''$$

If we let $R_1C_1 = \tau_1$,
$R_2C_2 = \tau_2$,
$R_5/R_6 = K_1$,
$R_5/R_7 = K_2$, and
$(R_3R_5)/(R_4R_8) = K_3$,

then we get $-K_1x + K_2v_1 - K_3\tau\,x' = \tau^2\,x''$

or $$x'' + (K_3 / \tau_1)\,x' + (K_1 / \tau_1\tau_2)\,x = (K_2 / \tau_1\tau_2)\,v_1.$$

Comparing this to one standard form for a second-order differential equation (see appendix), $x'' + (\omega_0 / Q)\,x' + \omega_0^2\,x = f(t)$,

we have $\omega_0 = (K_1 / \tau_1\tau_2)^{½}$ and $Q = 1 / 2\,\zeta = (K_1 / \tau_1\tau_2)^{½}\,\tau_1 / K_3$

where ω_0 is the critical (or undamped natural) frequency,

Q is the quality factor, and

ζ is the damping factor.

Note these equations simplify further if $\tau_1 = \tau_2$, as is the case in the circuit provided.

(a) Draft this circuit with the component values as shown. Run a transient sim to 35 seconds and plot x(t). Calculate the step response and check the simulation results against plots for the ideal responses expected from a second order system with the same ω_0 and damping factor. (Use Matlab, Wolfram Alpha, or some other means to get the ideal plots). Hint: See appendix for a discussion about calculating the natural and forced responses. Add the two and use the initial conditions to solve for the constants.
Report: Schematic, plots of x(t) (simulated & ideal), ω_0, ζ, Q, expression for step response.

(b) What does shorting R_3 do to the damping factor? Run the simulation again with R_3 shorted, plot x(t), and explain what you see. You may have to limit the rise time of the input source to 1µs or so to obtain convergence of the simulation. (Alternately, you can try to just make R_3 very small.)
Report: Plot of x(t), brief comment.

(c) Calculate the value required for R_3 to achieve critical damping (see appendix for the theory). Run simulations at the calculated value and at 10x and 0.1x the calculated value. Plot x(t) in each case, identifying overdamped, underdamped, and critically damped operation.
Report: Calculation of R_3 for critical damping, 3 plots for x(t) with labels.

(d) Increase ω_0 by a factor of 100, re-calculate the required R_3 to achieve critical damping, and rerun the simulations in (c) including the overdamped and

underdamped cases (with a reasonable time interval). Comment. Hint: to keep the resistor values reasonable for such op-amp circuits, you may want to modify both τ_1 and K_1 to achieve the new ω_0, and modify other element values too.
Report: Calculation of new parameters & element values, 3 plots with labels, and a brief comment.

(e) Keeping $\omega_0 = 100$, set R_3 for critical damping and run an AC simulation, plotting the frequency response at the point we have been calling x(t). Calculate the frequency response (Matlab, etc) and compare.
Report: Plots of simulated and calculated frequency response.

(f) Set R_3 to achieve Q=10 and rerun the simulation. Compare to (e).
Report: Calculation of R_3 & K_3, new simulation, brief comment.

(g) Simulate the frequency response at the output of U1. What kind of response is this?
Report: New simulation, brief comment.

(h) Verify the relationship between bandwidth, center frequency, and Q using the results of the previous step. (See appendix for the theory).
Report: Derivation of measured and calculated bandwidth.

(i) There is another frequency response hiding in this circuit. Find it.
Report: Brief comment.

2. Build it

Breadboard the circuit and repeat finding x(t) as done in parts (a) – (d), but this time using hardware measurements. Compare.

Practical notes:
step (a): to see the settling behavior on the scope, make v_1 a square wave of period around one minute, select an appropriate time scale on the scope, and use one-shot mode.
step (b): you have to excite the system to see it ring; if you don't, you'll just see DC at the output. Maybe the easiest way to do this is to set v_1 to a constant value (maybe 2 volts) and excite the system by briefly shorting C_2. Or you could add a switch at the input.
step (c): 411 opamps can't drive loads lower than 2kΩ, and feedback resistors greater than a few MΩ will cause you grief. Bear this in mind when setting the ratio R3/R4.
step (d): again, watch your loading. No resistors below 2kΩ.

Report: Photo of circuit, plots of responses, brief comments. For this part, making a reasonable attempt will be a significant component of the grade.

Appendix A: Background theory for second-order circuits

Consider a circuit whose behavior is governed by the following second-order differential equation, where y is a particular variable (e.g. element voltage or current) and y' represents $\frac{d}{dt}$y(t) :

$$y'' + 2\,\alpha\, y' + \omega_0^2\, y = f(t)$$

Two alternate forms that are sometimes used are:

$$y'' + 2\,\zeta\,\omega_0\, y' + \omega_0^2\, y = f(t)$$

$$y'' + (\omega_0 / Q)\, y' + \omega_0^2\, y = f(t)$$

Definitions:

f(t) - forcing function (determined by input)

ω_0 - undamped natural frequency

ζ - damping ratio

Q - quality factor

A.1 Natural Response

The solution can be separated into natural and forced responses. The natural response is a solution for f(t)=0, and it has three main cases:

Overdamped ($\zeta > 1$ or $\alpha > \omega_0$) : $y_n(t) = K_1\, e^{s_1 t} + K_2\, e^{s_2 t}$

Critically damped ($\zeta = 1$ or $\alpha = \omega_0$): $y_n(t) = K_1 t e^{-\alpha t} + K_2\, e^{-\alpha t}$

Underdamped ($\zeta < 1$ or $\alpha < \omega_0$) : $y_n(t) = e^{-\alpha t}\,(K_1 \cos \omega_d\, t + K_2 \sin \omega_d\, t)$

where

$$s_1, s_2 = -\,\alpha \pm (\alpha^2 - \omega_0^2)^{1/2}$$

$$\omega_d = (\omega_0^2 - \alpha^2)^{1/2}$$

K_1 , K_2 - constants determined by initial values for y and $\frac{d}{dt}$y

An underdamped response continually switches direction, a property that is often called *ringing*. A circuit is said to be *stable* if $y_n(t) \rightarrow 0$ as $t \rightarrow \infty$.

A.2 Forced Response

The forced response is a unique response determined by just the forcing function f(t). If the input f(t) is constant, the forced response is also a constant, and generally easy to solve for since all the time derivatives are zero. For stable circuits with a constant forcing function, the natural response fades and the forced response is the final value.

If the input f(t) is sinusoidal, the forced response has a sinusoidal solution that can be determined using phasors to represent the pure sinusoidal waveforms found throughout the circuit. Phasor analysis can be applied to the differential equation, or directly to the underlying circuit equations using element impedances. For stable circuits, after the natural response has faded this AC steady-state sinusoidal forced response is all that remains.

The ratio between the output and input phasor is a complex number $K(\omega)$ that is a function of angular frequency ω. At each frequency its magnitude tells us how much the amplitude is scaled from input to output, and its angle tells us how much the phase is shifted, assuming the input (and thus output) sinusoid has a frequency of ω. $K(\omega)$ is often called the frequency response.

Here we have $x'' + (K_3 / \tau_1)\, x' + (K_1 / \tau_1\tau_2)\, x = (K_2 / \tau_1\tau_2)\, v_1$, which for phasors becomes $-\omega^2 X(\omega)\, x'' + j\,\omega\, (K_3 / \tau_1)\, X(\omega) + (K_1 / \tau_1\tau_2)\, X(\omega) = (K_2 / \tau_1\tau_2)\, V_1(\omega)$. Since $K(\omega) = X(\omega) / V_1(\omega)$, it is straightforward to solve for the magnitude and phase of $K(\omega)$.

Linear circuits exhibit the property of superposition. Since input signals such as those representing speech or music can be broken down into a sum of sinusoidal components, the response of a linear circuit to such inputs can be determined by adding up the responses to each input component computed using the corresponding values of $K(\omega)$.

A.3 Resonant Filters

An underdamped second-order circuit has a frequency response $K(\omega)$ with a magnitude that exhibits a sharp peak or valley as shown in figure 2. Such circuits can thus be designed to pass or block signals within a particular range of frequencies. In this case they are often called *bandpass* or *band stop* filters, respectively. The center frequency is labeled ω_r and a common measure of the bandwidth is labeled B.

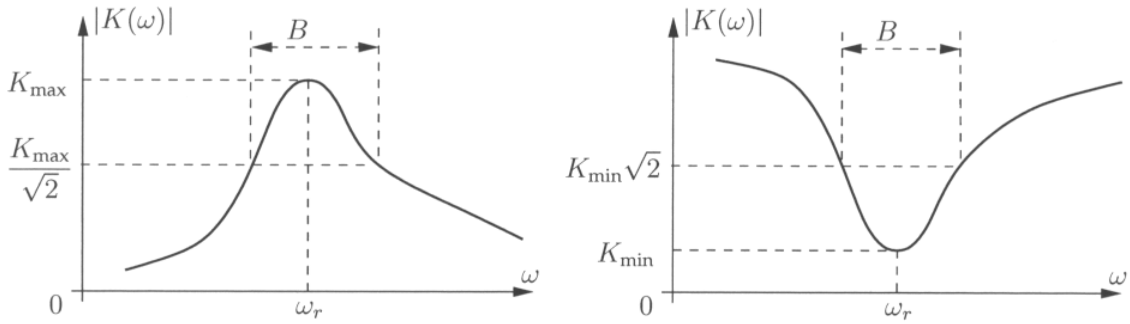


Figure 2. Magnitude frequency response of resonant bandpass and band stop filters

It turns out that $Q = (\omega_r) / B$, and thus Q can provide a measure of the frequency selectivity of the filter.

There is a fundamental connection between the frequency selectivity of a circuit and its "settling" behavior, e.g., its response to a step change in DC input. When such a change occurs, $y_n(t)$ fades more rapidly for larger values of α. Since $\alpha = \omega_0 / 2Q$, this means more rapid fading (e.g., less "ringing" for an underdamped circuit) corresponds with lower selectivity.