ALS (Alternating Least Squares)

$$\underbrace{\min_{X,Y} \sum_{(i,j)\in R} (r_{ij} - x_i^T y_j)^2 + \lambda \sum_i \|x_i\|^2 C_i + \lambda \sum_j \|y_j\|^2 C_j}_{P}$$

X – user matrix, Y – item matrix

$$\arg\min_{x_i} P = \arg\min_{x_i} \Big[\sum_{(i,j)\in R} \cancelto{0}{r_{ij}^2} - 2 \sum_{i,j} r_{ij} x_i^T y_j +$$

$$+ \sum_{i,j} (x_i^T y_j)^2 + \lambda (x_i, x_i) C_i + \lambda \cancelto{0}{(y_j, y_j) C_j} \Big] =$$

$$= \arg\min_{x_i} \Big[-2 x_i^T \sum_{i,j} r_{ij} y_j + \sum x_i^T y_j \cdot y_j^T x_i +$$

$$+ \lambda C_i x_i^T x_i \Big] =$$

$$= \arg\min_{x_i} \Big[\underbrace{-2 x_i^T \overbrace{\Big(\sum_{i,j} r_{ij} y_j \Big)}^{B_i}}_{-2x_i^T B_i} + \underbrace{x_i^T \overbrace{\Big(\sum_{i,j} y_j y_j^T + \lambda C_i \Big)}^{A_i} x_i}_{x_i^T A_i x_i} \Big]$$

$$\Rightarrow \arg\min_{x_i} (-2 x_i^T B_i + x_i^T A_i x_i)$$

$$\frac{\partial}{\partial x_i} (-2 x_i^T B_i + x_i^T A_i x_i) = 0 = -2B_i + 2A_i x_i$$

$$\frac{\partial}{\partial x_i} (-2 x_i^T B_i) = -2 B_i$$

$$\frac{\partial}{\partial x_i} (x_i^T A_i x_i) = 2 A_i x_i$$

$$\underbrace{A^{-1} A}_{=E} x = A^{-1} B \Rightarrow x = A^{-1} B$$

$$\Rightarrow \boxed{x_i = A_i^{-1} B_i = \Big(\sum y_j y_j^T + \lambda C_i \Big)^{-1} \Big(\sum_{i,j} r_{ij} y_j \Big)}$$

$$\hat{C}_i = \frac{(\sum_j c_{ij})^\alpha |\{i\}|}{\sum_i (\sum_j c_{ij})^\alpha}$$

iALS (IALS)

implicit

$$p_{ui} = \begin{cases} 1, & r_{ui} > 0 \\ 0, & r_{ui} \leq 0 \text{ or } r_{ui} \text{ - undefined} \end{cases}$$

p_{ui} – предпочтение (preference)

$$c_{ui} = 1 + \alpha |r_{ui}|$$

c_{ui} – степень увер-ти в p_{ui} (confidence)

$$\underset{x_i}{\operatorname{argmin}} \sum_{i,j} c_{ij} (p_{ij} - x_i^T y_j)^2 + \lambda \sum_i \|x_i\|^2 \hat{C}_i + \lambda \sum_j \|y_j\|^2 \hat{C}_j =$$

$$= \ldots \left(\sum_j c_{ij} y_j y_j^T + \lambda \hat{C}_i I \right)^{-1} \left(\sum_j c_{ij} p_{ij} y_j \right) =$$

$$= \ldots \Big(\sum_{j|p_{ij}=0} \underbrace{c_{ij}}_{=1} y_j y_j^T + \sum_{j|p_{ij}\neq 0} c_{ij} y_j y_j^T + \lambda \hat{C}_i I \Big) \times$$

$$\times \Big(\sum_{j|p_{ij}\neq 0} c_{ij} p_{ij} y_j \Big) =$$

$$= \ldots \Big(\sum_{j|p_{ij}=0} y_j y_j^T + \sum_{j|p_{ij}\neq 0} c_{ij} y_j y_j^T + \lambda \hat{C}_i I \Big) \times$$

$$\times \Big(\sum_{j|p_{ij}\neq 0} c_{ij} p_{ij} y_j \Big) =$$

$$\sum_j y_j y_j^T - \sum_{j|p_{ij}\neq 0} y_j y_j^T$$

$\cancel{j|p_{ij}\neq 0}$ j

$$= \left(\sum_{j} y_j y_j^T - \sum_{j \mid p_{ij} \neq 0} y_j y_j^T + \sum_{j \mid p_{ij} \neq 0} c_{ij} y_j y_j^T + \lambda \hat{C}_i I \right)^{-1} \times \left(\sum_{j \mid p_{ij} \neq 0} c_{ij} p_{ij} y_j \right) =$$

$$= \left(Y^T Y + \lambda \hat{C}_i I + \sum_{j \mid p_{ij} \neq 0} (c_{ij} - 1) y_j y_j^T \right)^{-1} \times \left(\sum_{j \mid p_{ij} \neq 0} c_{ij} p_{ij} y_j \right)$$